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Bilateral contact problem with adhesion and damage

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Abstract. We study a mathematical problem describing the frictionless adhesive contact between a viscoelastic material with damage and a foundation. The adhesion process is modeled by a bonding field on the contact surface. The contact is bilateral and the tangential shear due to the bonding field is included. We establish a variational formulation for the problem and prove the existence and uniqueness of the solution. The existence of a unique weak solution for the problem is established using arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result for parabolic inequalities, and Banach's fixed point theorem.

Keywords: dynamic process, viscoelastic material with damage, adhesion, bilateral frictionless contact, existence and uniqueness, fixed point.

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1 Introduction

Processes of adhesion are important in industry where parts, usually non metallic, are glued together. Recently, composite materials reached prominence, since they are very strong and light, and therefore, of considerable importance in aviation, space exploration and in the automotive industry. However, composite materials may undergo delamination under stress, in which different layers debond and move relative to each other. To model the process when bonding is not permanent, and debonding may take place, we need to describe the adhesion together with the contact. A number of recent publications deal with such models, see, e.g. [4, 5, 7, 13, 14, 18, 19] and references therein. The main new idea in these papers is the introduction of an internal variable, the bonding field, which has values between zero and one, and which describes the fractional density of active bonds on the contact surface. Reference [11] deals with the static and quasistatic problems, and their numerical approximations. A model for the process of dynamic, frictionless, adhesive contact between a viscoelastic body and a foundation was recently considered in [13]. There the contact was modeled with normal compliance and the material was assumed to be linearly viscoelastic.

The present paper represents a continuation of [13, 14] and deals with a model for the dynamic, adhesive and the frictionless contact between a viscoelastic body and a foundation.

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The difference consits in the fact that here we assume a bilateral contact and we use a nonlinear Kelvin–Voigt viscoelastic constitutive law with growth assumptions on the viscoelastic operator, which leads to a new and nonstandard mathematical model. As in [6, 11], we use the bonding field as an additional dependent variable, defined and evolving on the contact surface. Our purpose is to provide the existence of a unique weak solution to the model.

The subject of damage is extremely important in design engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [8, 9] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [10]. In all these papers the damage of the material is described by a damage function α restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$ the material is completely damaged, when $0 < \alpha < 1$ there is a partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [11, 17].

In this paper, the inclusion describing the evolution of damage field is

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K\,(\alpha) \ni \phi(\varepsilon(\mathbf{u}), \alpha),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\zeta \in H^1(\Omega)\ \mid\ 0 \le \zeta \le\ 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system. A general viscoelastic constitutive law with damage is given by

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}, \alpha),$$

where $\mathcal{A}$ is the nonlinear viscosity function, $\mathcal{G}$ is the nonlinear elasticity function which depends on the internal state variable describing the damage of the material caused by elastics deformations, and the dot represents the time derivative, i.e.,

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t}, \quad \ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial^2 t}.$$

The main aim of this paper is to couple a viscoelastic problem with damage and a frictionless contact problem with adhesion. We study a dynamic problem of frictional adhesive contact. We model the material behavior with a viscoelastic constitutive law with damage and the contact with normal compliance with adhesion. We derive a variational formulation and prove the existence and uniqueness of a weak solution.

The paper is organized as follows. In Section 2 we introduce the notation and give some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is based on the theory of evolution equations with monotone operators, a fixed point argument and a classical existence and uniqueness result for parabolic inequalities.

2 Notation and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [6].

We denote by $\mathbb{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d$, $(d = 2, 3)$ while $(\cdot, \cdot)$ and $|\cdot|$ represent the inner product and Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}_d$ respectively.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notations

$$\begin{aligned} H &= L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \\ \mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ H_1 &= \{\mathbf{u} = (u_i) \in H \mid \varepsilon(u) \in \mathcal{H}\}, \\ \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \mid \operatorname{Div} \sigma \in H\}, \end{aligned}$$

where $\boldsymbol{\varepsilon} \colon H_1 \to \mathcal{H}$ and $\operatorname{Div} \colon \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,i}).$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is assumed, and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i \cdot v_i \, dx, \quad \forall \mathbf{u}, \mathbf{v} \in H, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \cdot \tau_{ij} \, dx, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}, \\ (\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in H_1, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1. \end{aligned}$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma \colon H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$ we also write $\mathbf{v}$ for the trace $\gamma \mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v} \cdot \nu, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} \colon \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\boldsymbol{\sigma} \nu).\nu, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \nu - \sigma_\nu \nu. \tag{2.2}$$

We recall that the following Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma} \nu.\mathbf{v} \, da, \quad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Moreover, for a real number r, we use r_+ to represent its positive part, that is, $r_+ = \max\{0, r\}$.

3 Problem statement

The physical setting is as follows. A viscoelastic body occupies the domain $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$ with outer Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}\Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$ and, therefore, the displacement field vanishes there. A volume force density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$ and surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$. The body is in bilateral adhesive and frictionless contact with an obstacle, the so-called foundation, over the contact surface Γ_3. Moreover, the process is dynamic, and thus the inertial terms are included is the equation of motion. We use a viscoelastic constitutive law with damage to model the material's behavior and an ordinary differential equation to describe the evolution of the bonding field. The classical formulation of the problem may be stated as follows.

Problem P

Find a displacement field $\mathbf{u}\colon \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \times [0, T] \to \mathbb{S}^d$, a damage field $\alpha\colon \Omega \times [0, T] \to \mathbb{R}$, and a adhesion field $\beta\colon \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\rho\ddot{\mathbf{u}} = \operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega \times (0, T), \tag{3.1}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}(t), \alpha) \quad \text{in } \Omega \times (0, T), \tag{3.2}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi(\varepsilon(\mathbf{u}), \alpha), \tag{3.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{3.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.5}$$

$$\mathbf{u}_\nu = \mathbf{0} \quad \text{on } \Gamma_3 \times (0, T), \tag{3.6}$$

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta, \mathbf{u}_\tau) \quad \text{on } \Gamma_3 \times (0, T), \tag{3.7}$$

$$\dot{\beta} = H_{ad}(\beta, \mathbf{R}(|u_\tau|)) \quad \text{on } \Gamma_3 \times (0, T), \tag{3.8}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0, T), \tag{3.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0 \quad \text{in } \Omega, \tag{3.10}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{3.11}$$

Equation (3.1) represents the equation of motion in which ρ denotes the material mass density. The relation (3.2) represents the nonlinear viscoelastic constitutive law with damage introduced in Section 1, the evolution of the damage field is governed by the inclusion (3.3), where ϕ is the mechanical source of the damage growth, assumed to be a rather general function of the strains and damage itself, and $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Conditions (3.4) and (3.5) are the displacement and traction boundary conditions, respectively. Condition (3.6) shows that the contact is bilateral, i.e., there is no loss of the contact during the process, while condition (3.7) shows that the tangential traction depends on the intensity of adhesion and the tangential displacement. Equation (3.8) governs the evolution of the adhesion field, here H_{ad} is a general function discussed below and $\mathbf{R}\colon \mathbb{R}_+ \to [0, L]$ is the truncation function defined as

$$\begin{cases} s & \text{if } 0 \le s \le L, \\ L & \text{if } s > L, \end{cases} \tag{3.12}$$

where $L > 0$ is a characteristic lenght of the bonds (see, e.g., [2]). Equation (3.9) represents a homogeneous Neumann boundary condition where $\frac{\partial\alpha}{\partial\nu}$ represents the normal derivative of α.

In (3.10) we consider the initial conditions where $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ the initial velocity and α_0 the initial damage. Finally, (3.11) is the initial condition, in which β_0 denotes the initial bonding.

To obtain the variational formulation of the problem (3.1)–(3.11), we introduce subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \mid \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since meas$(\Gamma_1) > 0$ Korn's inequality holds and there exists a constant $C_k > 0$ which depends only Ω and Γ_1 such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geq C_k |\mathbf{v}|_{H_1}, \quad \forall \mathbf{v} \in V.$$

On V we consider the inner product and the associated norms given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

$$|\mathbf{v}|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}}, \quad \forall \mathbf{v} \in V.$$

It follows from Korn's inequality that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V and therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a constant C_0 depending only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0 |\mathbf{v}|_V, \quad \forall \mathbf{v} \in V. \tag{3.13}$$

In the study of the mechanical problem (3.1)–(3.11), we make the following assumptions. The *viscosity operator* $\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following assumptions.

$$\begin{cases} \text{(a)} & \text{There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ & |\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)| \leq L_{\mathcal{A}} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} & \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & (\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{A}} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d)} & \text{The mapping } \boldsymbol{\xi} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}) \text{ is continuous } \mathbb{S}^d \text{ a.e. } x \in \Omega. \end{cases} \tag{3.14}$$

The *elasticity operator* $\mathcal{G}\colon \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies the following assumptions.

$$\begin{cases} \text{(a)} & \text{There exists } M_{\mathcal{G}} > 0 \text{ such that} \\ & |\mathcal{G}(\mathbf{x}, \boldsymbol{\xi}_1, \alpha_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}_2, \alpha_2)| \leq M_{\mathcal{G}}(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|) \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega \\ & \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d, \ \alpha \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, 0, 0) \ \in \mathcal{H}. \end{cases} \tag{3.15}$$

The damage source function $\phi\colon \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions.

$$\begin{cases} \text{(a)} & \text{There exists } M_{\phi} > 0 \text{ such that} \\ & |\phi(\mathbf{x}, \boldsymbol{\xi}_1, \alpha_1) - \phi(\mathbf{x}, \boldsymbol{\xi}_2, \alpha_2)| \leq M_{\phi}(|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| + |\alpha_1 - \alpha_2|). \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} & \text{For any } \boldsymbol{\xi} \in \mathbb{S}^d, \ \alpha \in \mathbb{R}, \ \mathbf{x} \to \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto \phi(\mathbf{x}, 0, 0) \in L^2(\Omega). \end{cases} \tag{3.16}$$

The tangential contact function $p_\tau\colon \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following assumptions.

$$\begin{cases} \text{(a)} & \text{There exists } L_\tau > 0 \text{ such that} \\ & |p_\tau(\mathbf{x},\beta_1,r_1) - p_\tau(\mathbf{x},\beta_2,r_2)| \le L_\tau(|\beta_1-\beta_2| + |r_1-r_2|) \\ & \forall \beta_1,\beta_2 \in \mathbb{R}, \quad r_1,r_2 \in \mathbb{R}^d, \quad \text{a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(b)} & \text{The mapping } \mathbf{x} \mapsto p_\tau(\mathbf{x},\beta,r) \text{is Lebesgue measurable on } \Gamma_3 \\ & \forall \beta \in \mathbb{R}, r \in \mathbb{R}^d \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto p_\tau(\mathbf{x},0,0) \in L^\infty(\Gamma_3)^d \\ \text{(d)} & p_\tau(\mathbf{x},\beta,r).\nu(\mathbf{x}) = 0, \ \ \forall r \in \mathbb{R}^d \text{ such that } r.\nu(\mathbf{x}) = 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.17}$$

The adhesion function $H_{ad}\colon \Gamma_3 \times \mathbb{R} \times [0,L] \to \mathbb{R}$ satisfies the following assumptions.

$$\begin{cases} \text{(a)} & \text{There exists } L_{Had} > 0 \text{ such that} \\ & |H_{ad}(\mathbf{x},b_1,r) - H_{ad}(\mathbf{x},b_2,r)| \le L_{Had}|b_1-b_2| \\ & \forall b_1,b_2 \ \in \mathbb{R}, \quad r \in [0,L], \text{ a.e. } \mathbf{x} \in \Gamma_3, \text{ and} \\ & |H_{ad}(\mathbf{x},b_1,r_1) - H_{ad}(\mathbf{x},b_2,r_2)| \le L_{Had}(|b_1-b_2| + |r_1-r_2|) \\ & \forall b_1,b_2 \in [0,1], \quad r_1,r_2 \in [0,L], \text{ a.e. } \mathbf{x} \in \Gamma_3, \\ \text{(b)} & \text{The mapping } \mathbf{x} \mapsto H_{ad}(\mathbf{x},b,r) \text{ is Lebesgue measurable on } \Gamma_3, \\ & \forall b \in \mathbb{R}, \quad r \in [0,L], \\ \text{(c)} & \text{The mapping } (b,r) \mapsto H_{ad}(\mathbf{x},b,r) \text{ is continuous on } \mathbb{R}\times[0,L], \\ & \text{a.e. } \mathbf{x} \in \Gamma_3, \\ \text{(d)} & H_{ad}(\mathbf{x},0,r) = 0, \quad \forall r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_3, \\ \text{(e)} & H_{ad}(\mathbf{x},b,r) \ge 0, \quad \forall b \le 0, \quad r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_3 \text{ and} \\ & H_{ad}(\mathbf{x},b,r) \le 0, \quad \forall b \ge 1, \quad r \in [0,L], \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.18}$$

We suppose that the mass density satisfies

$$\rho \in L^\infty(\Omega), \text{ and there exixts } \ \rho^* > 0, \text{ such that } \ \rho(x) \ge \rho^*, \text{ a.e. } x \in \Omega. \tag{3.19}$$

We also suppose that the body forces and surface traction have the regularity

$$\mathbf{f}_0 \in L^2(0,T;H), \quad \mathbf{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d). \tag{3.20}$$

Finally, we assume that the initial data satisfy the following conditions

$$\mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in H, \tag{3.21}$$

$$\alpha_0 \in K, \tag{3.22}$$

$$\beta_0 \in L^\infty(\Gamma_3) \text{ and } 0 \le \beta_0 \le 1 \text{ a.e. on } \Gamma_3. \tag{3.23}$$

We define the bilinear form $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\zeta,\varphi) = k\int_\Omega \nabla\zeta.\nabla\varphi\, dx. \tag{3.24}$$

We will use a modified inner product on $H = L^2(\Omega)^d$, given by

$$((\mathbf{u},\mathbf{v}))_H = (\rho\mathbf{u},\mathbf{v})_H, \quad \forall \mathbf{u},\mathbf{v} \in H,$$

that is, weighted with ρ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$|\mathbf{v}|_H = (\rho\mathbf{v},\mathbf{v})_H^{\frac{1}{2}}, \quad \forall\, \mathbf{v} \in H.$$

Using assumption (3.19), it follows that $|\cdot|_V$ and $|\cdot|_H$ are equivalent norms on H. Moreover, the inclusion mapping of $(V, |\cdot|_V)$ into $(H, |\cdot|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(\cdot,\cdot)_{V'\times V}$ to represent the duality pairing between V' and V. We have

$$(\mathbf{u},\mathbf{v})_{V'\times V} = ((\mathbf{u},\mathbf{v}))_H, \quad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Finally, we denote by $|\cdot|_{V'}$ the norm on the dual space V'. Assumptions (3.20) allow us, for a.e. $t \in (0,T)$ to define $f(t) \in V'$ by

$$(\mathbf{f}(t),\mathbf{v})_{V'\times V} = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\,dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\,da, \quad \forall \mathbf{v} \in V, \tag{3.25}$$

and

$$\mathbf{f} \in L^2(0,T;V'). \tag{3.26}$$

Let $j\colon L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ be the functional

$$j(\beta,\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} p_\tau(\beta,\mathbf{u}_\tau).\mathbf{v}_\tau\,da, \quad \forall \beta \in L^\infty(\Gamma_3), \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{3.27}$$

Keeping in mind (3.17), we observe that the integrals in (3.27) are well defined. Using standard arguments based on Green's formula (2.3) we can derive the following variational formulation of the problem P.

Problem PV

Find a displacement field $\mathbf{u}\colon [0,T] \to V$ a stress field $\boldsymbol{\sigma}\colon [0,T] \to \mathcal{H}$ a damage field $\alpha\colon [0,T] \to H^1(\Omega)$ and an adhesion field $\beta\colon [0,T] \to L^\infty(\Gamma_3)$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}(t),\alpha) \quad \text{a.e. } t \in (0,T), \tag{3.28}$$

$$\begin{aligned} \alpha(t) \in K, \quad & (\dot\alpha(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \\ & \geq (\phi(\boldsymbol{\varepsilon}(\mathbf{u}(t)),\alpha(t)), \zeta - \alpha(t))_{L^2(\Omega)}, \quad \forall \zeta \in K, \end{aligned} \tag{3.29}$$

a.e. $t \in [0,T]$,

$$(\ddot{\mathbf{u}}(t),\mathbf{v})_{V'\times V} + (\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\beta(t),\mathbf{u}(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_{V'\times V}, \quad \forall \mathbf{v} \in V, \tag{3.30}$$

a.e. $t \in [0,T]$,

$$\dot\beta(t) = H_{ad}(\beta(t), \mathbf{R}(|\mathbf{u}_\tau(t)|)), \quad 0 \leq \beta(t) \leq 1 \quad \text{a.e.}\, t \in [0,T], \tag{3.31}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0. \tag{3.32}$$

We notice that the variational problem PV is formulated in terms of the displacement, stress field, damage field and adhesion field. The existence of a unique solution of problem PV is stated and proved in the next section.

Our main result, concerning the well-posedness of the problem PV is the following.

Theorem 3.1. *Assume that* (3.14)–(3.23) *hold. Then there exists a unique solution* $\{u, \sigma, \alpha, \beta\}$ *which satisfies*

$$\mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \qquad \ddot{\mathbf{u}} \in L^2(0,T;V'), \tag{3.33}$$
$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \qquad \operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;V'), \tag{3.34}$$
$$\alpha \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1,(\Omega)) \tag{3.35}$$
$$\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)). \tag{3.36}$$

A quadruple $\{\mathbf{u}, \sigma, \alpha, \beta\}$ which satisfies (3.28)–(3.32) is called a weak solution to the Problem P. We conclude that under the stated assumptions, problem (3.1)–(3.11) has a unique solution satisfying (3.33)–(3.36). The proof of Theorem (3.1) will be carried out in several steps and is based on the theory evolution equations with monotone operators, a fixed point argument and a classical existence and uniqueness result for parabolic inequalities. To this end, we assume in the following that (3.14)–(3.23) hold. Below, C denotes a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{A}$, $\mathcal{G}$, p_τ, L and T but does not depend on t nor on the rest of the input data, and whose value may change from place to place. Moreover, for the sake of simplicity, we suppress, in what follows, the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$.

The proof of Theorem 3.1 will be provided in the next section.

4 Existence and uniqueness result

Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given. In the first step we consider the following variational problem.

Problem P_V^η

Find a displacement field $\mathbf{u}_\eta \colon [0,T] \to V$ such that

$$\begin{gathered}(\ddot{\mathbf{u}}_\eta(t), \mathbf{v})_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(v))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \mathbf{v})_{V' \times V} = (\mathbf{f}(t), \mathbf{v})_{V' \times V}, \\ \forall \mathbf{v} \in V, \ \text{a.e. } t \in (0,T),\end{gathered} \tag{4.1}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{4.2}$$

To solve Problem P_V^η, we apply an abstract existence and uniqueness result which we recall for the convenience of the reader. Let V and H denote real Hilbert spaces such that V is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of V', i.e., $V \subset H \subset V'$ we say that these inclusions define a *Gelfand triple*. The notations $|\cdot|_V$, $|\cdot|_{V'}$ and $(\cdot,\cdot)_{V' \times V}$ represent the norms on V and on V' and the duality pairing between them, respectively. The following abstract result may be found in [20, page 48].

Theorem 4.1. *Let* V, H *be as above, and let* $A \colon V \to V'$ *be a hemicontinuous and monotone operator which satisfies*

$$(A\mathbf{v}, \mathbf{v})_{V' \times V} \geq \omega |\mathbf{v}|_V^2 + \lambda, \quad \forall \mathbf{v} \in V, \tag{4.3}$$
$$\|A\mathbf{v}\|_{V'} \leq C(|\mathbf{v}|_V + 1), \quad \forall \mathbf{v} \in V. \tag{4.4}$$

for some constants $\omega > 0$, $C > 0$ *and* $\lambda \in \mathbb{R}$. *Then, given* $\mathbf{u}_0 \in H$ *and* $f \in L^2(0,T;V')$ *there exists a unique* u *which satisfies*

$$\mathbf{u} \in L^2(0,T;V') \cap C([0,T];H), \qquad \dot{\mathbf{u}} \in L^2(0,T;V'),$$
$$\frac{d\mathbf{u}}{dt} + A\mathbf{u}(t) = f(t), \quad \textit{a.e. } t \in (0,T),$$
$$\mathbf{u}(0) = \mathbf{u}_0.$$

We apply this theorem to problem P_V^η.

Lemma 4.2. *There exists a unique solution to problem* P_V^η *possessing the regularity condition expressed in* (3.33).

Proof. We define the operator $A\colon V \to V'$ by

$$(A\mathbf{u},\mathbf{v})_{V'\times V} = (\mathcal{A}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{4.5}$$

it follows from (4.5) and (3.14)(a) that

$$|A\mathbf{u} - A\mathbf{v}|_{V'} \le L_{\mathcal{A}}|\mathbf{u}-\mathbf{v}|_V, \quad \forall \mathbf{u},\mathbf{v} \in V, \tag{4.6}$$

which shows that $A\colon V \to V'$ is continuous, and so hemicontinuous. Now, by (4.5) and (3.14)(b), we find

$$(A\mathbf{u} - A\mathbf{v},\mathbf{u}-\mathbf{v})_{V'\times V} \ge m_{\mathcal{A}}|\mathbf{u}-\mathbf{v}|_V^2, \quad \forall \mathbf{u},\mathbf{v} \in V, \tag{4.7}$$

i.e., $A\colon V \to V'$ is a monotone operator. Choosing $\mathbf{v} = \mathbf{0}_V$ in (4.7) we obtain

$$(A\mathbf{u},\mathbf{u})_{V'\times V} \ge m_{\mathcal{A}}|\mathbf{u}|_V^2 - |A0_V|_{V'}\,|\mathbf{u}|_V,$$
$$(A\mathbf{u},\mathbf{u})_{V'\times V} \ge m_{\mathcal{A}}|\mathbf{u}|_V^2 - |A0_V|_{V'}|\mathbf{u}|_V \ge \frac{1}{2}m_{\mathcal{A}}|\mathbf{u}|_V^2 - \frac{1}{2m_{\mathcal{A}}}|A0_V|_{V'}, \quad \forall \mathbf{u} \in V.$$

Thus, A satisfies condition (4.3) with $\omega = \frac{1}{2}m_{\mathcal{A}}$ and $\lambda = -\frac{1}{2m_{\mathcal{A}}}|A0_V|_{V'}$. Next by (4.6) we deduce that

$$|A\mathbf{u}|_{V'} \le L_{\mathcal{A}}|\mathbf{u}|_V + |A0_V|_{V'}, \quad \forall \mathbf{u} \in V.$$

This inequality implies that A satisfies condition (4.4). Finally, we recall that by (3.26) and (3.21) we have $\mathbf{f} - \eta \in L^2(0,T;V')$ and $\mathbf{v}_0 \in H$.

It now follows from Theorem 3.1 that there exists a unique function $\mathbf{v}_\eta$ which satisfies

$$\mathbf{v}_\eta \in L^2(0,T;V) \cap C([0,T];H), \quad \frac{d\mathbf{v}_\eta}{dt} \in L^2(0,T;V'), \tag{4.8}$$

$$\frac{d\mathbf{v}_\eta}{dt} + A\mathbf{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \quad \text{a.e. } t \in (0;T), \tag{4.9}$$

$$\mathbf{v}_\eta(0) = \mathbf{v}_0. \tag{4.10}$$

Let $\mathbf{u}_\eta\colon [0;T] \to V$ be defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\,ds + \mathbf{u}_0, \quad \forall t \in [0,T]. \tag{4.11}$$

It follows from (4.5) and (4.8)–(4.11) that $\mathbf{u}_\eta$ is a solution of the variational problem P_V^η and it has the regularity expressed in (3.33). This concludes the proof of the existence part of Lemma 4.2. The uniqueness of the solution to problem (4.9)–(4.10), guaranteed by Theorem 4.1. □

In the second step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and consider the following initial-value problem.

Problem P_V^{β}

Find the adhesion field $\beta_\eta\colon [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = H_{ad}(\beta_\eta(t), \mathbf{R}(|\mathbf{u}_{\eta\tau}(t)|)), \quad \text{a.e. } t \in (0,T), \tag{4.12}$$

$$\beta_\eta(0) = \beta_0. \tag{4.13}$$

We have the following result.

Lemma 4.3. *There exits a unique solution β_η of problem P_V^{β} and it satisfies $\beta_\eta \in W^{1,\infty}(0,T; L^\infty(\Gamma_3))$. Moreover,*

$$0 \le \beta_\eta(t) \le 1, \quad \forall t \in [0,T], \quad \text{a.e. } on\ \Gamma_3. \tag{4.14}$$

Proof. For the sake of simplicity we suppress the dependence of various functions on $\mathbf{x} \in \Gamma_3$. Notice that the equalities and inequalities below are valid a.e. $\mathbf{x} \in \Gamma_3$. We consider the mapping $F\colon (0,T) \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$F(t,\beta) = H_{ad}(\beta, \mathbf{R}(|\mathbf{u}_{\eta\tau}(t)|), \text{ a.e. } t \in (0,T), \quad \forall \beta \in L^\infty(\Gamma_3).$$

It easy to check that F is Lipschitz continuous with respect to the second variable, uniformly in time; also, for all $\beta \in L^\infty(\Gamma_3)$, $t \mapsto F(t,\beta)$ belongs to $L^\infty(0,T; L^\infty(\Gamma_3))$.
Thus, the existence of a unique function β_η which satisfies(4.12)–(4.13) follows from a version of the Cauchy–Lipschitz theorem.

To check (4.14), we suppose that $\beta_\eta(t_0) < 0$ for some $t_0 \in [0,T]$. By assumption ($\beta_0 \in L^\infty(\Gamma_3)$, $0 \le \beta_0(x) \le 1$ a.e. $x \in \Gamma_3$) we have $0 \le \beta_\eta(0) \le 1$ and therefore $t_0 > 0$, moreover, since the mapping $t \mapsto \beta(t)\colon [0,T] \to \mathbb{R}$ is continuous, we can find $t_1 \in [0, t_0)$ such that $\beta_\eta(t_1) = 0$.

Now, let $t_2 = \sup\{t \in [t_1, t_0],\ \beta_\eta(t) = 0\}$, then $t_2 < t_0$, $\beta_\eta(t_2) = 0$ and $\beta_\eta(t) < 0$ for $t \in (t_2, t_0]$. Assumption (3.18)(e) and equation (4.12) imply that $\dot{\beta}_\eta(t) \ge 0$ for $t \in (t_2, t_0]$, and therefore $\beta_\eta(t_0) \ge \beta_\eta(t_2) = 0$, which is a contradiction. We conclude that $\beta_\eta(t) \ge 0$ for all $t \in [0,T]$. A similar argument shows that $\beta_\eta(t) \le 1$ for all $t \in [0,T]$. □

We now study the dependence of the solution of problem P_V^{β} with respect to $\boldsymbol{\eta}$.

Lemma 4.4. *Let $\boldsymbol{\eta}_i \in L^2(0,T; V')$ and let β_{η_i}, $i = 1,2$, denote the solution of problem P_V^{β}, then*

$$|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)|^2_{L^2(\Gamma_3)} \le C \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|^2_{V}\, ds, \quad \forall t \in [0,T]. \tag{4.15}$$

Proof. Let $t \in [0,T]$ The equalities and inequalities below are valid a.e. $\mathbf{x} \in \Gamma_3$ and, as usual, we do not depict the dependence on $\mathbf{x}$ explicitly.

Using (4.12) and (4.13), we can write

$$\beta_i(t) = \beta_0 + \int_0^t H_{ad}(\beta_i(s), \mathbf{R}(|\mathbf{u}_{i\tau}(s)|))\, ds, \quad i = 1,2,$$

where $\beta_i = \beta_{\eta_i}$ and $\mathbf{u}_i = \mathbf{u}_{\eta_i}$. Using now the adhesion rate function H_{ad} and the definition (3.18)(a) of the truncation $\mathbf{R}$, we obtain

$$|\beta_1(t) - \beta_2(t)| \le L_{Had} \int_0^t |\beta_1(s) - \beta_2(s)|\, ds + L_{Had} \int_0^t |\mathbf{u}_{1\tau}(s) - \mathbf{u}_{2\tau}(s)|\, ds.$$

We apply now Gronwall's inequality to deduce that

$$|\beta_1(t) - \beta_2(t)| \leq C \int_0^t |\mathbf{u}_{1\tau}(s) - \mathbf{u}_{2\tau}(s)|\, ds,$$

which implies that

$$|\beta_1(t) - \beta_2(t)|^2 \leq C \int_0^t |\mathbf{u}_{1\tau}(s) - \mathbf{u}_{2\tau}(s)|^2\, ds.$$

Integrating the last inequality over Γ_3 we find

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)^d} \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_{L^2(\Gamma_3)^d}\, ds,$$

and from (3.13), we obtain

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)^d} \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds.$$

□

In the third step, let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem P_V^θ

Find a damage field $\alpha_\theta\colon [0,T] \to H^1(\Omega)$ such that

$$\alpha_\theta(t) \in K, \quad (\dot{\alpha_\theta}(t), \zeta - \alpha_\theta(t))_{L^2(\Omega)} + a(\alpha_\theta(t), \zeta - \alpha_\theta(t)) \geq (\phi(t), \zeta - \alpha_\theta(t))_{L^2(\Omega)}, \quad \forall \zeta \in K, \text{ a.e. } t \in (0,T), \tag{4.16}$$

$$\alpha_\theta(0) = 0. \tag{4.17}$$

To solve P_V^θ we recall the following standard result for parabolic variational inequalities (see, e.g., [1, page 124]).

Theorem 4.5. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed and convex set of V. Assume that $a(\cdot,\cdot)\colon V \times V' \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0*

$$a(\mathbf{v}, \mathbf{v}) + c_0\, |\mathbf{v}|^2_H \geq \zeta |\mathbf{v}|^2_V, \quad \forall \mathbf{v} \in V.$$

Then for every $\mathbf{u}_0 \in K$ and $\mathbf{f} \in L^2(0,T;H)$ there exists a unique function $\mathbf{u} \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$.

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V' \times V} + a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u}(t))_H, \quad \forall\, \mathbf{v} \in K.$$

We apply this theorem to problem P_V^θ.

Lemma 4.6. *Problem R_V^θ has a unique solution α_θ such that*

$$\alpha_\theta \in H^1(0,T;L^2(\Omega) \cap\; L^2(0,T;H^1(\Omega)). \tag{4.18}$$

The inclusion of $(H^1(\Omega), |\cdot|_{H^1(\Omega)})$ into $(L^2(\Omega), |\cdot|_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(\alpha, \zeta)_{(H^1(\Omega))' \times H^1(\Omega)}$ for the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We have

$$(\alpha, \zeta)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha, \zeta)_{L^2(\Omega)}, \quad \forall \alpha \in L^2(\Omega),\ \zeta \in H^1(\Omega),$$

and we note that K is closed convex set in $H^1(\Omega)$. Then, using the definition (3.24) of the bilinear form a and the fact that $\alpha_0 \in K$ in (3.22), it is easy to see that Lemma 4.6 is a straightforward consequence of Theorem 4.5.

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the functional j and the function ϕ for $t \in [0, T]$ we consider the element

$$\Lambda(\boldsymbol{\eta}, \theta)(t) = (\Lambda^1(\boldsymbol{\eta}, \theta)(t), \Lambda^2(\boldsymbol{\eta}, \theta)(t)) \in V' \times L^2(\Omega), \tag{4.19}$$

defined by the equalities

$$(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \mathbf{v})_{V' \times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t), \alpha_\theta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}), \quad \forall \mathbf{v} \in V, \tag{4.20}$$

$$\Lambda^2(\boldsymbol{\eta}, \theta)(t) = \phi(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \alpha_\theta(t)), \quad \forall \mathbf{v} \in V. \tag{4.21}$$

We have the following result.

Lemma 4.7. *For $(\boldsymbol{\eta}, \theta) \in L^2(0, T; V' \times L^2(\Omega))$, the function $\Lambda(\boldsymbol{\eta}, \theta)\colon [0, T] \to V' \times L^2(\Omega)$ is continuous, and there is a unique element $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0, T; V' \times L^2(\Omega))$ such that $\Lambda(\boldsymbol{\eta}^*, \theta^*) = (\boldsymbol{\eta}^*, \theta^*)$.*

Proof. Let $(\boldsymbol{\eta}, \theta) \in L^2(0, T; V' \times L^2(\Omega))$ and $t_1, t_2 \in [0, T]$. Using (3.15), (3.16) and (3.17), we have

$$\begin{aligned} |\Lambda^1(\boldsymbol{\eta}, \theta)(t_1) - \Lambda^1(\boldsymbol{\eta}, \theta)(t_2)|_{V'} &\le |\mathcal{G}\varepsilon(u_\eta(t_1), \alpha_\theta(t_1)) - \mathcal{G}\varepsilon(u_\eta(t_2) - \alpha_\theta(t_2))|_{\mathcal{H}} \\ &\quad + |j(\beta_{u_\eta}(t_1) - \beta_{u_\eta}(t_2), u_\eta(t_1) - u_\eta(t_2)| \\ &\le C(|u_\eta(t_1) - u_\eta(t_2)|_V + |\alpha_\theta(t_1) \ - \alpha_\theta(t_2)|_{L^2(\Omega)} \\ &\quad + |\beta_{u_\eta}(t_1) - \beta_{u_\eta}(t_2)|_{L^2(\Gamma_3)}). \end{aligned} \tag{4.22}$$

Recall that above $\mathbf{u}_{\eta\nu}$, $\mathbf{u}_{\eta\tau}$ denote the normal and tangential components of the function $\mathbf{u}_\eta$, respectively. Next, due to the regularities of $\mathbf{u}_\eta$, α_θ and β_η expressed in (3.33), (3.35) and (3.36) respectively, we deduce from (4.22) that $\Lambda^1(\boldsymbol{\eta}, \theta) \in C(0, T; V')$. By a similar argument, from (4.21) and (3.16) it follows that

$$|\Lambda^2(\boldsymbol{\eta}, \theta)(t_1) - \Lambda^2(\boldsymbol{\eta}, \theta)(t_2)|_{L^2(\Omega)} \le C(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\alpha_\theta(t_1) - \alpha_\theta(t_2)|_{L^2(\Omega)}). \tag{4.23}$$

Therefore, $\Lambda^2(\boldsymbol{\eta}, \theta) \in C(0, T; L^2(\Omega))$ and $\Lambda(\boldsymbol{\eta}, \theta) \in C(0, T; V' \times L^2(\Omega))$.

Let now $(\boldsymbol{\eta}_1, \theta_1), (\boldsymbol{\eta}_2, \theta_2) \in L^2(0, T; V' \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i$, $\alpha_{\theta_i} = \alpha_i$ and $\beta_{\eta_i} = \beta_i$ for $i = 1, 2$. Arguments similar to those used in the proof of (4.22) and (4.23) yield

$$\begin{aligned} &|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)|^2_{V' \times L^2(\Omega)} \\ &\quad \le C\left(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\alpha_1(t) - \alpha_2(t)|^2_{L^2(\Omega)} + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)}\right). \end{aligned} \tag{4.24}$$

Since

$$u_i(t) = \int_0^t v_i(s)\,ds + \mathbf{u}_0, \quad \forall t \in [0,T],$$

we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \le C \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2\,ds, \quad \forall t \in [0,T]. \tag{4.25}$$

Moreover, from (4.1) we infer that a.e. on $(0,T)$

$$(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2), \boldsymbol{\varepsilon}(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} + (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} = 0.$$

We integrate this equality with respect to time. We use the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$ and (3.14) to find that

$$m_{\mathcal{A}} \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2\,ds \le -\int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_{V'\times V}\,ds, \quad \forall t \in [0,T].$$

Then, using the inequality $2ab \le \frac{a^2}{\gamma} + \gamma b^2$ we obtain

$$\int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V^2\,ds \le C \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_{V'}^2\,ds, \quad \forall t \in [0,T]. \tag{4.26}$$

On the other hand, from the Cauchy problem adhesion we can write

$$\beta_i(t) = \beta_0 + \int_0^t H_{ad}(\beta_i(s), \mathbf{R}(|\mathbf{u}_{i\tau}(s))) \,ds, \quad i = 1,2,$$

and then

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \le \int_0^t |H_{ad}(\beta_1(s), \mathbf{R}(|\mathbf{u}_{1\tau}(s)|) - H_{ad}(\beta_2(s), \mathbf{R}(|\mathbf{u}_{2\tau}(s)|)|\,ds,$$

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \le L_{Had} \int_0^t |\beta_1(s) - \beta_2(s)| ds + L_{Had} \int_0^t |\mathbf{u}_{1\tau}(s) - \mathbf{u}_{2\tau}(s)|\,ds.$$

Using the definition of $\mathbf{R}(|\mathbf{u}_\tau|)$

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \le C\left(\int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)}\,ds + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{(L^2(\Gamma_3))^d}\,ds\right). \tag{4.27}$$

Next, we apply Gronwall's inequality to deduce

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \le C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{(L^2(\Gamma_3))^d}\,ds, \quad \forall t \in [0,T],$$

and from (3.13) we obtain

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)}^2 \le C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2\,ds, \quad \forall t \in [0,T]. \tag{4.28}$$

From (4.16) we deduce that

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2)) \le (\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \quad \text{a.e. } \in (0,T).$$

Integrating the inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and the inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \ge 0$ we find

$$\frac{1}{2}|\alpha_1(t) - \alpha_2(t)|_{L^2(\Omega)}^2 \le C \int_0^t (\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)}\,ds,$$

which implies that

$$|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)} \le C\int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds + \int_0^t |\alpha_1(s)-\alpha_2(s)|^2_{L^2(\Omega)}\,ds, \tag{4.29}$$

This inequality combined with Gronwall's inequality leads to

$$|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)} \le C\int_0^t |\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds, \quad \forall t\in[0,T]. \tag{4.30}$$

We substitute (4.28) in (4.24) and we use (4.25) to obtain

$$\begin{aligned}
&|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)}\\
&\quad\le C\left(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V + \int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V\,ds + |\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)}\right)\\
&\quad\le C\left(\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2_V ds + |\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Omega)}\right).
\end{aligned}$$

It now follows from the above and the estimates (4.26) and (4.30) that

$$|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)} \le C(\int_0^t |(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)|^2_{V'\times L^2(\Omega)}\,ds).$$

Reiterating this inequality m times leads to

$$|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))} \le \frac{(CT)^m}{m!}|(\eta_1,\theta_1)-(\eta_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))}.$$

Thus for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V'\times L^2(\Omega))$ and so Λ has a unique fixed point. □

Now, we have all the ingredients needed to prove Theorem 3.1

Existence

Let $(\boldsymbol{\eta}^*,\theta^*)\in L^2(0,T;V'\times L^2(\Omega))$ be the fixed point of Λ given by (4.19). Denote by $\mathbf{u}_{\eta^*}$ the solution of problem P_V^η for $\boldsymbol{\eta}=\eta^*$ and let α_{θ^*} be the solution of problem P_V^θ for $\theta=\theta^*$. Let β_{η^*} be the solution of problem P_V^β for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$. We denote by $\boldsymbol{\sigma}_{\eta^*}$ the function given by $\boldsymbol{\sigma}_{\eta^*}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_{\eta^*}(t))+\mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}_{\eta^*}(t),\alpha_{\eta^*}(t))$.

Using (4.20), (4.21), keeping in mind that $\Lambda^1(\boldsymbol{\eta}^*,\theta^*)=\boldsymbol{\eta}^*$; $\Lambda^2(\boldsymbol{\eta}^*,\theta^*)=\theta^*$, we find that the quadruplet $\{\mathbf{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\alpha_{\theta^*},\beta_{\eta^*}\}$ is a solution of problem PV. This solution has the regularity expressed in (3.33)–(3.36); this follows from the regularities of the solutions of problems P_V^η, P_V^θ and P_V^β. Moreover, it follows from (3.33), (3.14) and (3.15) that $\boldsymbol{\sigma}_{\eta^*}\in L^2(0,T;\mathcal{H})$. Choosing now $v=\varphi$ in (3.30), where $\varphi\in C_0^\infty(\Omega)^d$, and using (3.19) and (3.27) we find

$$\rho\ddot{\mathbf{u}}(t)=\mathrm{Div}\,\boldsymbol{\sigma}(t)+\mathbf{f}_0(t), \quad \text{a.e. } t\in(0,T).$$

Then assumptions (3.19) and (3.20), the regularity expressed in (3.33) and the above equality imply that $\mathrm{Div}\,\boldsymbol{\sigma}\in L^2(0,T;V')$ which shows that $\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H})$.

Uniqueness

Let $\{\mathbf{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}, \alpha_{\theta^*}, \beta_{\eta^*}\}$ be the solution of PV obtained above and let $\{\mathbf{u}, \boldsymbol{\sigma}, \alpha, \beta\}$ be another solution which satisfies (3.33)–(3.36).

We denote by $\boldsymbol{\eta} \in L^2(0,T;V')$ and $\theta \in L^2(0,T;L^2(\Omega))$ the functions

$$(\boldsymbol{\eta}(t), \mathbf{v})_{V'\times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}(t), \alpha(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\beta(t), \mathbf{u}(t), \mathbf{v}), \tag{4.31}$$

$$\theta(t) = \phi(\boldsymbol{\varepsilon}\mathbf{u}(t), \alpha(t)). \tag{4.32}$$

Equalities (3.28), (3.30) and (4.31) with the initial condition $\mathbf{u}(0) = \mathbf{u}_0$ imply that $\mathbf{u}$ is a solution of P_V^{η} and since it follows from Lemma 4.2 that this problem has a unique solution, denoted $\mathbf{u}_{\eta}$ we conclude that

$$u = u_{\eta}. \tag{4.33}$$

Next, (3.31), (4.33) and (4.32) and the initial condition $\beta(0) = \beta_0$ imply that β is a solution of P_V^{β} since Lemma 4.3 shows that the problem has a unique solution, denoted β_{η}, we obtain

$$\beta = \beta_{\eta}. \tag{4.34}$$

Equalities (3.29), (4.32) and the initial condition $\alpha(0) = \alpha_0$ now imply that α is a solution of P_V^{θ}. From Lemma 4.6 problem P_V^{θ} has a unique solution, denoted α_{θ} and it follows that

$$\alpha = \alpha_{\theta}. \tag{4.35}$$

Using (4.19) and (4.31)–(4.35), we conclude that $\Lambda(\boldsymbol{\eta}, \theta) = (\boldsymbol{\eta}, \theta)$ and by uniqueness of the fixed point of Λ it follows that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^*, \quad \theta = \theta^*. \tag{4.36}$$

The uniqueness of the solution is now a consequence of (4.33)–(4.36) together with the equality $\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}(\boldsymbol{\varepsilon}(\mathbf{u}), \alpha)$.

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