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Qualitative properties of a functional differential equation

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Abstract. The aim of this paper is to discuss some basic problems (existence and uniqueness, data dependence) of the fixed point theory for a functional differential equation with an abstract Volterra operator. In the end an application is given.

Keywords: functional differential equations, weakly Picard operators, data dependence.

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1 Introduction

It is well-known that differential equations appear in mathematical models of various phenomena in physics, economy, biology, engineering, and other fields of science. Many illustrative examples of such models can be found in the literature (see, e.g., [1, 5–8] and the references therein).

We consider the functional differential equation of the form

$$x'(t) = g(x)(t) + f(t, x(t)), \quad t \in [a, b] \tag{1.1}$$

$$x(a) = x_0, \tag{1.2}$$

where the following conditions hold:

(C$_1$) $x_0 \in \mathbb{R}$, $f\colon C([a, b] \times \mathbb{R}) \to \mathbb{R}$;

(C$_2$) there exists $L_f > 0$ such that

$$|f(t, u_1) - f(t, u_2)| \le L_f\, |u_1 - u_2|\,, \quad \forall t \in [a, b],\ u_1, u_2 \in \mathbb{R};$$

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(C$_3$) $g\colon C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is an abstract Volterra operator and there exists $L_g > 0$ and $\tau > 0$ with $\tau > L_g + L_f$, such that

$$\|g(x) - g(y)\|_\tau \le L_g \|x - y\|_\tau, \quad \forall x, y \in C([a,b],\mathbb{R}),$$

where $\|\cdot\|_\tau$ is the Bielecki norm defined by

$$\|x\|_\tau = \sup_{t\in[a,b]} (\|x(t)\| e^{-\tau(t-a)}), \quad \tau > 0.$$

The present paper is motivated by a recent paper [9] where the author studied a differential equation with abstract Volterra operator of the form

$$x'(t) = f(t, x(t), V(x)(t)),\ t \in [a,b].$$

The aim of our paper is to apply the technique from [2–4, 13, 14] to a functional differential equation that includes an abstract Volterra operator.

The equation involving abstract Volterra operators have been investigated by many authors. The results on the existence and uniqueness, continuous dependence of solutions of Cauchy's problem and even more specialized topics can be found in [2, 9, 14] and the references therein.

The novelty of our paper consist in applying the weakly Picard operators technique for an equation written as a sum of two operators.

The paper is organized as follows. In Section 2, we recall some definitions and results concerning the weakly Picard operator theory. In Section 3 we prove first the existence and uniqueness theorem and then we obtain some properties regarding the data dependence of the solution. In the last section an example is given.

2 Preliminaries

In this section we will use the terminologies and notations extracted from [10–12]. For the convenience of the reader some of them are recalled below.

Let (X,d) be a metric space and $A\colon X \to X$ an operator. We denote by:
$F_A := \{x \in X \mid A(x) = x\}$ the fixed points set of A;
$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ the family of the nonempty invariant subsets of A;
$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$ the iterate operators of the operator A.

Definition 2.1. Let (X,d) be a metric space. An operator $A\colon X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. Let (X,d) be a metric space. An operator $A\colon X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Definition 2.3. If A is weakly Picard operator then we consider the operator A^∞ defined by

$$A^\infty\colon X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. It is clear that $A^\infty(X) = F_A$.

The following results are very useful in the sequel.

Lemma 2.5. *Let $(X, d, \leq)$ be an ordered metric space and $A\colon X \to X$ an operator. We suppose that:*

(i) A is WPO;

(ii) A is increasing.

Then, the operator A^∞ is increasing.

Lemma 2.6 (Abstract Gronwall lemma). *Let $(X, d, \leq)$ be an ordered metric space and $A\colon X \to X$ an operator. We suppose that:*

(i) A is WPO;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then:

(a) $x \leq A(x) \Longrightarrow x \leq x_A^*$;

(b) $x \geq A(x) \Longrightarrow x \geq x_A^*$.

Lemma 2.7 (Abstract comparison lemma). *Let $(X, d, \leq)$ an ordered metric space and $A, B, C\colon X \to X$ be such that:*

(i) the operator A, B, C are WPOs;

(ii) $A \leq B \leq C$;

(iii) the operator B is increasing.

Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

Another important notion is the following.

Definition 2.8. Let (X, d) be a metric space, $A\colon X \to X$ be a weakly Picard operator and $c \in \mathbb{R}_+^*$. The operator A is c-weakly Picard operator iff

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall x \in X.$$

For the c-PO$_s$ and c-WPO$_s$ we have the following lemma.

Lemma 2.9. *Let (X, d) be a metric space and $A, B\colon X \to X$ be two operators. We suppose that:*

(i) A is c-PO with $F_A = \{x_A^*\}$;

(ii) there exists $\eta \in \mathbb{R}_+^*$ such that $d(A(x), B(x)) \leq \eta$, $\forall x \in X$.

If $x_B^* \in F_B$, then $d(x_B^*, x_A^*) \leq c\eta$.

Lemma 2.10. *Let (X, d) be a metric space and $A, B\colon X \to X$ be two operators. We suppose that:*

(i) the operators A and B are c-WPO$_s$;

(ii) *there exists* $\eta \in \mathbb{R}_+^*$ *such that* $d(A(x), B(x)) \leq \eta$, $\forall x \in X$.

Then $H_d(F_A, F_B) \leq c\eta$, *where* H_d *stands for the Pompeiu–Hausdorff functional with respect to* d.

The following result is the characterization theorem of weakly Picard operators.

Theorem 2.11. *An operator* A *is a weakly Picard operator if and only if there exists a partition of* X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, *such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

For some examples of WPOs see [10–12].

3 Main result

We remark that if $x \in C^1([a,b], \mathbb{R})$ is a solution of the problem (1.1)–(1.2), then x is a solution of

$$x(t) = x_0 + \int_a^t g(x)(s)\, ds + \int_a^t f(s, x(s))\, ds, \ t \in [a,b] \tag{3.1}$$

and if $x \in C([a,b], \mathbb{R})$ is a solution of (3.1), then $x \in C^1([a,b], \mathbb{R})$ and is a solution of (1.1)–(1.2). Also, if $x \in C^1([a,b], \mathbb{R})$ is a solution of (1.1), then x is a solution of

$$x(t) = x(a) + \int_a^t g(x)(s)\, ds + \int_a^t f(s, x(s))\, ds, \ t \in [a,b] \tag{3.2}$$

and if $x \in C([a,b], \mathbb{R})$ is a solution of (3.2), then $x \in C^1([a,b], \mathbb{R})$ and is a solution of (1.1).

Let us consider the following operators $B_f, E_f \colon C([a,b], \mathbb{R}) \to C([a,b], \mathbb{R})$ defined by $B_f(x)(t) :=$ the right-hand side of (3.1) and $E_f(x)(t) :=$ the right-hand side of (3.2).

The first result of the paper is the following:

Theorem 3.1. *We suppose that the conditions* (C_1), (C_2), *and* (C_3) *are satisfied. Then*

(a) *the problem* (1.1)–(1.2) *has in* $C([a,b], \mathbb{R})$ *a unique solution;*

(b) *the operator* B_f *is PO in* $C([a,b], \mathbb{R})$;

(c) *the operator* E_f *is WPO in* $C([a,b], \mathbb{R})$.

Proof. Consider on $X = C([a,b], \mathbb{R})$ the Bielecki norm $\|\cdot\|_\tau$ defined by

$$\|x\|_\tau = \sup_{t \in [a,b]} \left(\|x(t)\|\, e^{-\tau(t-a)} \right), \quad \tau > 0.$$

For $x_0 \in \mathbb{R}$, we consider

$$X_{x_0} := \{x \in C[a,b] \mid x(a) = x_0\}.$$

We remark that $X = \cup_{x_0 \in \mathbb{R}} X_{x_0}$ is a partition of $C[a,b]$ and

(1) $B_f(X) \subset X_{x_0}$ and $E_f(X_{x_0}) \subset X_{x_0}$, $\forall x_0 \in \mathbb{R}$;

(2) $B_f|_{X_{x_0}} = E_f|_{X_{x_0}}$, $\forall x_0 \in \mathbb{R}$.

We have

$$\left\|B_f(x) - B_f(y)\right\|_\tau \le \tfrac{1}{\tau}(L_g + L_f)\left\|x - y\right\|_\tau, \quad \forall x, y \in X.$$

On the other hand, for a suitable choice of $\tau > 0$ such that $\frac{1}{\tau}(L_g + L_f) < 1$, we have that B_f is a contraction in $(X, \|\cdot\|_\tau)$. So, we obtain (a) and (b). Moreover the operator $E_f|_{X_{x_0}} : X_{x_0} \to X_{x_0}$ is a contraction and from the characterization theorem of WPO (Theorem 2.11) we have that E_f is c-WPO with $c = \left[1 - \frac{1}{\tau}(L_g + L_f)\right]^{-1}$. □

Next we study the relation between the solution of the problem (1.1)–(1.2) and the subsolution of the same problem. We have the following theorem.

Theorem 3.2 (Theorem of Čaplygin type)**.** *We suppose that:*

(a) the conditions (C_1), (C_2) *and* (C_3) *are satisfied;*

(b) $f(x, \cdot) \colon \mathbb{R} \to \mathbb{R}$ *is increasing;*

(c) $g \colon C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ *is increasing.*

Let x be a solution of equation (1.1) *and y a solution of the inequality*

$$y'(t) \le g(y)(t) + f(t, y(t)), \quad t \in [a, b].$$

Then $y(a) \le x(a)$ implies that $y \le x$.

Proof. We have the following two relations

$$x = E_f(x) \quad \text{and} \quad y \le E_f(y).$$

From the conditions (C_1), (C_2), and (C_3) follows that the operator E_f is WPO. Also, from conditions (b) and (c) we have that E_f is an increasing operator. Applying Lemma 2.5 we obtain that E_f^∞ is increasing. Let $x_0 \in \mathbb{R}$, then we denote by $\widetilde{x}_0$ the following function

$$\widetilde{x}_0 \colon [a, b] \to \mathbb{R}, \ \widetilde{x}_0(t) = x_0, \quad \forall t \in [a, b]. \tag{3.3}$$

From Theorem 3.1 we have that $E_f(X_{x_0}) \subset X_{x_0}$, $\forall x_0 \in \mathbb{R}$. $E_f|_{X_{x_0}}$ is a contraction and since $\widetilde{x}_0 \in X_{x_0}$ then

$$E_f^\infty(\widetilde{x}_0) = E_f^\infty(y), \quad \forall y \in X_{x_0}.$$

Let $y \le E_f(y)$, since E_f is increasing, from the Gronwall lemma (Lemma 2.6) we get $y \le E_f^\infty(y)$. Also, $y, \widetilde{y}(a) \in X_{y(a)}$, so $E_f^\infty(y) = E_f^\infty(\widetilde{y}(a))$. But $y(a) \le x(a)$, E_f^∞ is increasing and $E_f^\infty(\widetilde{x}(a)) = E_f^\infty(x) = x$. So

$$y \le E_f^\infty(y) = E_f^\infty(\widetilde{y}(a)) \le E_f^\infty(\widetilde{x}(a)) = x.$$

So the proof is completed. □

Now we study the monotony of the system (1.1)–(1.2) with respect to f. For this we use Lemma 2.7.

Theorem 3.3 (Comparison theorem)**.** *We suppose that* $f_i \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $i = 1, 2, 3$ *satisfy the conditions* (C_1), (C_2), *and* (C_3). *Furthermore, we suppose that:*

(i) $f_1 \le f_2 \le f_3$ *and* $g_1 \le g_2 \le g_3$;

(ii) $f_2(t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ *is increasing;*

(iii) $g_2\colon C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ *is increasing.*

Let $x_i \in C^1([a,b],\mathbb{R})$ *be a solution of the equation*

$$x_i'(t) = g_i(x)(t) + f_i(t,x(t)), \quad t \in [a,b] \text{ and } i = 1,2,3.$$

If $x_1(a) \le x_2(a) \le x_3(a)$, *then* $x_1 \le x_2 \le x_3$.

Proof. From Theorem 3.1 we have that the operators $E_{f_i}, i = 1,2,3$, are WPO$_s$. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that $E_{f_1} \le E_{f_2} \le E_{f_3}$.

Let $\widetilde{x}_i(a) \in C([a,b],\mathbb{R})$ be defined by $\widetilde{x}_i(a)(t) = x_i(a)$, $\forall t \in [a,b]$. It is clear that

$$\widetilde{x}_1(a)(t) \le \widetilde{x}_2(a)(t) \le \widetilde{x}_3(a)(t), \quad \forall t \in [a,b].$$

From Lemma 2.7 we have that $E^{\infty}_{f_1}(\widetilde{x}_1(a)) \le E^{\infty}_{f_2}(\widetilde{x}_2(a)) \le E^{\infty}_{f_3}(\widetilde{x}_3(a))$.

But $x_i = E^{\infty}_{fi}(\widetilde{x}_i(a))$, $i = 1,2,3$ and therefore applying Lemma 2.7 we get that $x_1 \le x_2 \le x_3$. □

Consider the Cauchy problem (1.1)–(1.2) and suppose the conditions of Theorem 3.1 are satisfied. Denote by $x^*(\cdot; x_0, g, f)$, the solution of this problem. We have the following result.

Theorem 3.4 (Data dependence theorem)**.** *We suppose that* x_{0i}, g_i, f_i, $i = 1,2$ *satisfy the conditions* (C_1), (C_2), *and* (C_3). *Furthermore, we suppose that there exist* $\eta_i > 0$, $i = 1,2,3$ *such that*

(i) $|x_{01}(t) - x_{02}(t)| \le \eta_1$, $\forall t \in [a,b]$;

(ii) $|g_1(u) - g_2(u)| \le \eta_2$, $\forall t \in [a,b]$, $u \in C([a,b],\mathbb{R})$;

(iii) $|f_1(t,v) - f_2(t,v)| \le \eta_3$, $\forall t \in [a,b]$, $v \in \mathbb{R}$.

Then

$$\|x_1^*(t; x_{01}, g_1, f_1) - x_2^*(t; x_{02}, g_2, f_2)\| \le \frac{\eta_1 + (b-a)(\eta_2 + \eta_3)}{1 - \frac{1}{\tau}(L_g + L_f)},$$

where $x_i^*(t; x_{0i}, g_i, f_i)$, $i = 1,2$ *are the solution of the problem* (1.1)–(1.2) *with respect to* x_{0i}, g_i, f_i, $L_f = \max\{L_{f_1}, L_{f_2}\}$ *and* $L_g = \max\{L_{g_1}, L_{g_2}\}$.

Proof. Consider the operators $B_{x_{0i},g_i,f_i} = x_{0i} + \int_a^t g_i(x)(s)ds + \int_a^t f_i(s,x(s))\,ds, i = 1,2$. From Theorem 3.1 these operators are c_i-PO$_s$ with $c_i = \left[1 - \frac{1}{\tau}(L_g + L_f)\right]^{-1}$. On the other hand

$$\left\|B_{x_{01},g_1,f_1}(x) - B_{x_{02},g_2,f_2}(x)\right\| \le \eta_1 + (b-a)(\eta_2 + \eta_3), \quad \forall x \in C[a,b].$$

Now the proof follows from Lemma 2.9. □

Applying Lemma 2.10 we have the theorem:

Theorem 3.5. *We suppose that* f_1 *and* f_2 *satisfy the conditions* (C_1), (C_2), *and* (C_3). *Let* $S_{E_{f_1}}, S_{E_{f_2}}$ *be the solution set of system* (1.1) *corresponding to* f_1 *and* f_2. *Suppose that there exist* $\eta_i > 0$, $i = 1,2$, *such that*

$$|g_1(u) - g_2(u)| \le \eta_1 \quad \text{and} \quad |f_1(t,v) - f_2(t,v)| \le \eta_2 \tag{3.4}$$

for all $t \in [a,b]$, $u \in C([a,b],\mathbb{R})$, $v \in \mathbb{R}$. *Then*

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \le \frac{(b-a)(\eta_1 + \eta_2)}{1 - \frac{1}{\tau}(L_g + L_f)},$$

where $L_f = \max\{L_{f_1}, L_{f_2}\}, L_g = \max\{L_{g_1}, L_{g_2}\}$ *and* $H_{\|\cdot\|_C}$ *denotes the Pompeiu–Hausdorff functional with respect to* $\|\cdot\|_C$ *on* $C[a,b]$.

4 Application

Next we give an application concerning the results from the main section.

Example 4.1. Consider the following functional-differential equation (see [6])

$$x'(t) = \int_0^t K(t,s,x(s),x(\lambda s))\,ds + f(t,x(t)), \quad t \in [0,1]. \tag{4.1}$$

For this equation the conditions (C_1)–(C_3) have the form

(a) $\lambda \in (0,1),\ K : C([0,1]\times[0,1]\times\mathbb{R}^2) \to \mathbb{R},\ f\colon C([0,1]\times\mathbb{R}) \to \mathbb{R}$;

(b) there exists $L_f > 0$ such that

$$|f(t,u_1) - f(t,u_2)| \le L_f\,|u_1 - u_2|\,, \quad \forall t \in [0,1],\ u_1,u_2 \in \mathbb{R};$$

(c) there exists $L_K > 0$ such that

$$|K(t,s,u_1,u_2) - K(t,s,v_1,v_2)| \le L_g(|u_1 - v_1| + |u_2 - v_2|), \\ \forall t,s \in [0,1], u_1,v_1,u_2,v_2 \in \mathbb{R};$$

(d) there exists $\tau > 0$ such that $\frac{2L_K}{\tau^2} + \frac{L_f}{\tau} \le 1$.

If $x \in C^1([0,1],\mathbb{R})$ is a solution of (4.1), then x is a solution of

$$x(t) = x(0) + \int_0^t \int_0^p K(p,s,x(s),x(\lambda s))\,ds\,dp + \int_0^t f(s,x(s))\,ds, \quad t \in [0,1] \tag{4.2}$$

and if $x \in C([0,1],\mathbb{R})$ is a solution of (4.2) then $x \in C^1([0,1],\mathbb{R})$ and is a solution of (4.1).

Let us consider the following operator $E_f\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$E_f(x)(t) := x(0) + \int_0^t \int_0^p K(p,s,x(s),x(\lambda s))\,ds\,dp + \int_0^t f(s,x(s))\,ds, \quad t \in [0,1].$$

Consider on $X = C([0,1],\mathbb{R})$ the Bielecki norm $\|\cdot\|_\tau$ defined by

$$\|x\|_\tau = \sup_{t\in[0,1]} (\|x(t)\|\,e^{-\tau t}),$$

with $\tau > 0$ from (d). For $\alpha \in \mathbb{R}$, we consider $X_\alpha := \{x \in C[0,1] \mid x(0) = \alpha\}$.

We remark that $X = \cup_{\alpha\in\mathbb{R}} X_\alpha$ is a partition of $C[0,1]$ and $E_f(X_\alpha) \subset X_\alpha$, $\forall \alpha \in \mathbb{R}$. From the conditions of Theorem 3.1 we have that the operator E_f is WPO in $C([0,1],\mathbb{R})$. Also one can apply the Theorems 3.2, 3.3 and 3.5 for the study of Čaplygin inequalities, monotony and data dependence of the solution of equation (4.1).

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