



## Boundedness character of a max-type system of difference equations of second order

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Received 13 April 2014, appeared 20 September 2014

Communicated by Josef Diblík

**Abstract.** The boundedness character of positive solutions of the next max-type system of difference equations

$$x_{n+1} = \max \left\{ A, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0,$$

with  $\min\{A, p, q\} > 0$ , is characterized.

**Keywords:** max-type system of difference equations, positive solutions, bounded solutions, unbounded solutions.

**2010 Mathematics Subject Classification:** 39A10, 39A20.

### 1 Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in last few decades (see, e.g., [1–47]). Some of the systems that are of interest are symmetric or those obtained from symmetric by modifications of their parameters (see, for example, [5, 9, 13–19, 22, 23, 36, 39–44] and the related references therein). Another subarea, of interest deals with *max-type* difference equations and systems (see, for example, [1, 7, 10–12, 17, 19, 21, 28–35, 38, 40, 42, 43, 45–47] and the related references therein). However, there are only a few papers which belong to both areas (see [17, 19, 21, 40, 42, 43]). Although majority of the papers in the area treat equations or systems with integer powers of their variables, there are some papers on equations or systems with non-integer powers of their variables (see, for example, [3, 4, 8, 20, 27–33, 35, 47]). Paper [29] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

$$x_{n+1} = \max \left\{ a, \frac{x_n^p}{x_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

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with  $\min\{a, p\} > 0$ .

Motivated by [29], in [43], S. Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equations

$$x_{n+1} = \max \left\{ a, \frac{y_n^p}{x_{n-1}^p} \right\}, \quad y_{n+1} = \max \left\{ a, \frac{x_n^p}{y_{n-1}^p} \right\}, \quad n \in \mathbb{N}_0, \quad (1.2)$$

with  $\min\{a, p\} > 0$ .

For related max-type difference equations see also [28,30,33,35].

Here we continue the line of investigations by studying the boundedness character of positive solutions of the next system of max-type difference equations

$$x_{n+1} = \max \left\{ A, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0, \quad (1.3)$$

where  $\min\{A, p, q\} > 0$ .

Two of our results (Theorem 2.2 and Theorem 2.4) are natural extensions of the results on the boundedness character of positive solutions of system (1.2) appearing in [43]. For the other two results (Theorem 2.1 and Theorem 2.3) we need some other methods, different from the ones used in studying system (1.2). Generally speaking, the paper is also a continuation of studying special cases of the next systems of difference equations

$$x_{n+1} = \max \left\{ A_n, \frac{y_{n-k}^p}{x_{n-l}^q} \right\}, \quad y_{n+1} = \max \left\{ A_n, \frac{x_{n-k}^p}{y_{n-l}^q} \right\}, \quad n \in \mathbb{N}_0,$$

$$x_{n+1} = A_n + \frac{y_{n-k}^p}{x_{n-l}^q}, \quad y_{n+1} = A_n + \frac{x_{n-k}^p}{y_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

where  $k, l \in \mathbb{N}$ ,  $\min\{p, q\} > 0$  and  $(A_n)_{n \in \mathbb{N}_0}$  is a sequence of positive numbers, as well as special cases of their scalar counterparts

$$x_{n+1} = \max \left\{ A_n, \frac{x_{n-k}^p}{x_{n-l}^q} \right\}$$

$$x_{n+1} = A_n + \frac{x_{n-k}^p}{x_{n-l}^q}, \quad n \in \mathbb{N}_0,$$

where  $k, l \in \mathbb{N}$ ,  $\min\{p, q\} > 0$  and  $(A_n)_{n \in \mathbb{N}_0}$  is a sequence of positive numbers.

For some results in the area see, for example, [2,4,6,8,14,20,24,25,28–30,33] and the related references therein.

Solution  $(x_n, y_n)_{n \geq -1}$  of system (1.3) is bounded if there is an  $M \geq 0$  such that

$$\|(x_n, y_n)\|_2 = \sqrt{x_n^2 + y_n^2} \leq M, \quad n \geq -1. \quad (1.4)$$

If

$$\sup_{n \geq -1} \sqrt{x_n^2 + y_n^2} = +\infty$$

we say that the solution is unbounded.

## 2 Boundedness character of positive solutions of system (1.3)

In this section we prove the main results of this paper, which give a complete picture for the boundedness character of positive solutions of system (1.3).

**Theorem 2.1.** *Assume that  $A > 0$ ,  $2\sqrt{q} \leq p < 1 + q$  and  $q \in (0, 1)$ . Then all positive solutions of system (1.3) are bounded.*

*Proof.* First note that from (1.3) we have

$$\min\{x_n, y_n\} \geq A, \quad n \in \mathbb{N}. \quad (2.1)$$

It is not difficult to see that the conditions  $2\sqrt{q} \leq p < 1 + q$  and  $q \in (0, 1)$  imply that the polynomial  $P(t) = t^2 - pt + q$  has zeroes  $t_1$  and  $t_2$  such that  $0 < t_2 < t_1 < 1$ .

We have

$$\begin{aligned} x_{n+1} &= \max \left\{ A, \frac{y_n^{t_1+t_2}}{x_{n-1}^{t_1 t_2}} \right\}, \\ y_{n+1} &= \max \left\{ A, \frac{x_n^{t_1+t_2}}{y_{n-1}^{t_1 t_2}} \right\}, \quad n \in \mathbb{N}_0, \end{aligned}$$

which along with (2.1) implies that

$$\frac{x_{n+1}}{y_n^{t_1}} = \max \left\{ \frac{A}{y_n^{t_1}}, \frac{y_n^{t_2}}{x_{n-1}^{t_1 t_2}} \right\} \leq \max \left\{ A^{1-t_1}, \left( \frac{y_n}{x_{n-1}^{t_1}} \right)^{t_2} \right\} \quad (2.2)$$

$$\frac{y_{n+1}}{x_n^{t_1}} = \max \left\{ \frac{A}{x_n^{t_1}}, \frac{x_n^{t_2}}{y_{n-1}^{t_1 t_2}} \right\} \leq \max \left\{ A^{1-t_1}, \left( \frac{x_n}{y_{n-1}^{t_1}} \right)^{t_2} \right\} \quad (2.3)$$

for every  $n \in \mathbb{N}$ , and consequently

$$\max \left\{ \frac{x_{n+1}}{y_n^{t_1}}, \frac{y_{n+1}}{x_n^{t_1}} \right\} \leq \max \left\{ A^{1-t_1}, \max \left\{ \frac{x_n}{y_{n-1}^{t_1}}, \frac{y_n}{x_{n-1}^{t_1}} \right\}^{t_2} \right\}, \quad n \in \mathbb{N}. \quad (2.4)$$

Let

$$u_n = \max \left\{ \frac{x_n}{y_{n-1}^{t_1}}, \frac{y_n}{x_{n-1}^{t_1}} \right\}, \quad n \in \mathbb{N},$$

and

$$v_{n+1} = \max \left\{ A^{1-t_1}, v_n^{t_2} \right\}, \quad n \in \mathbb{N}, \quad (2.5)$$

with

$$v_1 = u_1.$$

By induction, we have

$$u_n \leq v_n, \quad n \in \mathbb{N}. \quad (2.6)$$

The fact  $t_2 \in (0, 1)$  implies that the equation  $g(x) = x$ , where

$$g(x) = \max \left\{ A^{1-t_1}, x^{t_2} \right\}, \quad x \in (0, \infty), \quad (2.7)$$

has a unique fixed point  $\bar{x} \geq 1$  and

$$(g(x) - x)(x - \bar{x}) < 0, \quad x \in \mathbb{R}_+ \setminus \{\bar{x}\}. \quad (2.8)$$

Hence, for  $v_1 \in (0, \bar{x}]$ , we have

$$v_n \leq v_{n+1} \leq \bar{x}, \quad n \in \mathbb{N},$$

and for  $v_1 \geq \bar{x}$ , we have

$$v_n \geq v_{n+1} \geq \bar{x}, \quad n \in \mathbb{N}.$$

Hence,  $(v_n)_{n \in \mathbb{N}}$  is bounded, which along with (2.6) implies that

$$u_n \leq L_1, \quad n \in \mathbb{N}_0,$$

for some  $L_1 \geq \bar{x} \geq 1$ .

Therefore

$$x_{n+1} \leq L_1 y_n^{t_1}, \quad y_{n+1} \leq L_1 x_n^{t_1}, \quad n \in \mathbb{N}_0. \quad (2.9)$$

From (2.9) we easily get

$$x_n + y_n \leq 2L_1 (x_{n-1} + y_{n-1})^{t_1}, \quad n \in \mathbb{N}, \quad (2.10)$$

from which it easily follows that

$$x_n + y_n \leq (2L_1)^{\frac{1-t_1^n}{1-t_1}} (x_0 + y_0)^{t_1^n} \leq (2L_1)^{\frac{1}{1-t_1}} \max\{1, x_0 + y_0\}. \quad (2.11)$$

From (2.1) and (2.11) the boundedness of sequences  $(x_n)_{n \geq -1}$  and  $(y_n)_{n \geq -1}$ , and consequently the theorem follows.  $\square$

**Theorem 2.2.** Assume that  $A > 0$ ,  $p > 0$  and  $p^2 < 4q$ . Then all positive solutions of system (1.3) are bounded.

*Proof.* Let sequence  $(p_n)_{n \in \mathbb{N}_0}$  be defined as follows

$$p_{k+1} = \frac{q}{p - p_k}, \quad p_0 = 0. \quad (2.12)$$

Using (1.3) and (2.12) we have

$$\begin{aligned} x_{n+1} &= \max \left\{ A, \frac{y_n^p}{x_{n-1}^q} \right\} = \max \left\{ A, \left( \frac{y_n}{x_{n-1}^{q/p}} \right)^p \right\} \\ &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \frac{x_{n-1}^{p-\frac{q}{p}}}{y_{n-2}^q} \right\}^p \right\} \\ &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \left( \frac{x_{n-1}}{y_{n-2}^{q/(p-\frac{q}{p})}} \right)^{p-\frac{q}{p}} \right\}^p \right\} \\ &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \frac{y_{n-2}^{p-\frac{q}{p}}}{x_{n-3}^q} \right\}^{p-\frac{q}{p}} \right\}^p \right\} \end{aligned} \quad (2.13)$$

$$\begin{aligned}
 &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \left( \frac{y_{n-2}}{x_{n-3}^{q/(p-\frac{q}{p})}} \right)^{p-\frac{q}{p}} \right\}^{p-\frac{q}{p}} \right\}^p \right\} \\
 &= \dots \\
 &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \dots, \max \left\{ \frac{A}{y_{n-2k}^{p_{2k}}}, \frac{y_{n-2k}^{p-p_{2k}}}{x_{n-(2k+1)}^q} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\}, \quad (2.14)
 \end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \dots, \max \left\{ \frac{A}{y_{n-2k}^{p_{2k}}}, \left( \frac{y_{n-2k}}{x_{n-(2k+1)}^{q/(p-p_{2k})}} \right)^{p-p_{2k}} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\}, \\
 &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \dots, \max \left\{ \frac{A}{y_{n-2k}^{p_{2k}}}, \max \left\{ \frac{A}{x_{n-(2k+1)}^{q/(p-p_{2k})}}, \frac{x_{n-(2k+1)}^{p-\frac{q}{p-p_{2k}}}}{y_{n-(2k+2)}^q} \right\}^{p-p_{2k}} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\}, \quad (2.15)
 \end{aligned}$$

$$= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \dots, \max \left\{ \frac{A}{x_{n-(2k+1)}^{p_{2k+1}}}, \left( \frac{x_{n-(2k+1)}}{y_{n-(2k+2)}^{q/(p-p_{2k+1})}} \right)^{p-p_{2k+1}} \right\}^{p-p_{2k}} \right\}^{p-\frac{q}{p}} \right\}^p \right\}.$$

If  $p^2 \leq q$ , then by using (2.1) in (2.13), for  $n \geq 3$ , we get

$$x_{n+1} = \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \frac{x_{n-1}^{p-\frac{q}{p}}}{y_{n-2}^q} \right\}^p \right\} \leq \max \left\{ A, A^{p-q}, \frac{1}{A^{pq+q-p^2}} \right\},$$

so  $(x_n)_{n \geq -1}$  is bounded, in this case.

The monotonicity of  $g(x) = q/(p-x)$  on the interval  $(0, p)$  along with the fact  $0 = p_0 < p_1 = q/p$  implies that  $p_k$  is increasing as far as  $p_k < p$ . If  $p_k < p$  for every  $k \in \mathbb{N}_0$ , then there would exist  $\lim_{k \rightarrow \infty} p_k := \hat{p}$  and  $(\hat{p})^2 - p\hat{p} + q = 0$ , but the equation does not have real roots because of the condition  $p^2 < 4q$ .

Therefore, there is an  $l_0 \in \mathbb{N}$  such that

$$p_{l_0-1} < p \quad \text{and} \quad p_{l_0} \geq p.$$

If  $l_0 = 2k$ , then by using (2.1) in (2.14), we get

$$\begin{aligned}
 x_{n+1} &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \frac{A}{y_{n-2}^{q/(p-\frac{q}{p})}}, \dots, \max \left\{ \frac{A}{y_{n-2k}^{p_{2k}}}, \frac{y_{n-2k}^{p-p_{2k}}}{x_{n-(2k+1)}^q} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\} \\
 &\leq \max \left\{ A, \max \left\{ \frac{A}{A^{q/p}}, \max \left\{ \frac{A}{A^{q/(p-\frac{q}{p})}}, \dots, \max \left\{ \frac{A}{A^{p_{2k}}}, \frac{1}{A^{q-p+p_{2k}}} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\},
 \end{aligned}$$

for  $n \geq 2k + 2$ , from which the boundedness of  $(x_n)_{n \geq -1}$  follows in this case.

If  $l_0 = 2k + 1$ , then by using (2.1) in (2.15), we get

$$\begin{aligned}
x_{n+1} &= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}^{q/p}}, \max \left\{ \dots \max \left\{ \frac{A}{y_{n-2k}^{p_{2k}}}, \max \left\{ \frac{A}{x_{n-(2k+1)}^{p_{2k+1}}}, \frac{x_{n-(2k+1)}^{p-p_{2k+1}}}{y_{n-(2k+2)}^q} \right\}^{p-p_{2k}} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\} \\
&\leq \max \left\{ A, \max \left\{ \frac{A}{A^{q/p}}, \max \left\{ \dots \max \left\{ \frac{A}{A^{p_{2k}}}, \max \left\{ \frac{A}{A^{p_{2k+1}}}, \frac{1}{A^{q-p+p_{2k+1}}} \right\}^{p-p_{2k}} \right\}^{p-p_{2k-1}} \right\}^{p-\frac{q}{p}} \right\}^p \right\},
\end{aligned}$$

for  $n \geq 2k + 3$ , from which the boundedness of  $(x_n)_{n \geq -1}$  follows in this case.

Since the system (1.3) is symmetric, the boundedness of  $(x_n)_{n \geq -1}$  imply the boundedness of  $(y_n)_{n \geq -1}$ , finishing the proof of the theorem.  $\square$

**Theorem 2.3.** *Assume that  $A > 0$ ,  $p = 1 + q$ , and  $q \in (0, 1)$ . Then all positive solutions of system (1.3) are bounded.*

*Proof.* First note that by using the change of variables

$$x_n = A\hat{x}_n, \quad y_n = A\hat{y}_n, \quad n \in \mathbb{N}_0,$$

system (1.3), in this case, is reduced to the same system with  $A = 1$ . Hence we may assume that  $A = 1$ .

Assume that the sequences  $(a_n)_{n \in \mathbb{N}_0}$  and  $(b_n)_{n \in \mathbb{N}_0}$  are defined by

$$\begin{aligned}
a_0 &= q, & b_0 &= q + 1, \\
a_{2n+1} &= (q + 1)b_{2n} - a_{2n}, & b_{2n+1} &= qb_{2n}, \quad n \in \mathbb{N}_0, \\
b_{2n+2} &= (q + 1)a_{2n+1} - b_{2n+1}, & a_{2n+2} &= qa_{2n+1}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{2.16}$$

From this, by using (1.3) and a simple inductive argument, we have

$$\begin{aligned}
x_{n+1} &= \max \left\{ 1, \frac{y_n^{q+1}}{x_{n-1}^q} \right\} = \max \left\{ 1, \frac{y_n^{b_0}}{x_{n-1}^{a_0}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{x_{n-1}^{(q+1)b_0 - a_0}}{y_{n-2}^{qb_0}} \right\} = \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{x_{n-1}^{a_1}}{y_{n-2}^{b_1}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{y_{n-2}^{(q+1)a_1 - b_1}}{x_{n-3}^{qa_1}} \right\} = \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{y_{n-2}^{b_2}}{x_{n-3}^{a_2}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{1}{x_{n-3}^{a_2}}, \frac{x_{n-3}^{(q+1)b_2 - a_2}}{y_{n-4}^{qb_2}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{1}{x_{n-3}^{a_2}}, \frac{x_{n-3}^{a_3}}{y_{n-4}^{b_3}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{1}{x_{n-3}^{a_2}}, \frac{1}{y_{n-4}^{b_3}}, \frac{y_{n-4}^{(q+1)a_3 - b_3}}{x_{n-5}^{qa_3}} \right\} \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \frac{1}{x_{n-3}^{a_2}}, \frac{1}{y_{n-4}^{b_3}}, \frac{y_{n-4}^{b_4}}{x_{n-5}^{a_4}} \right\} \\
&= \dots \\
&= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{y_{n-2k}^{b_{2k}}}{x_{n-2k-1}^{a_{2k}}} \right\}
\end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{x_{n-2k-1}^{(q+1)b_{2k}-a_{2k}}}{y_{n-2k-2}^{qb_{2k}}} \right\} \\
 &= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{x_{n-2k-1}^{a_{2k+1}}}{y_{n-2k-2}^{b_{2k+1}}} \right\} \tag{2.17}
 \end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{1}{y_{n-2k-2}^{b_{2k+1}}}, \frac{y_{n-2k-2}^{(q+1)a_{2k+1}-b_{2k+1}}}{x_{n-2k-3}^{qa_{2k+1}}} \right\} \\
 &= \max \left\{ 1, \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-2}^{b_1}}, \dots, \frac{1}{y_{n-2k-2}^{b_{2k+1}}}, \frac{y_{n-2k-2}^{b_{2k+2}}}{x_{n-2k-3}^{a_{2k+2}}} \right\}, \tag{2.18}
 \end{aligned}$$

for every  $k \in \mathbb{N}$ .

From (2.16) we have that

$$b_{2n} = \frac{a_{2n+1} + a_{2n}}{q+1}, \quad n \in \mathbb{N}_0.$$

Applying this to the following relation

$$b_{2n+2} = (q+1)a_{2n+1} - qb_{2n}, \quad n \in \mathbb{N}_0,$$

we get

$$a_{2n+3} + a_{2n+2} - (q^2 + q + 1)a_{2n+1} + qa_{2n} = 0, \quad n \in \mathbb{N}_0.$$

From this and the relation  $a_{2n+2} = qa_{2n+1}$ , we get

$$a_{2n+3} - (q^2 + 1)a_{2n+1} + q^2a_{2n-1} = 0, \quad n \in \mathbb{N}. \tag{2.19}$$

It is easy to see that the general solution of difference equation (2.19) is

$$a_{2n+1} = c_1 + c_2q^{2n}, \quad n \in \mathbb{N}_0.$$

From this and since

$$\begin{aligned}
 a_1 &= (q+1)b_0 - a_0 = q^2 + q + 1, & b_1 &= qb_0 = q^2 + q, \\
 a_2 &= qa_1 = q^3 + q^2 + q, & b_2 &= (q+1)a_1 - b_1 = (q+1)(q^2 + 1), \\
 a_3 &= (q+1)b_2 - a_2 = q^4 + q^3 + q^2 + q + 1,
 \end{aligned}$$

we have that

$$c_1 + c_2 = q^2 + q + 1, \quad c_1 + c_2q^2 = q^4 + q^3 + q^2 + q + 1$$

and consequently

$$c_1 = \frac{1}{1-q}, \quad c_2 = \frac{q^4 + q^3}{q^2 - 1} = \frac{q^3}{q-1}.$$

Hence

$$a_{2n+1} = \frac{1 - q^{2n+3}}{1 - q}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

Letting  $n \rightarrow +\infty$  we get

$$\lim_{n \rightarrow +\infty} a_{2n+1} = \frac{1}{1-q}. \tag{2.21}$$

From this and (2.16) we also have that

$$\lim_{n \rightarrow +\infty} a_{2n} = \lim_{n \rightarrow +\infty} b_{2n+1} = \frac{q}{1-q} = q \lim_{n \rightarrow +\infty} b_{2n}. \quad (2.22)$$

Now note that from (2.17) and (2.18) we have that

$$\begin{aligned} x_{2n+1} &= \max \left\{ 1, \frac{1}{x_{2n-1}^{a_0}}, \frac{1}{y_{2n-2}^{b_1}}, \dots, \frac{1}{y_0^{b_{2n-1}}}, \frac{y_0^{b_{2n}}}{x_{-1}^{a_{2n}}} \right\} \\ x_{2n} &= \max \left\{ 1, \frac{1}{x_{2n-2}^{a_0}}, \frac{1}{y_{2n-3}^{b_1}}, \dots, \frac{1}{x_0^{a_{2n-2}}}, \frac{x_0^{a_{2n-1}}}{y_{-1}^{b_{2n-1}}} \right\}, \end{aligned}$$

for  $n \in \mathbb{N}$ .

From this, since  $\min\{x_n, y_n\} \geq 1$  for  $n \in \mathbb{N}$ , and by using (2.21) and (2.22) the boundedness of the sequence  $(x_n)_{n \geq -1}$  easily follows.

Since system (1.3) is symmetric, the boundedness of  $(x_n)_{n \geq -1}$  imply the boundedness of  $(y_n)_{n \geq -1}$ , finishing the proof of the theorem.  $\square$

The following theorem shows that positive solutions of system (1.3) are unbounded in the other cases.

**Theorem 2.4.** *Assume that  $A > 0$ . If  $p^2 \geq 4q \geq 4$ , or  $p > 1 + q$  and  $q \in (0, 1)$ , then system (1.3) has positive unbounded solutions.*

*Proof.* Assume that  $p^2 \geq 4q \geq 4$  and  $p \neq 2$ . From (1.3) we have

$$x_{n+1} \geq \frac{y_n^p}{x_{n-1}^q}, \quad y_{n+1} \geq \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0. \quad (2.23)$$

Let  $a_n = \ln(x_n y_n)$ ,  $n \geq -1$ . Then from (2.23), it follows that

$$a_{n+1} - p a_n + q a_{n-1} \geq 0, \quad n \in \mathbb{N}_0. \quad (2.24)$$

The polynomial  $P(t) = t^2 - pt + q$  has the zeroes  $t_{1,2} = (p \pm \sqrt{p^2 - 4q})/2$ , and  $t_1 > 1$ , and  $t_2 > 0$ .

From (2.24) we get

$$a_{n+1} - t_1 a_n - t_2 (a_n - t_1 a_{n-1}) \geq 0, \quad n \in \mathbb{N}_0, \quad (2.25)$$

that is,

$$\frac{x_{n+1} y_{n+1}}{(x_n y_n)^{t_1}} \geq \left( \frac{x_n y_n}{(x_{n-1} y_{n-1})^{t_1}} \right)^{t_2}, \quad n \in \mathbb{N}_0, \quad (2.26)$$

which implies that

$$\frac{x_{n+1} y_{n+1}}{(x_n y_n)^{t_1}} \geq \left( \frac{x_0 y_0}{(x_{-1} y_{-1})^{t_1}} \right)^{t_2^{n+1}}, \quad n \in \mathbb{N}_0. \quad (2.27)$$

Let  $x_i, y_i$ ,  $i \in \{-1, 0\}$  be chosen such that

$$x_0 y_0 > 1 \quad \text{and} \quad x_0 y_0 = (x_{-1} y_{-1})^{t_1}. \quad (2.28)$$



This, along with (2.27), yields

$$x_n y_n \geq \left( \frac{x_0 y_0}{(x_{-1} y_{-1})^{t_1}} \right)^{t_2^n} (x_{n-1} y_{n-1})^{t_1} = (x_{n-1} y_{n-1})^{t_1}, \quad n \in \mathbb{N}_0, \quad (2.29)$$

from which we get

$$x_n y_n \geq (x_0 y_0)^{t_1^n}, \quad n \in \mathbb{N}_0. \quad (2.30)$$

Letting  $n \rightarrow \infty$  in (2.30), using the first assumption in (2.28) and  $t_1 > 1$ , it follows that

$$x_n y_n \rightarrow +\infty \quad \text{as } n \rightarrow \infty, \quad (2.31)$$

which along with the inequality between arithmetic and geometric means implies

$$\sqrt{x_n^2 + y_n^2} \rightarrow +\infty \quad \text{as } n \rightarrow \infty, \quad (2.32)$$

from which it follows that  $(x_n, y_n)_{n \geq -1}$  is unbounded.

The proof in the case  $p > 1 + q$  and  $q \in (0, 1)$  is similar, since then

$$t_1 = (p + \sqrt{p^2 - 4q})/2 > 1.$$

If  $p = q + 1 = 2$ , then  $t_1 = t_2 = 1$ . If we choose  $x_i, y_i, i \in \{-1, 0\}$  such that

$$x_0 y_0 > x_{-1} y_{-1} > 0, \quad (2.33)$$

then from (2.27) we get

$$x_n y_n \geq \frac{x_0 y_0}{x_{-1} y_{-1}} x_{n-1} y_{n-1}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n y_n \geq \left( \frac{x_0 y_0}{x_{-1} y_{-1}} \right)^n x_0 y_0, \quad n \in \mathbb{N}_0. \quad (2.34)$$

Letting  $n \rightarrow \infty$  in (2.34) we get (2.31) and consequently (2.32), which implies that  $(x_n, y_n)_{n \geq -1}$  is unbounded, finishing the proof of the theorem.  $\square$

## Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (22-130-1435-HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

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