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Existence of positive solutions for *n*th-order boundary value problem with sign changing nonlinearity

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Abstract

In this paper, we investigate the existence of positive solutions for singular *n*th-order boundary value problem

$$u^{(n)}(t) + a(t)f(t, u(t)) = 0, \quad 0 \le t \le 1,$$
$$u^{(i)}(0) = u^{(n-2)}(1) = 0, \quad 0 \le i \le n-2,$$

where $n \ge 2$, $a \in C((0,1),[0,+\infty))$ may be singular at $t = 0$ and (or) $t = 1$ and the nonlinear term f is continuous and is allowed to change sign. Our proofs are based on the method of lower solution and topology degree theorem.

Keywords: Existence; Singular; Positive solution; Cone; Sign changing nonlinearity.

1. Introduction

Boundary value problems for higher order differential equations play a very important role in both theories and applications. Existence of positive solutions for nonlinear higher order has been studied in the literature by using the Krasnosel'skii and Guo fixed point theorem, Leggett-Williams fixed point theorem, Lower- and upper- solutions method and so on. We refer the reader to [2-9] for some recent results. However, to the best of our knowledge, few papers can be found in the literature for *n*th-order boundary value problem with sign changing nonlinearity, most papers are dealing with the existence of positive solutions when the nonlinear term f is nonnegative. For example, in [3], by using the Krasnosel'skii and Guo fixed point theorem, Eloe and Henderson studied the existence of positive solutions for the following boundary value problem

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$$\begin{cases} u^{(n)}(t) + a(t)f(u(t)) = 0, & t \in (0,1), \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, & 0 \le i \le n-2, \end{cases} \tag{1.1}$$

where

(A_1) $f : [0, \infty] \to [0, \infty)$ is continuous;

(A_2) $a : (0, 1) \to [0, \infty)$ is continuous and does not vanish identically on any subinterval;

(A_3) f is either superlinear or sublinear.

Motivated by the above works, in this paper, we study the existence of positive solutions for singular nth-order boundary value problem with sign changing nonlinearity as follows

$$\begin{cases} u^{(n)}(t) + a(t)f(t, u(t)) = 0, & t \in (0,1), \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, & 0 \le i \le n-2. \end{cases} \tag{1.2}$$

Throughout this paper, we assume the following conditions hold.

(C_1) $f : [0, 1] \times [0, \infty) \to (-\infty, +\infty)$ is continuous;

(C_2) $a : (0, 1) \to [0, \infty)$ is continuous, and $0 < \int_0^1 a(t)dt < \infty$, if $n = 2$; $0 < \int_0^1 (1-t)a(t)dt < \infty$, if $n \ge 3$.

The purpose of this paper is to establish the existence of positive solutions for BVP (1.2) by constructing available operator and combining the method of lower solution with the method of topology degree.

The rest of this paper is organized as follows: in section 2, we present some preliminaries and lemmas. Section 3 is devoted to prove the existence of positive solutions for BVP (1.2). An example is considered in section 4 to illustrate our main results.

2. Preliminary Lemmas

Lemma 2.1. *Suppose that* $y(t) \in C[0, 1]$*, then boundary value problem*

$$\begin{cases} u^{(n)}(t) + y(t) = 0, & 0 \le t \le 1, \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, & 0 \le i \le n-2, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t,s)=\frac{1}{(n-1)!}\begin{cases}(1-s)t^{n-1}-(t-s)^{n-1}, & 0\le s\le t\le 1,\\ (1-s)t^{n-1}, & 0\le t<s\le 1.\end{cases}$$

Proof. The proof follows by direct calculations.

Lemma 2.2. $G(t,s)$ *has the following properties.*

(1) $0\le G(t,s)\le k(s)$, $t,s\in[0,1]$, *where*

$$k(s)=\frac{1-s}{(n-1)!};$$

(2) $\frac{\partial^i}{\partial t^i}G(t,s)>0$ *on* $(0,1)\times(0,1)$, $0\le i\le n-2$*;*

(3) $G''_{tt}(t,s):=\frac{\partial^2}{\partial t^2}G(t,s)\le(n-1)(n-2)k(s),\quad n\ge 3.$

Proof. It is easy to check that (1) and (3) hold. The proof of (2), please see [1].

Remark 2.1. By (C_2) and Lemma 2.2, we have

$$0<\int_0^1 G(t,s)a(s)ds<\infty, n\ge 2,\quad \text{and}\quad 0<\int_0^1\int_0^1 G''_{\tau\tau}(\tau,s)a(s)d\tau ds<\infty, n\ge 3.$$

By the definition of completely continuous operator, we can check that the following lemma holds.

Lemma 2.3. *Let* P *is a cone of* $X=C[0,1]$*. Suppose* $T:P\to X$ *is completely continuous. Define* $A:TX\to P$ *by*

$$(Ay)(t)=\max\{y(t),0\},\quad y\in TX.$$

Then, $A\circ T:P\to P$ *is also a completely continuous operator.*

3. Main results

Let $X=C[0,1]$, $P=\{u\in X:\ u(t)\ge 0,\ t\in[0,1]\}$ with $\|u\|=\max_{t\in[0,1]}|u(t)|$. Set

$$f_1(t,u)=\max\{0,f(t,u)\},\quad f_2(t,u)=\max\{0,-f(t,u)\},$$

and

$$\delta=\int_0^1 k(s)a(s)ds,\quad w(t)=\int_0^1 G(t,s)a(s)ds.$$

For convenience, we introduce the following notations

$$\overline{f}_\infty = \limsup_{u\to\infty} \max_{0\le t\le 1} \frac{f(t,u)}{u}, \quad \overline{f}_0 = \limsup_{u\to 0^+} \max_{0\le t\le 1} \frac{f(t,u)}{u},$$

$$\overline{f}_{1\infty} = \limsup_{u\to\infty} \max_{0\le t\le 1} \frac{f_1(t,u)}{u}, \quad \overline{f}_{10} = \limsup_{u\to 0^+} \max_{0\le t\le 1} \frac{f_1(t,u)}{u},$$

$$\overline{f}_{2\infty} = \limsup_{u\to\infty} \max_{0\le t\le 1} \frac{f_2(t,u)}{u}, \quad \overline{f}_{20} = \limsup_{u\to 0^+} \max_{0\le t\le 1} \frac{f_2(t,u)}{u}.$$

Theorem 3.1. *Suppose that* (C_1) *and* (C_2) *hold, in addition assume* $\overline{f}_{1\infty} = A < +\infty$, $\overline{f}_{2\infty} = B < +\infty$ *(or* $\overline{f}_{10} = A < +\infty$, $\overline{f}_{20} = B < +\infty$*) with* $A + B < \frac{1}{\delta}$ *and there exist* r, λ *with* $r > \lambda > 0$ *such that*

$$\begin{cases} \lambda \le \min_{t\in[0,1]} f(t,\lambda w(t)), & n = 2, \\ \lambda \ge \max_{t\in[0,1]} f(t,\lambda w(t)), & n \ge 3. \end{cases} \tag{3.1}$$

Then BVP (1.2) *has a positive solution* $u^*(t)$ *satisfying*

$$0 < \lambda w(t) \le u^*(t), \quad 0 < t < 1 \quad \textit{and} \quad \|u^*\| \le r.$$

Proof. Let

$$g(t,u(t)) = \begin{cases} f(t,u(t)), & u(t) \ge \lambda w(t), \\ f(t,\lambda w(t)), & u(t) \le \lambda w(t). \end{cases} \tag{3.2}$$

Define the operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)a(s)g(s,u(s))ds, \quad 0 \le t \le 1. \tag{3.3}$$

Similar to the proof of Lemma 2.1 in [7], we can easily check that T is a completely continuous operator.
Define the operator $A : X \to P$ by

$$(Au)(t) = \max\{u(t), 0\}. \tag{3.4}$$

By Lemma 2.3, we get $A \circ T : P \to P$ is also completely continuous.

If $\overline{f}_{1\infty} = A < +\infty$, $\overline{f}_{2\infty} = B < +\infty$, then by hypothesis $A + B < \frac{1}{\delta}$, we may take $A_0 > A$, $B_0 > B$ such that $A_0 + B_0 < \frac{1}{\delta}$, $\overline{f}_{1\infty} < A_0$ and $\overline{f}_{2\infty} < B_0$. Thus, we choose $L > 0$ such that

$$f_1(t,u) < A_0 u, \quad f_2(t,u) < B_0 u, \quad \text{if } u \ge L, \quad t \in [0,1], \tag{3.5}$$

and there exists $r > L$ such that

$$f_1(t,u) < A_0 r, \quad f_2(t,u) < B_0 r, \quad \text{if } \lambda w(t) \le u(t) \le L, \quad t \in [0,1]. \tag{3.6}$$

Let $\Omega = \{u \in P : \ \|u\| < r\}$. Then, for $u \in \partial\Omega$, we have by (3.5) and (3.6) that

$$\max_{\substack{t\in[0,1]\\ u\in[\lambda w(t),r]}} f_1(t,u) = \max\left\{ \max_{\substack{t\in[0,1]\\ u\in[\lambda w(t),L]}} f_1(t,u), \quad \max_{\substack{t\in[0,1]\\ u\in[L,r]}} f_1(t,u)\right\} < A_0 r. \tag{3.7}$$

Similarly,

$$\max_{\substack{t\in[0,1]\\ u\in[\lambda w(t),r]}} f_2(t,u) < B_0 r. \tag{3.8}$$

Thus, for each $u \in \partial\Omega$, from (3.7) and (3.8), we have

$$\begin{aligned}
(A\circ T)u(t) &= \max\left\{\int_0^1 G(t,s)a(s)g(s,u(s))ds, 0\right\}\\
&\le \int_0^1 G(t,s)a(s)|g(s,u(s))|ds\\
&\le \max_{\substack{t\in[0,1]\\ u\in[0,r]}} |g(t,u)| \int_0^1 k(s)a(s)ds\\
&= \delta \max_{\substack{t\in[0,1]\\ u\in[\lambda w(t),r]}} |f(t,u)|\\
&= \delta \max_{\substack{t\in[0,1]\\ u\in[\lambda w(t),r]}} (f_1(t,u) + f_2(t,u))\\
&< \delta(A_0 + B_0)r < r = \|u\|,
\end{aligned} \tag{3.9}$$

which implies

$$\|(A\circ T)u\| < \|u\|, \quad \forall u \in \partial\Omega.$$

Thus, we have

$$\deg_P\{I - A\circ T,\ \Omega,\ 0\} = 1,$$

where $\deg_P$ means the degree on cone P. Hence, $A\circ T$ has a fixed point u^* in $\overline{\Omega}$, i.e., $(A\circ T)(u^*) = u^*$, $u^* \in \overline{\Omega}$.

If $\overline{f}_{10} = A < +\infty$, $\overline{f}_{20} = B < +\infty$, then we take $A_0 > A$, $B_0 > B$ such that $A_0 + B_0 < \frac{1}{\delta}$, $\overline{f}_{10} < A_0$ and $\overline{f}_{20} < B_0$. Take $r > 0$ such that

$$f_1(t,u) < A_0 u, \quad f_2(t,u) < B_0 u, \quad \text{if } 0 < u(t) \le r, \quad t \in [0,1].$$

Then, for $u \in \partial\Omega$, similar to the proof of (3.9), we have that $A \circ T$ has a fixed point u^* in $\overline{\Omega}$. Hence, in any case we always have that $A \circ T$ has a fixed point u^* in $\overline{\Omega}$.

In the following, we shall show the following relation holds

$$(Tu^*)(t) \ge \lambda w(t), \quad t \in [0,1]. \tag{3.10}$$

Assume the contrary, then there exists $t_0 \in [0,1]$ such that

$$(Tu^*)(t_0) - \lambda w(t_0) = \min_{t\in[0,1]}\{(Tu^*)(t) - \lambda w(t)\} = M < 0. \tag{3.11}$$

Obviously, $t_0 \ne 0$, so $t_0 \in (0,1]$ and

$$(Tu^*)'(t_0) - \lambda w'(t_0) = 0. \tag{3.12}$$

There are two cases to consider.

Case 1. $t_0 = 1$. It is obvious that $Tu^*(t) - \lambda w(t)$ on $[0,1]$ is continuous. From (3.11), we have that there exists $t_1 \in [0,1)$ such that $(Tu^*)(t_1) - \lambda w(t_1) = 0$ and $(Tu^*)(t) - \lambda w(t) < 0$ for $t \in (t_1, 1]$.

If $n = 2$, then by (3.1), (3.3) and (3.12), one has

$$\begin{aligned}
(Tu^*)'(t) - \lambda w'(t) &= (Tu^*)'(1) - \lambda w'(1) - \int_t^1 [(Tu^*)'(s) - \lambda w'(s)]' ds \\
&= \int_t^1 a(s)[g(s,u^*(s)) - \lambda] ds \\
&= \int_t^1 a(s)[f(s,\lambda w(s)) - \lambda] ds \\
&\ge [\min_{t\in[0,1]} f(t,\lambda w(t)) - \lambda] \int_t^1 a(s) ds \ge 0.
\end{aligned}$$

Then, we have that $(Tu^*)'(t) - \lambda w'(t) \ge 0$.

If $n \ge 3$, from (3.1), (3.3) and (3.12), we get

$$\begin{aligned}
(Tu^*)'(t) - \lambda w'(t) &= (Tu^*)'(1) - \lambda w'(1) - \int_t^1 [(Tu^*)'(\tau) - \lambda w'(\tau)]' d\tau \\
&= \int_t^1 \int_0^1 G''_{\tau\tau}(\tau,s) a(s)[\lambda - g(s,u^*(s))] ds d\tau \\
&= \int_t^1 \int_0^1 G''_{\tau\tau}(\tau,s) a(s)[\lambda - f(s,\lambda w(s))] ds d\tau
\end{aligned}$$

$$\geq [\lambda - \max_{t\in[0,1]} f(t, \lambda w(t))] \int_t^1 \int_0^1 G''_{\tau\tau}(\tau, s)a(s)dsd\tau \geq 0.$$

Then, we have $(Tu^*)'(t) - \lambda w'(t) \geq 0$. Therefore, in any case we always have that $(Tu^*)'(t) - \lambda w'(t) \geq 0$, which implies

$$(Tu^*)(t_0) - \lambda w(t_0) = (Tu^*)(1) - \lambda w(1) \geq (Tu^*)(t_1) - \lambda w(t_1) = 0.$$

It contradicts (3.11), so (3.10) holds.

Case 2. $t_0 \in (0, 1)$. Obviously, $Tu^*(t) - \lambda w(t)$ on $[0, 1]$ is continuous. By (3.11) and $t_0 \in (0, 1)$, we have that there exists $t_2 \in [0, t_0) \cup (t_0, 1]$ such that $(Tu^*)(t_2) - \lambda w(t_2) = 0$ and $(Tu^*)(t) - \lambda w(t) < 0$ for $t \in (t_2, t_0]$ or $t \in [t_0, t_2)$. Without loss of generality, we assume that $t \in (t_2, t_0]$.

If $n = 2$, we have by (3.1), (3.3) and (3.12) that

$$\begin{aligned}(Tu^*)'(t) - \lambda w'(t) &= (Tu^*)'(t_0) - \lambda w'(t_0) - \int_t^{t_0} [(Tu^*)'(s) - \lambda w'(s)]' ds \\ &= \int_t^{t_0} a(s)[g(s, u^*(s)) - \lambda]ds \\ &= \int_t^{t_0} a(s)[f(s, \lambda w(s)) - \lambda]ds \\ &\geq [\min_{t\in[0,1]} f(t, \lambda w(t)) - \lambda] \int_t^{t_0} a(s)ds \geq 0.\end{aligned}$$

Thus, $(Tu^*)'(t) - \lambda w'(t) \geq 0$.

If $n \geq 3$, from (3.1), (3.3), (3.12) and Lemma 2.2 (2), we obtain

$$\begin{aligned}(Tu^*)'(t) - \lambda w'(t) &= (Tu^*)'(t_0) - \lambda w'(t_0) - \int_t^{t_0} [(Tu^*)'(\tau) - \lambda w'(\tau)]' d\tau \\ &= \int_t^{t_0} \int_0^1 G''_{\tau\tau}(\tau, s)a(s)[\lambda - g(s, u^*(s)]dsd\tau \\ &= \int_t^{t_0} \int_0^1 G''_{\tau\tau}(\tau, s)a(s)[\lambda - f(s, \lambda w(s))]dsd\tau \\ &\geq [\lambda - \max_{t\in[0,1]} f(t, \lambda w(t))] \int_t^{t_0} \int_0^1 G''_{\tau\tau}(\tau, s)a(s)dsd\tau \geq 0.\end{aligned}$$

Thus, $(Tu^*)'(t)-\lambda w'(t) \geq 0$. Hence, in any case we always have that $(Tu^*)'(t)-\lambda w'(t) \geq 0$, which implies

$$(Tu^*)(t_0) - \lambda w(t_0) \geq (Tu^*)(t_2) - \lambda w(t_2) = 0.$$

It contradicts (3.11), so (3.10) holds. Thus, $(A \circ T)u^* = Tu^* = u^*$, $u^* \in \overline{\Omega}$, i.e., BVP

(1.2) has a positive solution $u^*(t)$ satisfying $0 < \lambda w(t) \le u^*(t)$, $0 < t < 1$, and $\|u^*\| \le r$.

Corollary 3.1 *Suppose that* (C_1) *and* (C_2) *hold, in addition assume* $\overline{f}_{1\infty} = 0$*,* $\overline{f}_{2\infty} = 0$ *(or* $\overline{f}_{10} = 0$*,* $\overline{f}_{20} = 0$ *), if there exists a constant* $\lambda > 0$ *such that*

$$\begin{cases} \lambda \le \min\limits_{t\in[0,1]} f(t, \lambda w(t)), & n = 2, \\ \lambda \ge \max\limits_{t\in[0,1]} f(t, \lambda w(t)), & n \ge 3. \end{cases}$$

Then BVP (1.2) *has a positive solution.*

Theorem 3.2. *Suppose that* (C_2) *holds, in addition assume* $f(t, 0) \ge 0$*,* $a(t)f(t, 0) \not\equiv 0$ *and* $\overline{f}_{1\infty} = A < +\infty$ *,* $\overline{f}_{2\infty} = B < +\infty$ *(or* $\overline{f}_{10} = A < +\infty$*,* $\overline{f}_{20} = B < +\infty$ *) with* $A + B < \frac{1}{\delta}$*. Then BVP* (1.2) *has a positive solution.*

Proof: Similar to the proof of Theorem 3.1, we can complete the proof of Theorem 3.2, so we omit it here.

Corollary 3.4. *Suppose that* (C_2) *holds, in addition assume* $f(t, 0) \ge 0$*,* $a(t)f(t, 0) \not\equiv 0$*,* $\overline{f}_{1\infty} = 0$ *and* $\overline{f}_{2\infty} = 0$ *(or* $\overline{f}_{10} = 0$ *and* $\overline{f}_{20} = 0$ *). Then BVP* (1.2) *has a positive solution.*

Corollary 3.3. *Suppose that* (C_2) *holds, in addition assume* $f : [0, 1]\times[0, \infty) \to [0, \infty)$*,* $a(t)f(t, 0) \not\equiv 0$ *and* $\overline{f}_\infty = A < +\infty$ *(or* $\overline{f}_0 = A < +\infty$ *) with* $A < \frac{1}{\delta}$*. Then BVP* (1.2) *has a positive solution.*

Remark 3.1. Suppose that (C_2) holds, in addition assume $f : [0, 1] \times [0, \infty) \to [0, \infty)$, $a(t)f(t, 0) \not\equiv 0$ and $\overline{f}_\infty = 0$ (or $\overline{f}_0 = 0$). Then BVP (1.2) has a positive solution.

4. An example

Example 4.1. Let $a(t) = \dfrac{(n-1)!}{2(1-t)}$, $n \ge 3$, and

$$f(t, u) = \begin{cases} [1 + \ln(1 + (e-1)t)](u + u^{\frac{1}{2}} + 1), & (t, u) \in [0, 1] \times [0, 1], \\ [1 + \ln(1 + (e-1)t)][3 - (2e^{-2} + 6)(u - 1)], & (t, u) \in [0, 1] \times [1, 2], \\ -[1 + \ln(1 + (e-1)t)](e^{-u} + \frac{3}{2} + \sin \pi u)u, & (t, u) \in [0, 1] \times [2, \infty). \end{cases}$$

By simple calculation, we have

$$0 < \delta = \int_0^1 k(s)a(s)ds = \int_0^1 \tfrac{1-s}{(n-1)!} \cdot \tfrac{(n-1)!}{2(1-s)} ds = \tfrac{1}{2} < \infty.$$

It is easy to see that $f \in C([0,1]\times[0,\infty), R)$ and $f(t,0) > 0$, $a(t)f(t,0) \not\equiv 0$, $\forall t \in [0,1]$. Owing to $f(t,u) > 0$, $(t,u) \in [0,1]\times[0,1]$, and $f(t,u) < 0$, $(t,u) \in [0,1]\times[2,\infty)$, we have

$$f_1(t,u) = f(t,u), \quad f_2(t,u) = 0, \quad (t,u) \in [0,1]\times[0,1],$$

and

$$f_1(t,u) = 0, \quad f_2(t,u) = -f(t,u), \quad (t,u) \in [0,1]\times[2,\infty).$$

By calculating, we obtain that

$$A = \overline{f}_{1\infty} = \frac{1}{2}, \quad B = \overline{f}_{2\infty} = \frac{1}{2},$$

and so $A + B = 1 < 2 = \frac{1}{\delta}$. Therefore, by Theorem 3.2, BVP (1.2) has a positive solution.

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