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Implicit first order differential systems with nonlocal conditions

Octavia Bolojan and **Radu Precup**✉

Babeş–Bolyai University, 1 M. Kogălniceanu Street, Cluj-Napoca, RO–400084, Romania

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Abstract. The present paper is devoted to the existence of solutions for implicit first order differential systems with nonlocal conditions expressed by continuous linear functionals. The lack of complete continuity of the associated integral operators, due to the implicit form of the equations, is overcome by using Krasnoselskii's fixed point theorem for the sum of two operators. Moreover, a vectorial version of Krasnoselskii's theorem and the technique based on vector-valued norms and matrices having the spectral radius less than one are likely to allow the system nonlinearities to behave independently as much as possible. In addition, the connection between the support of the nonlocal conditions and the constants from the growth conditions is highlighted.

Keywords: first order differential system, implicit differential equation, nonlocal condition, fixed point, vector-valued norm, spectral radius of a matrix.

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1 Introduction and preliminaries

The purpose of this paper is to obtain the existence of solutions to the nonlocal problem for a class of first order implicit differential systems

$$\begin{cases} x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)) \\ y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)) \\ x(0) = \alpha[x] \\ y(0) = \beta[y], \end{cases} \qquad (\text{on } [0,1]) \tag{1.1}$$

where $g_i, h_i \colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $\alpha, \beta \colon C[0,1] \to \mathbb{R}$ are continuous linear functionals with $\alpha[1] \neq 1$ and $\beta[1] \neq 1$.

In the recent years much attention has been given to different types of problems with nonlocal conditions. We refer to the bibliographies of the papers [2–4, 11, 13, 17–19, 25, 31, 32] for references. The motivation is that such problems arise from the mathematical modeling of

✉Corresponding author. Email: r.precup@math.ubbcluj.ro

real processes, such as heat, fluid, chemical or biological flow, where the nonlocal conditions can be seen as feedback controls, see for example [7] and the recent survey paper [28].

One can distinguish between discrete nonlocal conditions, or multi-point boundary conditions, and continuous conditions given by continuous linear functionals. For both types, it is important to take into consideration the interval on which a given condition really acts, that is, the support of that condition. More exactly, the *support* associated to the condition $x(0) = \alpha[x]$, where $\alpha : C[0,1] \to \mathbb{R}$ is linear, is the minimal closed subinterval $[0, t_0]$ of $[0,1]$ with the property

$$\alpha[x] = \alpha[y] \text{ whenever } x = y \text{ on } [0, t_0].$$

This notion was first introduced in [4] and has become essential for the existence theory of nonlocal problems. Indeed, as shown in [4,19,25], stronger conditions have to be satisfied by the nonlinear terms of the equations on $[0, t_0]$, compared to the hypotheses asked on $[t_0, 1]$. One may assert that the "integral" equation equivalent to a nonlocal problem on the interval $[0,1]$, is of Fredholm type on the support $[0, t_0]$ of the nonlocal condition, and of Volterra type on the remaining interval $[t_0, 1]$. In the present paper again we shall exploit this idea, even pregnantly. We shall do this, by considering a special norm on $C[0,1]$, namely

$$|x|_* = \max\left\{|x|_{C[0,t_0]}, |x|_{C_\theta[t_0,1]}\right\},$$

where $|\cdot|_{C[0,t_0]}$ is the usual max norm on $C[0, t_0]$,

$$|x|_{C[0,t_0]} = \max_{t\in[0,t_0]} |x(t)|,$$

while $|x|_{C_\theta[t_0,1]}$ denotes the Bielecki type norm on $C[t_0, 1]$,

$$|x|_{C_\theta[t_0,1]} = \max_{t\in[t_0,1]} |x(t)| e^{-\theta(t-\eta)}.$$

Here $\eta < t_0$ and $\theta > 0$ are given numbers. As we shall see, the joint role of the parameters η (any fixed number with $\eta < t_0$) and θ (chosen large enough) is to weaken the assumptions on $g_1(t, x, y), g_2(t, x, y)$ when $t \in [t_0, 1]$.

Note a key property of the functional α in connection with its corresponding support,

$$|\alpha[x]| \le \|\alpha\| \, |x|_{C[0,t_0]},$$

for every $x \in C[0,1]$, when normally, for any continuous linear functional $\alpha\colon C[0,1] \to \mathbb{R}$, we have $|\alpha[x]| \le \|\alpha\| \, |x|_{C[0,1]}$, where the notation $\|\alpha\|$ is used to denote the norm of the continuous linear functional α.

A standard technique for nonlocal problems, as like for boundary value problems in general, is the reduction of the problem to a fixed point problem for a suitable integral type operator. Then a fixed point theorem guarantees the existence of a solution. Topological fixed point theorems, as well as index theory, are essentially based on compactness, which in case of explicit equations usually holds, while for implicit equations it becomes a problem. In order to overcome this difficulty, one may think to use Krasnoselskii's fixed point theorem for a sum of two operators, a contraction and a completely continuous mapping. Krasnoselskii's theorem [14,15] (see also [10]) has become a basic result of the nonlinear analysis with a large number of applications to nonlinear operator, integral, and differential equations. Combining Banach's contraction principle and Schauder's fixed point theorem, it can be considered a bridge between metrical and topological fixed point theories. There is a rich literature concerning different generalizations and applications of this theorem. Here are some of them: [1,5,6,9,12,20,21,27,30].

Nonlocal problems for implicit differential equations have been, by our knowledge, less investigated. We just mention the paper [16] on the monotone iterative technique for an implicit first order equation subject to a two-point boundary condition, and the paper [8] whose aim is to obtain a Carathéodory solution for an implicit first order equation under a nonlocal condition, via Schauder's fixed point theorem and Kolmogorov's compactness criterion in L^1.

Since we are interested here in systems of equations, we have opted for a vectorial approach based on the use of vector-valued norms, inverse-positive matrices and of a vectorial version of Krasnoselskii's fixed point theorem for sums of two operators. The vectorial approach allows the system nonlinearities to behave independently as much as possible.

We recall now some basic notions which are involved in our vectorial setting. By a *vector-valued metric* on a set X we mean a mapping $d\colon X \times X \to \mathbb{R}^n_+$ such that (i) $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$ for all $x, y \in X$ and (iii) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$. Here by $\le$ we mean the natural componentwise order relation of $\mathbb{R}^n$, more exactly, if $r, s \in \mathbb{R}^n$, $r = (r_1, r_2, \dots, r_n)$, $s = (s_1, s_2, \dots, s_n)$, then by $r \le s$ one means that $r_i \le s_i$ for $i = 1, 2, \dots, n$.

A set X together with a vector-valued metric d is called a *generalized metric space*. For such a space, the notions of Cauchy sequence, convergence, completeness, open and closed set are similar to those in usual metric spaces.

Similarly, we speak about a *vector-valued norm* on a linear space X, as being a mapping $\|\cdot\| : X \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda| \, \|x\|$ for $x \in X$, $\lambda \in \mathbb{R}$, and $\|x + y\| \le \|x\| + \|y\|$ for every $x, y \in X$. To any vector-valued norm $\|\cdot\|$ one can associate the vector-valued metric $d\,(x,y) := \|x - y\|$, and one says that $(X, \|\cdot\|)$ is a *generalized Banach space* if X is complete with respect to d.

If (X, d) is a generalized metric space and $T\colon X \to X$ is any mapping, we say that T is a *generalized contraction* (in Perov's sense) provided that a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ exists such that its powers M^k tend to the zero matrix 0 as $k \to \infty$, and

$$d(T(x), T(y)) \le M d(x,y) \quad \text{for all } x, y \in X.$$

Here and throughout the paper, the vectors in $\mathbb{R}^n$ are seen as column matrices.

There are several characterizations known of the matrices M with $M^k \to 0$ as $k \to \infty$ (see [23] and [29, pp. 12, 88]). More exactly, for a matrix $M \in M_{n\times n}(\mathbb{R}_+)$, the following statements are equivalent:

(a) $M^k \to 0$ as $k \to \infty$;

(b) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \cdots$ (where I stands for the unit matrix of the same order as M);

(c) the eigenvalues of M are located inside the unit disc of the complex plane, i.e. $\rho(M) < 1$, where $\rho(M)$ is the *spectral radius* of M;

(d) $I - M$ is nonsingular and inverse-positive, i.e. $(I - M)^{-1}$ has nonnegative entries.

Let us note that for a square matrix $M \in M_{2\times 2}(\mathbb{R}_+)$ of order 2,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

one has $\rho(M) < 1$ if and only if

$$a + d < \min\{2,\ 1 + ad - bc\}. \tag{1.2}$$

The following almost obvious lemma will be used in the sequel.

Lemma 1.1. *If $A \in M_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(A) < 1$, then $\rho(A+B) < 1$ for every matrix $B \in M_{n\times n}(\mathbb{R}_+)$ whose elements are small enough.*

The role of matrices with spectral radius less than one in the study of semilinear operator systems was pointed out in [24], in connection with several abstract results from nonlinear functional analysis. Thus, Banach's contraction principle admits a vectorial version in terms of generalized contractions in Perov's sense.

Following Perov's approach, in [30] (see also [22]) the following vectorial version of Krasnoselskii's fixed point theorem for a sum of two operators was obtained.

Theorem 1.2 ([30]). *Let $(X, \|\cdot\|)$ be a generalized Banach space, D a nonempty closed bounded convex subset of X and $T : D \to X$ such that:*

(i) $T = G + H$ with $G\colon D \to X$ completely continuous and $H\colon D \to X$ a generalized contraction, i.e. there exists a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ with $\rho(M) < 1$, such that $\|H(x) - H(y)\| \le M\|x - y\|$ for all $x, y \in D$;

(ii) $G(x) + H(y) \in D$ for all $x, y \in D$.

Then T has at least one fixed point in D.

The proofs of the vectorial versions of the Banach and Krasnoselskii theorems follow the same ideas as for the original results. However, for applications to systems, these versions allow nonlinearities to behave independently one to each other, and differently with respect to the system variables.

2 Main result

In order to obtain the equivalent integral form of the problem (1.1), denote

$$u(t) = x'(t), \qquad v(t) = y'(t).$$

Then, using the nonlocal conditions we obtain

$$\begin{aligned} x(t) &= \frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds, \\ y(t) &= \frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^t v(s)\,ds. \end{aligned}$$

Let

$$G_1(u,v)(t) = g_1\Big(t,\ \frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds,\ \frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)ds\right] + \int_0^t v(s)\,ds\Big),$$

$$G_2(u,v)(t) = g_1\Big(t,\ \frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds,\ \frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^t v(s)\,ds\Big).$$

Also define

$$H_1(u,v)(t) = h_1(t,u(t),v(t)),$$
$$H_2(u,v)(t) = h_2(t,u(t),v(t)).$$

Then the problem (1.1) is equivalent to the system

$$\begin{cases} u = G_1(u,v) + H_1(u,v) \\ v = G_2(u,v) + H_2(u,v). \end{cases} \tag{2.1}$$

Note that we look for solutions with $x,y \in C^1[0,1]$, i.e. $(x,y) \in C^1([0,1],\mathbb{R}^2)$, and so $u,v \in C[0,1]$, that is $(u,v) \in C([0,1],\mathbb{R}^2)$. The system (2.1) appears as a fixed point problem for the operator

$$T\colon C([0,1],\mathbb{R}^2) \to C([0,1],\mathbb{R}^2), \qquad T = (T_1,T_2),$$

where T_1, T_2 are given by

$$\begin{aligned} T_1(u,v) &= G_1(u,v) + H_1(u,v), \\ T_2(u,v) &= G_2(u,v) + H_2(u,v). \end{aligned} \tag{2.2}$$

We can rewrite (2.2) in a vectorial form, as a sum of two operators, namely

$$T(u,v) = G(u,v) + H(u,v),$$

where

$$T(u,v) = \begin{bmatrix} T_1(u,v) \\ T_2(u,v) \end{bmatrix}, \qquad G(u,v) = \begin{bmatrix} G_1(u,v) \\ G_2(u,v) \end{bmatrix}, \qquad H(u,v) = \begin{bmatrix} H_1(u,v) \\ H_2(u,v) \end{bmatrix}.$$

We shall assume that the nonlocal conditions expressed by the functionals α, β have the same support $[0,t_0]$, and that the growth of $g_1(t,u,v)$, $g_2(t,u,v)$ with respect to u and v is at most linear, on each of the two subintervals $[0,t_0]$ and $[t_0,1]$, that is

$$\begin{aligned} |g_1(t,u,v)| &\le \begin{cases} a_1|u| + b_1|v| + c_1 & \text{for } t \in [0,t_0) \\ A_1|u| + B_1|v| + C_1 & \text{for } t \in [t_0,1] \end{cases} \\ |g_2(t,u,v)| &\le \begin{cases} a_2|u| + b_2|v| + c_2 & \text{for } t \in [0,t_0) \\ A_2|u| + B_2|v| + C_2 & \text{for } t \in [t_0,1], \end{cases} \end{aligned} \tag{2.3}$$

for all $(u,v) \in \mathbb{R}^2$, and the functions $h_1,\ h_2$ satisfy the Lipschitz conditions

$$\begin{aligned} |h_1(t,u,v) - h_1(t,\overline{u},\overline{v})| &\le \overline{a}_1|u-\overline{u}| + \overline{b}_1|v-\overline{v}| \\ |h_2(t,u,v) - h_2(t,\overline{u},\overline{v})| &\le \overline{a}_2|u-\overline{u}| + \overline{b}_2|v-\overline{v}|, \end{aligned} \tag{2.4}$$

for all $(u,v), (\overline{u},\overline{v}) \in \mathbb{R}^2$ and $t \in [0,1]$. Here, for $i = 1,2$, $a_i, b_i, c_i > 0$ and $A_i, B_i, C_i, \overline{a}_i, \overline{b}_i$ are nonnegative numbers.

Denote

$$A_\alpha = \frac{\|\alpha\|}{|1-\alpha[1]|} + 1, \qquad B_\beta = \frac{\|\beta\|}{|1-\beta[1]|} + 1,$$

and consider the matrices

$$M_0 = \begin{bmatrix} a_1 t_0 A_\alpha & b_1 t_0 B_\beta \\ a_2 t_0 A_\alpha & b_2 t_0 B_\beta \end{bmatrix}, \qquad M_1 = \begin{bmatrix} \overline{a}_1 & \overline{b}_1 \\ \overline{a}_2 & \overline{b}_2 \end{bmatrix}.$$

With those notations, we can state and prove our main existence result.

Theorem 2.1. *Assume that g_1, g_2 satisfy (2.3) and h_1, h_2 satisfy (2.4). If the spectral radius of the matrix $M_0 + M_1$ is less than one, then problem (2.1) has at least one solution $(x,y) \in C^1([0,1], \mathbb{R}^2)$.*

Proof. We shall apply the vectorial version of Krasnoselskii's fixed point theorem to the space $X = C([0,1], \mathbb{R}^2)$, endowed with the vector-valued norm $\|\cdot\|_{C([0,1],\mathbb{R}^2)}$ defined by

$$\|w\|_{C([0,1],\mathbb{R}^2)} = \begin{bmatrix} |x|_* \\ |y|_* \end{bmatrix},$$

for $w = (x,y) \in C([0,1], \mathbb{R}^2)$.

Step 1. The operator G is completely continuous. This follows from the continuity of g_1, g_2 and the fact that the terms $\int_0^t u(s)\,ds$, $\int_0^t v(s)\,ds$ guarantee the equicontinuity in the Arzelà–Ascoli theorem.

Step 2. The operator H is a generalized contraction. To show this, let $(u,v), (\overline{u},\overline{v}) \in C([0,1], \mathbb{R}^2)$ be arbitrary. Using the assumption (2.4), for $t \in [0,t_0]$, we deduce that

$$\begin{aligned} |H_1(u,v)(t) - H_1(\overline{u},\overline{v})(t)| &= |h_1(t,u(t),v(t)) - h_1(t,\overline{u}(t),\overline{v}(t))| \\ &\le \overline{a}_1 |u(t) - \overline{u}(t)| + \overline{b}_1 |v(t) - \overline{v}(t)| \\ &\le \overline{a}_1 |u - \overline{u}|_{C[0,t_0]} + \overline{b}_1 |v - \overline{v}|_{C[0,t_0]} \end{aligned}$$

and taking the supremum for $t \in [0,t_0]$, we obtain

$$|H_1(u,v) - H_1(\overline{u},\overline{v})|_{C[0,t_0]} \le \overline{a}_1 |u - \overline{u}|_{C[0,t_0]} + \overline{b}_1 |v - \overline{v}|_{C[0,t_0]}. \tag{2.5}$$

Next, for $t \in [t_0,1]$, we obtain

$$\begin{aligned} &|H_1(u,v)(t) - H_1(\overline{u},\overline{v})(t)| \\ &\quad \le \overline{a}_1 |u(t) - \overline{u}(t)| + \overline{b}_1 |v(t) - \overline{v}(t)| \\ &\quad = \overline{a}_1 |u(t) - \overline{u}(t)| e^{-\theta(t-\eta)} e^{\theta(t-\eta)} + \overline{b}_1 |v(t) - \overline{v}(t)| e^{-\theta(t-\eta)} e^{\theta(t-\eta)} \\ &\quad \le \overline{a}_1 e^{\theta(t-\eta)} |u - \overline{u}|_{C_\theta[t_0,1]} + \overline{b}_1 e^{\theta(t-\eta)} |v - \overline{v}|_{C_\theta[t_0,1]}. \end{aligned}$$

Dividing by $e^{\theta(t-\eta)}$ and taking the supremum when $t \in [t_0,1]$, we have

$$|H_1(u,v) - H_1(\overline{u},\overline{v})|_{C_\theta[t_0,1]} \le \overline{a}_1 |u - \overline{u}|_{C_\theta[t_0,1]} + \overline{b}_1 |v - \overline{v}|_{C_\theta[t_0,1]}. \tag{2.6}$$

Taking into consideration (2.5) and (2.6), we see that

$$|H_1(u,v) - H_1(\overline{u},\overline{v})|_* \le \overline{a}_1 |u - \overline{u}|_* + \overline{b}_1 |v - \overline{v}|_*. \tag{2.7}$$

Similarly,

$$|H_2(u,v) - H_2(\overline{u},\overline{v})|_* \leq \overline{a}_2 |u - \overline{u}|_* + \overline{b}_2 |v - \overline{v}|_* . \tag{2.8}$$

The inequalities (2.7) and (2.8) can be put together under the vectorial form

$$\begin{bmatrix} |H_1(u,v) - H_1(\overline{u},\overline{v})|_* \\ |H_2(u,v) - H_2(\overline{u},\overline{v})|_* \end{bmatrix} \leq M_1 \begin{bmatrix} |u - \overline{u}|_* \\ |v - \overline{v}|_* \end{bmatrix}$$

or equivalently,

$$\|H(z) - H(\overline{z})\|_{C([0,1],\mathbb{R}^2)} \leq M_1 \|z - \overline{z}\|_{C([0,1],\mathbb{R}^2)} , \tag{2.9}$$

for $z = (u,v)$, $\overline{z} = (\overline{u},\overline{v})$. Since by our assumption, $(M_0 + M_1)^k \to 0$ as $k \to \infty$, and $M_1 \leq M_0 + M_1$, one also has that $M_1^k \to 0$ as $k \to \infty$. Hence H is a generalized contraction in Perov's sense.

Step 3. In what follows, we look for a nonempty, bounded, closed and convex subset D of $C\left([0,1], \mathbb{R}^2\right)$ such that $G(D) + H(D) \subset D$. To this end, we shall first estimate G. Let (u,v) be any element of $C\left([0,1], \mathbb{R}^2\right)$. For $t \in [0,t_0]$, using (2.3) we obtain

$$\begin{aligned}
&|G_1(u,v)(t)| \\
&\quad= \left| g_1\left(t, \frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds, \frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^t v(s)\,ds\right)\right| \\
&\quad\leq a_1\left|\frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds\right| + b_1\left|\frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^t v(s)\,ds\right| + c_1 \\
&\quad\leq a_1\left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1\right)\int_0^{t_0} |u(s)|\,ds + b_1\left(\frac{\|\beta\|}{|1-\beta[1]|} + 1\right)\int_0^{t_0} |v(s)|\,ds + c_1 \\
&\quad\leq a_1 t_0 A_\alpha |u|_{C[0,t_0]} + b_1 t_0 B_\beta |v|_{C[0,t_0]} + c_1.
\end{aligned}$$

Taking the supremum, we have

$$|G_1(u,v)|_{C[0,t_0]} \leq a_1 t_0 A_\alpha |u|_{C[0,t_0]} + b_1 t_0 B_\beta |v|_{C[0,t_0]} + c_1. \tag{2.10}$$

Furthermore, for $t \in [t_0, 1]$, we have that

$$\begin{aligned}
&|G_1(u,v)(t)| \\
&\quad\leq A_1\left|\frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^t u(s)\,ds\right| \\
&\qquad + B_1\left|\frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^t v(s)\,ds\right| + C_1 \\
&\quad\leq A_1\left|\frac{1}{1-\alpha[1]}\alpha\left[\int_0^{\cdot} u(s)\,ds\right] + \int_0^{t_0} u(s)\,ds\right| \\
&\qquad + B_1\left|\frac{1}{1-\beta[1]}\beta\left[\int_0^{\cdot} v(s)\,ds\right] + \int_0^{t_0} v(s)\,ds\right| + C_1 \\
&\qquad + A_1\left|\int_{t_0}^t u(s)\,ds\right| + B_1\left|\int_{t_0}^t v(s)\,ds\right|.
\end{aligned}$$

Then

$$\begin{aligned}
&|G_1(u,v)(t)|\\
&\quad\leq A_1 t_0 A_\alpha \, |u|_{C[0,t_0]} + B_1 t_0 B_\beta \, |v|_{C[0,t_0]} + C_1\\
&\qquad + A_1 \int_{t_0}^{t} |u(s)| \, e^{-\theta(s-\eta)} e^{\theta(s-\eta)} \, ds + B_1 \int_{t_0}^{t} |v(s)| \, e^{-\theta(s-\eta)} e^{\theta(s-\eta)} \, ds\\
&\quad\leq A_1 t_0 A_\alpha \, |u|_{C[0,t_0]} + B_1 t_0 B_\beta \, |v|_{C[0,t_0]} + C_1\\
&\qquad + \frac{A_1}{\theta} e^{\theta(t-\eta)} \, |u|_{C_\theta[t_0,1]} + \frac{B_1}{\theta} e^{\theta(t-\eta)} \, |v|_{C_\theta[t_0,1]} .
\end{aligned}$$

Dividing by $e^{\theta(t-\eta)}$ and taking the supremum when $t \in [t_0, 1]$, we obtain

$$\begin{aligned}
|G_1(u,v)|_{C_\theta[t_0,1]} \leq & \left(A_1 t_0 A_\alpha \, |u|_{C[0,t_0]} + B_1 t_0 B_\beta \, |v|_{C[0,t_0]} + C_1 \right) e^{-\theta(t_0-\eta)}\\
&+ \frac{A_1}{\theta} \, |u|_{C_\theta[t_0,1]} + \frac{B_1}{\theta} \, |v|_{C_\theta[t_0,1]} .
\end{aligned} \tag{2.11}$$

Now we can take advantage from the special choice of the norm $|\cdot|_{C_\theta[t_0,1]}$, more exactly from the choice of $\eta < t_0$, to assume (choosing large enough $\theta > 0$) that

$$A_1 e^{-\theta(t_0-\eta)} \leq a_1, \qquad B_1 e^{-\theta(t_0-\eta)} \leq b_1 \qquad \text{and} \qquad C_1 e^{-\theta(t_0-\eta)} \leq c_1, \tag{2.12}$$

and this way to eliminate from the first part of our estimation the growth constants A_1, B_1, C_1. Indeed, (2.11) and (2.12) give

$$|G_1(u,v)|_{C_\theta[t_0,1]} \leq a_1 t_0 A_\alpha \, |u|_{C[0,t_0]} + b_1 t_0 B_\beta \, |v|_{C[0,t_0]} + c_1 + \frac{A_1}{\theta} \, |u|_{C_\theta[t_0,1]} + \frac{B_1}{\theta} \, |v|_{C_\theta[t_0,1]} . \tag{2.13}$$

Now, (2.10) and (2.13) imply that

$$|G_1(u,v)|_* \leq \left(a_1 t_0 A_\alpha + \frac{A_1}{\theta} \right) |u|_* + \left(b_1 t_0 B_\beta + \frac{B_1}{\theta} \right) |u|_* + c_1. \tag{2.14}$$

Similarly,

$$|G_2(u,v)|_* \leq \left(a_2 t_0 A_\alpha + \frac{A_2}{\theta} \right) |u|_* + \left(b_2 t_0 B_\beta + \frac{B_2}{\theta} \right) |u|_* + c_2. \tag{2.15}$$

The inequalities (2.14) and (2.15) can be put under the vectorial form

$$\begin{bmatrix} |G_1(u,v)|_* \\ |G_2(u,v)|_* \end{bmatrix} \leq M_\theta \begin{bmatrix} |u|_* \\ |v|_* \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

or, using the vector-valued norm, equivalently,

$$\|G(u,v)\|_{C([0,1],\mathbb{R}^2)} \leq M_\theta \, \|(u,v)\|_{C([0,1],\mathbb{R}^2)} + c, \tag{2.16}$$

where $c := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and

$$M_\theta := \begin{bmatrix} a_1 t_0 A_\alpha + \frac{A_1}{\theta} & b_1 t_0 B_\beta + \frac{B_1}{\theta} \\ a_2 t_0 A_\alpha + \frac{A_2}{\theta} & b_2 t_0 B_\beta + \frac{B_2}{\theta} \end{bmatrix}.$$

Clearly $M_\theta = M_0 + M_2$, where

$$M_2 = \begin{bmatrix} \frac{A_1}{\theta} & \frac{B_1}{\theta} \\ \frac{A_2}{\theta} & \frac{B_2}{\theta} \end{bmatrix}.$$

On the other hand, from (2.9), we deduce that

$$\|H(\overline{u},\overline{v})\|_{C([0,1],\mathbb{R}^2)} \leq M_1 \|(\overline{u},\overline{v})\|_{C([0,1],\mathbb{R}^2)} + d, \tag{2.17}$$

for every $(\overline{u},\overline{v}) \in C\left([0,1],\mathbb{R}^2\right)$, where

$$d = \|H(0,0)\|_{C([0,1],\mathbb{R}^2)}.$$

Now we look for the set

$$D = \left\{(u,v) \in C\left([0,1],\mathbb{R}^2\right) : \|(u,v)\|_{C([0,1],\mathbb{R}^2)} \leq R\right\},$$

with $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, $R_1 \geq 0$, $R_2 \geq 0$. According to the estimations (2.16) and (2.17), the condition $G(D) + H(D) \subset D$ is satisfied provided that

$$(M_\theta + M_1) R + c + d \leq R,$$

equivalently

$$c + d \leq (I - M_\theta - M_1) R. \tag{2.18}$$

Since $M_\theta + M_1 = M_0 + M_1 + M_2$, $\rho(M_0 + M_1) < 1$ and the entries of M_2 are as small as desired for large enough $\theta > 0$, from Lemma 1.1, we can choose θ such that

$$\rho(M_\theta + M_1) < 1.$$

Then, according to the property (d) of matrices with spectral radius less than one, the inequality (2.18) is equivalent to

$$R \geq (I - M_\theta - M_1)^{-1}(c + d).$$

This proves the existence of radii $R_1, R_2 \geq 0$ for which the inwardness condition $G(D) + H(D) \subset D$ is satisfied. Thus Theorem 1.2 applies and guarantees the existence in D of at least one fixed point for T. □

Remark 2.2. It is worth to underline the exclusive contribution to the matrix $M_0 + M_1$ of the growth constants a_1, a_2, b_1, b_2 corresponding to the support interval $[0, t_0]$, in contrast to the constants A_1, A_2, B_1, B_2 which are not involved in any conditions.

Remark 2.3. In view of (1.2), the spectral radius of the matrix $M_0 + M_1$ is less than one if the following inequality holds:

$$\begin{aligned} &a_1 t_0 A_\alpha + \overline{a}_1 + b_2 t_0 B_\beta + \overline{b}_2 \\ &\quad < \min\left\{2, 1 + (a_1 t_0 A_\alpha + \overline{a}_1)\left(b_2 t_0 B_\beta + \overline{b}_2\right) - \left(b_1 t_0 B_\beta + \overline{b}_1\right)(a_2 t_0 A_\alpha + \overline{a}_2)\right\}. \end{aligned}$$

We note that Theorem 2.1 can be easily extended to general n-dimensional systems. In that case, the assumption about the spectral radius of the corresponding matrix of order n can be checked using computer algebra programs such as Maple and Mathematica.

We conclude this paper by an example illustrating our main result.

Example 2.4. Consider the problem

$$\begin{cases} x' = a\,(t)\,x \sin\,(x+y) + b\,(t)\,y\cos\,(x-y) + m\,(t)\sin x' + n\,(t)\,y' + f_1(t) \\ y' = c\,(t)\,x\cos\,(x+y) + d\,(t)\,y\sin\,(x-y) + \cos(p\,(t)\,x' + q\,(t)\,y') + f_2(t) \\ x(0) = \int_0^{1/2} x(s)\,ds, \quad y(0) = \int_0^{1/2} y(s)\,ds \end{cases} \tag{2.19}$$

where $a, b, c, d, m, n, p, q, f_1, f_2 \in C[0,1]$. In this case,

$$\begin{aligned} g_1(t,u,v) &= a\,(t)\,u\sin\,(u+v) + b\,(t)\,v\cos\,(u-v) + f_1(t) \\ g_2(t,u,v) &= c\,(t)\,u\cos\,(u+v) + d\,(t)\,v\sin\,(u-v) + f_2(t) \\ h_1(t,u,v) &= m\,(t)\sin u + n\,(t)\,v \\ h_2(t,u,v) &= \cos(p\,(t)\,u + q\,(t)\,v) \end{aligned}$$

and we have $t_0 = 1/2$ and g_1, g_2 satisfy (2.3) with

$$\begin{aligned} a_1 &= |a|_{C[0,1/2]}\,, & b_1 &= |b|_{C[0,1/2]}\,, & c_1 &= |f_1|_{C[0,1/2]}\,, \\ a_2 &= |c|_{C[0,1/2]}\,, & b_2 &= |d|_{C[0,1/2]}\,, & c_2 &= |f_2|_{C[0,1/2]}\,, \\ A_1 &= |a|_{C[1/2,1]}\,, & B_1 &= |b|_{C[1/2,1]}\,, & C_1 &= |f_1|_{C[1/2,1]}\,, \\ A_2 &= |c|_{C[1/2,1]}\,, & B_2 &= |d|_{C[1/2,1]}\,, & C_2 &= |f_2|_{C[1/2,1]}\,. \end{aligned}$$

Also, h_1, h_2 satisfy (2.4) with

$$\overline{a}_1 = |m|_{C[0,1]}\,, \qquad \overline{b}_1 = |n|_{C[0,1]}\,, \qquad \overline{a}_2 = |p|_{C[0,1]}\,, \qquad \overline{b}_2 = |q|_{C[0,1]}\,.$$

In addition,

$$\|\alpha\| = \|\beta\| = \alpha[1] = \beta\,[1] = 1/2.$$

For this example

$$\begin{aligned} M_0 + M_1 &= \begin{bmatrix} a_1 + \overline{a}_1 & b_1 + \overline{b}_1 \\ a_2 + \overline{a}_2 & b_2 + \overline{b}_2 \end{bmatrix} \\ &= \begin{bmatrix} |a|_{C[0,1/2]} + |m|_{C[0,1]} & |b|_{C[0,1/2]} + |n|_{C[0,1]} \\ |c|_{C[0,1/2]} + |p|_{C[0,1]} & |d|_{C[0,1/2]} + |q|_{C[0,1]} \end{bmatrix}. \end{aligned}$$

Therefore, according to Theorem 2.1 and Remark 2.3, if

$$a_1 + \overline{a}_1 + \; b_2 + \overline{b}_2 < \min\left\{2,\; 1 + (a_1 + \overline{a}_1)\left(b_2 + \overline{b}_2\right) - (a_2 + \overline{a}_2)\left(\,b_1 + \overline{b}_1\right)\right\}, \tag{2.20}$$

then the problem (2.19) has at least one solution.

Here are three particular cases:

(1^0) Assume that $a_1 = a_2$, $\overline{a}_1 = \overline{a}_2$ and $b_1 = b_2$, $\overline{b}_1 = \overline{b}_2$. Then the sufficient condition of existence (2.20) reduces to

$$a_1 + \overline{a}_1 + b_1 + \overline{b}_1 < 1,$$

that is

$$|a|_{C[0,1/2]} + |b|_{C[0,1/2]} + |m|_{C[0,1]} + |n|_{C[0,1]} < 1.$$

(2^0) Assume that $a_1 = b_2,\ \overline{a}_1 = \overline{b}_2$ and $b_1 = a_2,\ \overline{b}_1 = \overline{a}_2$. Then (2.20) becomes

$$a_1 + \overline{a}_1 + b_1 + \overline{b}_1 < 1,$$

that is

$$|a|_{C[0,1/2]} + |b|_{C[0,1/2]} + |m|_{C[0,1]} + |n|_{C[0,1]} < 1.$$

(3^0) Assume that $a_1 = b_1 = b_2$ and $\overline{a}_1 = \overline{b}_1 = \overline{b}_2$. Then (2.20) is equivalent to

$$a_1 + \overline{a}_1 + \sqrt{(a_1 + \overline{a}_1)\,(a_2 + \overline{a}_2)} < 1,$$

or more explicitly

$$|a|_{C[0,1/2]} + |m|_{C[0,1]} + \sqrt{\left(|a|_{C[0,1/2]} + |m|_{C[0,1]}\right)\left(|c|_{C[0,1/2]} + |p|_{C[0,1]}\right)} < 1.$$

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