Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **69**, 1–24; **http://www.math.u-szeged.hu/ejqtde/**

POSITIVE NON-SYMMETRIC SOLUTIONS OF A NON-LINEAR BOUNDARY VALUE PROBLEM

Sámuel Peres

Abstract: This paper deals with a non-linear second order ordinary differential equation with symmetric non-linear boundary conditions, where both of the non-linearities are of power type. It provides results concerning the existence and multiplicity of positive non-symmetric solutions for values of parameters not considered before. The main tool is the shooting method.

Keywords: second order ordinary differential equation; non-linear boundary condition; existence and multiplicity of solutions; shooting method; time map.

2010 Mathematics Subject Classification: 34B18, 34B15, 34B08.

1 Introduction

In this paper we study positive non-symmetric (i.e. non-even) solutions of the problem

$$\begin{cases} u''(x) = au^p(x), & x \in (-l,l), \\ u'(\pm l) = \pm u^q(\pm l) \end{cases} \tag{1}$$

for $p \in (-1,1)$, $q > \frac{p+1}{2}$, $a, l > 0$. (The choice of these conditions will soon be clarified.)

The first systematic study of positive solutions of (1) was done by M. Chipot, M. Fila and P. Quittner in [5]. They also studied the N-dimensional version of (1), but they were interested mainly in global existence and boundedness or blow-up of positive solutions of the corresponding N-dimensional parabolic problem

$$\begin{cases} u_t = \Delta u - au^p & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = u^q & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, n is the unit outer normal vector to $\partial\Omega$, $u_0 : \overline{\Omega} \to [0,\infty)$, $p, q > 1$ and $a > 0$. The same questions were independently studied in [13] for $N = 1$.

The reader can find the complete answer to the question of the existence and multiplicity of positive symmetric solutions of (1) for $p, q > 1$ in [5]. It was also proved there that (1) can possess positive non-symmetric solutions only for

$q > \frac{p+1}{2}$, but their existence and multiplicity was determined only under some additional condition. The solvability of (1) in the class of positive symmetric solutions was examined in [15] for all $p > -1$, $q \geq 0$ and $p = -1$, $q = 0$. The results of [5] concerning positive non-symmetric solutions were extended in [16] to all $p \geq 1$, $q > \frac{p+1}{2}$.

In view of the cited studies it is natural to ask the question of the existence and number of positive non-symmetric solutions of (1) for $p \in (-1, 1)$, $q > \frac{p+1}{2}$. This is what we investigate in this article.

It is known from [16] that given any $p \geq 1$, $q > \frac{p+1}{2}$ and $a > 0$, (1) has either two or no positive non-symmetric solutions, depending on the value of $l > 0$. Here we prove that (1) possesses at least four positive non-symmetric solutions for certain $p \in (-1, 1)$, $q > \frac{p+1}{2}$ and $a, l > 0$, and even infinitely many for some special choices of p, q, a and l. Moreover, the sets of (p, q) for which (1) has different multiplicity of solutions are separated by line segments and also some implicitly given curves. See Theorems 3.2 and 3.10 for the exact formulations.

Some further extensions and generalisations of the results from [5] can be found in the following studies: In [17], the behaviour of positive solutions of (2) was determined for all $p, q > 1$. Sign-changing solutions of the parabolic problem were considered in [6] for $p \geq 1$, $q > 1$—in that case, u^p and u^q are replaced by $|u|^{p-1}u$ and $|u|^{q-1}u$ respectively. The results from [6] regarding sign-changing stationary solutions for $N = 1$ were completed in [16]. Positive solutions of the elliptic problem with $-\lambda u + u^p$ on the right-hand side of the equation were dealt with in [14] for $\lambda \in \mathbb{R}$, $p, q > 1$, and later in [11] for $\lambda \in \mathbb{R}$, $p, q > 0$, $(p, q) \notin (0, 1)^2$. In [12] and [18], positive and sign-changing solutions of the parabolic problem with more general non-linearities $f(u)$, $g(u)$ instead of au^p, u^q were studied, while $f(x, u)$, $g(x, u)$ were considered in [2]. Many results concerning elliptic problems with non-linear boundary conditions were summarised in [19]. See also [1, 3, 4, 7, 8, 9, 10].

2 Preliminaries

In this section we recall the shooting method from [5] and [15].

Let $p, q \in \mathbb{R}$, $a, l > 0$. If u is a positive solution of (1), then $u'(-l) < 0 < u'(l)$, therefore u has a stationary point $x_0 \in (-l, l)$. So the function $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = au^p, \\ u(0) = m, \\ u'(0) = 0 \end{cases} \tag{3}$$

for some $m > 0$. Since $u \mapsto au^p$ is locally Lipschitz continuous on $(0, \infty)$, (3) has a unique maximal solution, which is apparently even and strictly convex. We will denote it by $u_{m,p,a}$ and its domain by $(-\Lambda_{m,p,a}, \Lambda_{m,p,a})$.

Let us also introduce the notation $\mathcal{N}(l) = \mathcal{N}(l; p, q, a)$ for the set of all positive non-symmetric (i. e. non-even) solutions of (1). Obviously, $\mathcal{N}(l)$ consists of all such functions $u_{m,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]}$ that $l_1 + l_2 = 2l$, $l_1 \neq l_2$ and $0 < l_i < \Lambda_{m,p,a}$, $u'_{m,p,a}(l_i) = u^q_{m,p,a}(l_i)$ for $i = 1, 2$.

2.1 Lemma (see [5, pp. 53–55] for $p, q > 1$ or [15, Lemma 2.4] for $p, q \in \mathbb{R}$)**.** *Let $p, q \in \mathbb{R}$, $a > 0$. Then the following statements are equivalent for arbitrary $m, l > 0$:*

(i) $l < \Lambda_{m,p,a}$ *and* $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$,

(ii) *the equation*

$$0 = \mathcal{F}(m,x) := \mathcal{F}_{p,q,a}(m,x) := \begin{cases} \dfrac{x^{2q}}{2a} - \dfrac{x^{p+1}}{p+1} + \dfrac{m^{p+1}}{p+1} & \text{if } p \neq -1, \\ \dfrac{x^{2q}}{2a} - \ln x + \ln m & \text{if } p = -1 \end{cases} \tag{4}$$

with the unknown $x > 0$ has some solution $R > m$, and

$$l = \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R}{m}\right),$$

where

$$I_p(y) = \begin{cases} \displaystyle\int_1^y \sqrt{\frac{p+1}{V^{p+1}-1}}\,\mathrm{d}V & \text{if } p \neq -1, \\ \displaystyle\int_1^y \frac{\mathrm{d}V}{\sqrt{\ln V}} & \text{if } p = -1 \end{cases} \tag{5}$$

for $y \geq 1$.

From now on we will consider only

$$p > -1,\ q > \frac{p+1}{2},\ a > 0. \tag{6}$$

However, the definition and the properties of I_{-1} will be needed for the proofs of Lemmata 3.8 and 3.9—that is the reason why we formulated Lemma 2.1 for $p \in \mathbb{R}$.

2.2 Lemma (see [5, pp. 57–58] for $p > 1$ or [15, Lemma 2.5 (iv)] for $p > -1$)**.** *Assume* (6) *and $m > 0$, and let us introduce*

$$M := M_{p,q,a} := \left(\frac{2q-p-1}{2q}\right)^{\frac{1}{p+1}} \left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}}.$$

If $m > M$, then $\mathcal{F}(m,\cdot)$ has no zero. If $m = M$, then the only zero of $\mathcal{F}(m,\cdot)$ is

$$\left(\frac{a}{q}\right)^{\frac{1}{2q-p-1}} =: R_{p,q,a}(M) =: R(M) > M.$$

If $m < M$, then $\mathcal{F}(m,\cdot)$ has two zeros, which will be denoted by $R_{i;p,q,a}(m) =: R_i(m)$, $i = 1, 2$, and which satisfy

$$m < R_1(m) < R(M) < R_2(m).$$

2.3 Definiton. Let (6) hold and put

$$L_i(m) := L_{i;p,q,a}(m) := \frac{m^{\frac{1-p}{2}}}{\sqrt{2a}} I_p\left(\frac{R_{i;p,q,a}(m)}{m}\right)$$

for $i = 1, 2$ and $m \in (0, M)$. We introduce $L_{p,q,a}(M) =: L(M)$ analogously. Functions L, L_1 and L_2 will be called **time maps** (associated with (3)).

Using Lemmata 2.1 and 2.2, we can describe $\mathcal{N}(l)$ by means of the time maps:

2.4 Lemma. *If* (6) *holds, then*

$$\mathcal{N}(l) = \left\{ u_{m,p,a}\left(\cdot \pm \frac{L_2(m) - L_1(m)}{2}\right)\Big|_{[-l,l]} \;:\; L_1(m) + L_2(m) = 2l \right\}$$

for all $l > 0$.

Thus, to determine the number of positive non-symmetric solutions of (1) for given p, q, a, l, one needs to calculate the limits of $L_1 + L_2$ at 0 and M, to examine its monotonicity and to estimate its possible relative extrema. In doing so, we will use

$$I_0(y) = 2\sqrt{y-1}, \tag{7}$$

$$I_{-1/2}(y) = \frac{2\sqrt{2}}{3}\sqrt{\sqrt{y}-1}\big(\sqrt{y}+2\big) \tag{8}$$

((8) can be obtained by substituting $\sqrt{V} - 1$ in (5)) and other properties of I_p from [15], as well as the following theorem.

2.5 Theorem. *The function* $(y, p) \mapsto I_p(y)$ *is continuously differentiable on the set* $(1, \infty) \times (-1, \infty)$*, while*

$$\frac{\partial}{\partial p}\frac{I_p(y)}{\sqrt{p+1}} = -\frac{1}{2}\int_1^y \frac{V^{p+1}\ln V}{(V^{p+1}-1)^{3/2}}\,\mathrm{d}V =: J_p(y) \tag{9}$$

for all $y > 1$, $p > -1$.

Proof: Firstly, we prove that $p \mapsto I_p(y)/\sqrt{p+1}$ is continuously differentiable on $(-1, \infty)$ for any $y > -1$, and fulfils (9). So chose arbitrary $y > 1$ and $p_0 > -1$. We have

$$\frac{I_p(y)}{\sqrt{p+1}} = \int_1^y \underbrace{\frac{1}{\sqrt{V^{p+1}-1}}}_{=:\mu(V,p)}\,\mathrm{d}V, \qquad p \geq p_0$$

with

$$\frac{\partial\mu}{\partial p}(V,p) = -\frac{V^{p+1}\ln V}{2(V^{p+1}-1)^{3/2}} < 0, \qquad V \in (1,y),\ p \geq p_0.$$

Since

$$\frac{\partial^2\mu}{\partial p^2}(V,p) = \frac{V^{p+1}\big(V^{p+1}+2\big)\ln^2 V}{4(V^{p+1}-1)^{5/2}} > 0, \qquad V \in (1,y),\ p \geq p_0,$$

$-\frac{\partial\mu}{\partial p}(\cdot,p_0)$ is a majorant of $\{\frac{\partial\mu}{\partial p}(\cdot,p)\}_{p\geq p_0}$. And it is also integrable because

$$\frac{\partial\mu}{\partial p}(V,p_0) = -\frac{V^{p_0+1}\ln V}{2(V^{p_0+1}-1)^{3/2}} \sim \frac{1}{2(p_0+1)\sqrt{V-1}}, \qquad V\to 1$$

(Taylor polynomials can be used). Consequently, $p\mapsto I_p(y)/\sqrt{p+1}$ is differentiable on (p_0,∞), and (9) holds. Moreover, $p\mapsto J_p(y)$ is continuous on (p_0,∞) due to the continuity of $\frac{\partial\mu}{\partial p}(V,\cdot)$ for all $V\in(1,y)$.

In order to obtain the continuous differentiability of $(y,p)\mapsto I_p(y)/\sqrt{p+1}$ (or equivalently of $(y,p)\mapsto I_p(y)$), we have to validate the continuity of its partial derivatives: Since $J_p(y)$ is continuous in p, and is apparently continuous and decreasing in y, it is indeed continuous. And the continuity of

$$\frac{\partial}{\partial y}\frac{I_p(y)}{\sqrt{p+1}} = \frac{1}{\sqrt{y^{p+1}-1}}$$

is obvious. □

3 The results

The following is known about $p\geq 1$:

3.1 Lemma (see [16, Lemmata 2.5 and 2.8])**.** *If* (6) *holds with* $p\geq 1$*, then* $\lim_{m\to 0}(L_1+L_2)(m)=\infty$*,* $(L_1+L_2)'<0$ *on* $(0,M)$ *and* $\lim_{m\to M}(L_1+L_2)(m) = 2L(M)$*.*

In view of Lemma 2.4, it means that supposing (6) and $p\geq 1$, $|\mathcal{N}(l)|=2$ for $l>L(M)$, and $\mathcal{N}(l)=\emptyset$ for $l\leq L(M)$.

The situation is much more complicated for $p<1$, and we have succeeded only in describing the behaviour of L_1+L_2 near 0 and M, except two special cases dealt with in the following theorem.

3.2 Theorem

(i) *If* $p=0$*,* $q=1$*,* $a>0$*, then* (1) *has infinitely many positive non-symmetric solutions for* $l=1$ *and none for* $l\neq 1$*.*

(ii) *If* $p=-\frac{1}{2}$*,* $q=\frac{1}{2}$*,* $a>0$*, then* (1) *possesses infinitely many positive non-symmetric solutions for* $l=\frac{8a}{3}$ *and none for* $l\neq\frac{8a}{3}$*.*

Proof: We have to calculate L_1+L_2, and the statement of the theorem will follow from Lemma 2.4.

In the case of $q=p+1>0$, (4) is quadratic in x^q, so one can solve it explicitly, obtaining

$$R_{1,2}(m) = \left(\frac{a}{q}\right)^{\frac{1}{q}}\left(1\mp\sqrt{1-\frac{2q}{a}m^q}\right)^{\frac{1}{q}}, \qquad m\in(0,M)=\left(0,\left(\frac{a}{2q}\right)^{\frac{1}{q}}\right). \quad (10)$$

(i) If $p = 0$ and $q = 1$, then by virtue of (7) and (10), we have

$$L_{1,2}(m) = \sqrt{2 - \frac{2a}{m} \mp 2\sqrt{1 - \frac{2m}{a}}} = 1 \mp \sqrt{1 - \frac{2m}{a}}$$

for $m \in (0, M) = (0, \frac{a}{2})$. Consequently, $L_1 + L_2 \equiv 2$.

(ii) Similarly, if $p = -\frac{1}{2}$ and $q = \frac{1}{2}$, then

$$\begin{aligned} L_{1,2}(m) &= \frac{4a}{3}\left(1 \mp \sqrt{1 - \frac{\sqrt{m}}{a}}\right)\left(1 \mp \sqrt{1 - \frac{\sqrt{m}}{a}} + \frac{\sqrt{m}}{a}\right) \\ &= \frac{8a}{3} \mp \frac{4a}{3}\sqrt{1 - \frac{\sqrt{m}}{a}}\left(2 + \frac{\sqrt{m}}{a}\right) \end{aligned}$$

for $m \in (0, M) = (0, a^2)$ due to (8) and (10), ensuring that $L_1 + L_2 \equiv \frac{16a}{3}$. □

3.3 Lemma (see [15, Lemmata 8.3 and 8.4])**.** *Assume that* (6) *holds with* $p < 1$. *Then*

$$\begin{aligned} &\lim_{m \to 0} L_1(m) = 0, \\ &\lim_{m \to 0} L_2(m) = \frac{2}{1-p}\left(\frac{p+1}{2a}\right)^{\frac{q-1}{2q-p-1}} =: L_{2;p,q,a}(0) =: L_2(0), \\ &\lim_{m \to M} L_i(m) = L(M), \qquad i = 1, 2. \end{aligned}$$

In the rest of this article we determine the values of (p, q) for which $L_1 + L_2$ is greater than $\lim_{m \to 0}(L_1 + L_2)(m)$ near 0 and for which it is less. The same will be done for the neighbourhood of M.

Standard asymptotic notations will be used: If f, g are functions defined in some punctured neighbourhood of a point $a \in \mathbb{R} \cup \{\pm\infty\}$, then

$$\begin{aligned} f(x) \sim g(x),\ x \to a \quad &\text{means} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = 1, \\ f(x) = o(g(x)),\ x \to a \quad &\text{means} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = 0, \\ f(x) = O(g(x)),\ x \to a \quad &\text{means} \quad \limsup_{x \to a} \left|\frac{f(x)}{g(x)}\right| < \infty. \end{aligned}$$

3.4 Lemma. *Assume that* (6) *holds with* $p < 1$. *Then*

(i) *if* $p > 0$ *or* $p = 0$, $q > 1$ *or* $q < -p$ *or* $p > -\frac{1}{2}$, $q = -p$, *then* $L_1 + L_2 < L_2(0)$ *in some neighbourhood of* 0,

(ii) *and if* $p = 0$, $q < 1$ *or* $p < 0$, $q > -p$ *or* $p < -\frac{1}{2}$, $q = -p$, *then* $L_1 + L_2 > L_2(0)$ *in some neighbourhood of* 0.

See Figure 1 showing these two sets in the (p, q)*-plane.*

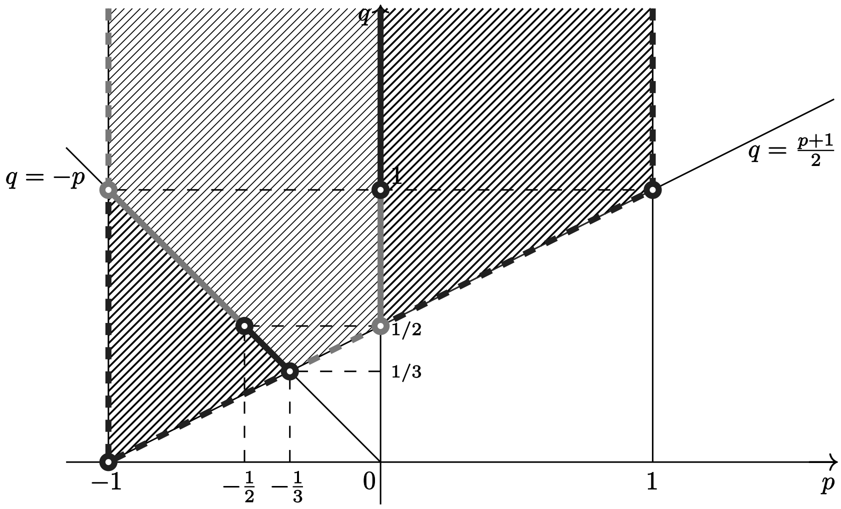

Figure 1: The two sets from Lemma 3.4 (i), (ii).

Proof: It is clear from Lemma 3.3 and [15, Lemma 8.5] that $L_1+L_2 > L_2(0)$ near 0 if either $p<0$, $q>-p$ or $p<-\frac{1}{2}$, $q=-p$. The statement of the lemma for the remaining pairs (p,q) can be verified finding the second term of the asymptotic expansion of $(L_1+L_2)(m)$ for $m \to 0$ and determining its sign. For this purpose, we will join

$$L_1(m) = \frac{1}{a} m^{q-p} + o\big(m^{q-p}\big), \qquad m \to 0$$

from [5, Lemma 3.3] (its proof was done only for $p>1$, but it holds for all $p>-1$) with several equalities from the proof of [15, Lemma 8.5].

All the asymptotic expansions will concern $m \to 0$.

- If $p \in (0,1)$, then $m^{q-p} = o(m^{(1-p)/2})$, so by means of step *1.* of the proof of [15, Lemma 8.5] we have

$$(L_1+L_2)(m) = L_2(0) + \sqrt{\frac{p+1}{2a}} B_p m^{\frac{1-p}{2}} + o\Big(m^{\frac{1-p}{2}}\Big), \tag{11}$$

  while $B_p < 0$ (see [15, Lemma 3.4]).

- If $p=0$, then according to step *1.* of the proof of [15, Lemma 8.5],

$$(L_1+L_2)(m) = L_2(0) - \frac{2q}{2q-1}\left(\frac{1}{2a}\right)^{\frac{q}{2q-1}} m + o(m)$$

  for $q>1$, and

$$(L_1+L_2)(m) = L_2(0) + \frac{1}{a} m^q + o(m^q)$$

  for $q<1$.

- Now consider $q<-p$ (and consequently, $p<-\frac{1}{3}$). Using the asymptotic expansion of $L_2(m)$ from step *3.* of the proof of [15, Lemma 8.5] and realising that $m^{q-p} = o(m^{p+1})$, we obtain

$$(L_1+L_2)(m) = L_2(0) + \underbrace{C_{p,q,a}}_{<0} m^{p+1} + o\big(m^{p+1}\big).$$

- Finally, if $-q = p \in (-\frac{1}{2}, -\frac{1}{3})$, then the equality $m^{q-p} = o(m^{(1-p)/2})$ and step *3.*(a) of the proof of [15, Lemma 8.5] yield the asymptotic expansion of the form as in (11) with $B_p < 0$ due to [15, Lemma 3.4]. $\square$

To determine the behaviour of $L_1 + L_2$ near M is much more difficult. For this purpose, the second term of the corresponding asymptotic expansion will be investigated, the finding of which requires the following lemma:

3.5 Lemma. *If* (6) *holds, then*

$$\frac{R_{1,2}(m)}{R(M)} = 1 \mp \frac{\sqrt{M-m}}{\sqrt{qM}} - \frac{p+2q-2}{6qM}(M-m) + o(M-m), \qquad m \to M-.$$

Proof: Assume (6). From [15, Lemma 8.1] we already know the first term of the asymptotic expansion of $R_{1,2}(m)/R(M)$ for $m \to M-$. Before calculating the next two terms, let us notice that (4), as an equation in m, has the explicit solution

$$m = x\left(1 - \frac{p+1}{2a}x^{2q-p-1}\right)^{\frac{1}{p+1}} =: r_{p,q,a}(x) =: r(x), \qquad x \in (0, R_2(0)),$$

which determines the inverse functions of R_1 and R_2, and will be an important tool of this proof.

All the asymptotic expansions appearing below will concern $m \to M-$ or $z \to 0$.

1. We search for such $d_1, d_2 > 0$ and $c_1 < 0$, $c_2 > 0$ that

$$\frac{R_i(m)}{R(M)} - 1 \sim c_i(M-m)^{d_i}$$

for $i = 1, 2$. (According to [15, Lemma 8.1], $R_i/R(M)$ is increasing for $i = 1$ and decreasing for $i = 2$, which explains the choice of the sign of c_i.) Using the substitution

$$\frac{R_i(m)}{R(M)} - 1 =: z, \tag{12}$$

one obtains

$$A_i := \lim_{m \to M-} \frac{\frac{R_i(m)}{R(M)} - 1}{(M-m)^{d_i}} = \lim_{z \to 0\mp} \frac{z}{\big(M - r\big(R(M)(1+z)\big)\big)^{d_i}},$$

where $z \to 0\mp$ means $z \to 0-$ for $i = 1$ and $z \to 0+$ for $i = 2$. This limit (which should be finite and non-negative, determining the value of c_i) will be calculated using the asymptotic expansion of the denominator of the last fraction. Therefore, it is convenient to derive the equality

$$\begin{aligned} &M - r\big(R(M)(1+z)\big) \\ &= M - M(1+z)\left(\frac{2q}{2q-p-1} - \frac{p+1}{2q-p-1}(1+z)^{2q-p-1}\right)^{\frac{1}{p+1}} \\ &= M\underbrace{\left[1 - (1+z)\left(1 - \frac{p+1}{2q-p-1}\Big((1+z)^{2q-p-1} - 1\Big)\right)^{\frac{1}{p+1}}\right]}_{=:h(z)}. \end{aligned}$$

Approximating $(1+z)^{2q-p-1}$ with its 2nd order Maclaurin polynomial, one obtains

$$h(z) = qz^2 + o(z^2),$$

which results in

$$A_i = \lim_{z\to 0\mp} \frac{z}{(qM)^{d_i}|z|^{2d_i}}.$$

Consequently, $d_i = \frac{1}{2}$ and $c_i = A_i = \mp 1/\sqrt{qM}$.

2. Now we seek $c_i \neq 0$ and $d_i > \frac{1}{2}$ fulfilling

$$\frac{R_i(m)}{R(M)} - 1 \pm \frac{\sqrt{M-m}}{\sqrt{qM}} \sim c_i(M-m)^{d_i}$$

for $i = 1, 2$. So we have to calculate the corresponding limit

$$B_i := \lim_{m\to M-} \frac{\frac{R_i(m)}{R(M)} - 1 \pm \frac{\sqrt{M-m}}{\sqrt{qM}}}{(M-m)^{d_i}} = \lim_{z\to 0\mp} \frac{z \pm \sqrt{\frac{h(z)}{q}}}{(qM)^{d_i}|z|^{2d_i}}$$

((12) was used again), which requires the knowledge of one more term of the asymptotic expansion of $h(z)$. Therefore, we derive that

$$\begin{aligned} h(z) &= 1 - (1+z)\Big(1 - z + (1-q)z^2 - \frac{pq + 2q^2 - 5q + 3}{3} z^3 + o\big(z^3\big)\Big) \\ &= qz^2\Big(1 + \frac{p+2q-2}{3} z + o(z)\Big), \end{aligned}$$

which yields

$$B_i = \lim_{z\to 0\mp} \frac{-\frac{p+2q-2}{6} z^2 + o\big(z^2\big)}{(qM)^{d_i}|z|^{2d_i}},$$

meaning that $d_i = 1$ and $c_i = -\frac{p+2q-2}{6qM}$. □

The next step is to calculate the expansion of $L_1 + L_2$.

3.6 Lemma. *If* (6) *holds, then*

$$\begin{aligned} &(L_1+L_2)(m) \\ &= 2L(M) + \left(\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}}\left(\frac{R(M)}{M}\right)^{\frac{1-p}{2}} + (p-1)I_p\left(\frac{R(M)}{M}\right)\right)\frac{M-m}{\sqrt{2aM^{p+1}}} + o(M-m) \end{aligned}$$

for $m \to M-$. Recall that

$$\frac{R(M)}{M} = \left(\frac{2q}{2q-p-1}\right)^{\frac{1}{p+1}}.$$

Proof: Assume (6). Unless otherwise stated, all the asymptotic expansions within this proof will concern $x := \frac{M-m}{m} \to 0+$. So we have

$$L_i(m) = \frac{M^{\frac{1-p}{2}}}{\sqrt{2a}}\Big(1 + \frac{p-1}{2}x + o(x)\Big) I_p\Big(\frac{R_i(m)}{M(1-x)}\Big) \tag{13}$$

for $i = 1, 2$. By means of Lemma 3.5 and

$$I_p(y) = I_p(y_0) + \sqrt{2q-p-1}(y-y_0) - \frac{(2q-p-1)^{3/2}}{4} y_0^p (y-y_0)^2 + o\big((y-y_0)^2\big),$$

which holds for $y \to y_0 := \frac{R(M)}{M}$ (and follows from the definition of the Taylor polynomial), we obtain

$$I_p\left(\frac{R_i(m)}{M(1-x)}\right) = I_p\left(y_0\left(1 \mp \frac{\sqrt{x}}{\sqrt{q}} + \frac{4q-p+2}{6q}x + o(x)\right)\right)$$
$$= I_p(y_0) \mp \sqrt{\frac{2q-p-1}{q}} y_0 \sqrt{x} + \frac{\sqrt{2q-p-1}(q-p+2)}{6q} y_0 x + o(x).$$

It can be inserted in (13), resulting in

$$L_i(m) = L(M) \mp \frac{\sqrt{x}}{\sqrt{aR^{p-1}(M)}}$$
$$+ \left(\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}} y_0^{\frac{1-p}{2}} + (p-1)I_p(y_0)\right) \frac{x}{2\sqrt{2aM^{p-1}}} + o(x),$$

which confirms the conclusion of the lemma. □

3.7 Lemma. *Assume that* (6) *holds with* $p < 1$. *There exist continuously differentiable functions* $\widehat{q} : (-1, 1) \to \mathbb{R}$ *and* $\overline{q} : (-1, -\frac{1}{7}) \to \mathbb{R}$ *such that* $\widehat{q} > 1$ *on* $(-1, 0)$, $\widehat{q}(p) > \frac{p+1}{2}$ *for* $p \in [0, 1)$, $\frac{p+1}{2} < \overline{q}(p) < p + \sqrt{2p(p-1)}$ *for* $p \in (-1, -\frac{1}{7})$, *and the following holds:*

(i) *If* $q > \widehat{q}(p)$ *or* $p < -\frac{1}{7}$, $q < \overline{q}(p)$, *then* $L_1 + L_2 > 2L(M)$ *in some neighbourhood of* M.

(ii) *If* $p \geq -\frac{1}{7}$, $q < \widehat{q}(p)$ *or* $p < -\frac{1}{7}$, $\overline{q}(p) < q < \widehat{q}(p)$, *then* $L_1 + L_2 < 2L(M)$ *in some neighbourhood of* M.

In addition, for all $p \in [-\frac{1}{7}, 1)$, $q = \widehat{q}(p)$ *is given as the only solution of*

$$\frac{\sqrt{2}(q-p+2)}{3\sqrt{q}} g^{\frac{1-p}{2}}(p,q) + (p-1)I_p\big(g(p,q)\big) =: f(p,q) = 0 \tag{14}$$

in $(\frac{p+1}{2}, \infty)$, *where*

$$g(p,q) = \left(\frac{2q}{2q-p-1}\right)^{\frac{1}{p+1}}.$$

Similarly, for all $p \in (-1, -\frac{1}{7})$, $q = \overline{q}(p)$ *and* $q = \widehat{q}(p)$ *are the only solutions of* (14) *in* $[p + \sqrt{2p(p-1)}, \infty)$ *and* $(\frac{p+1}{2}, p + \sqrt{2p(p-1)}]$ *respectively.*

See Figure 2 showing the graphs of $\widehat{q}$ *and* $\overline{q}$, *as obtained by numerical solution of* (14).

Proof: It is clear from Lemma 3.6 that $L_1 + L_2 > 2L(M)$ near M if $f(p,q) > 0$, while $L_1 + L_2 < 2L(M)$ near M if $f(p,q) < 0$. Obviously,

$$\lim_{q\to\infty} f(p,q) = \infty, \qquad p \in (-1, 1). \tag{15}$$

In the sequel we

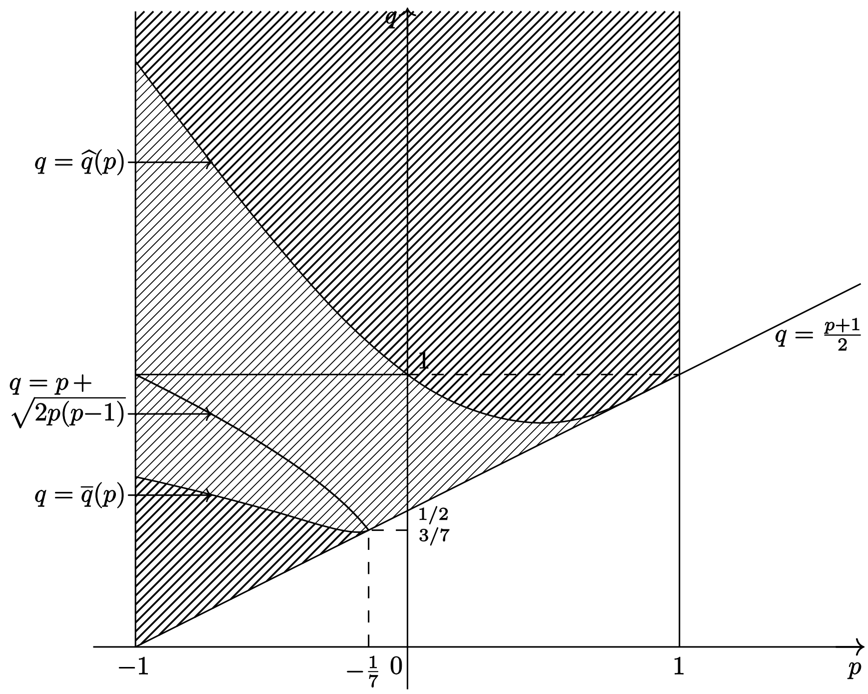

Figure 2: The graphs of $\widehat{q}$, $\overline{q}$ and the two sets from Lemma 3.7 (i), (ii).

1. find $\lim_{q\to\frac{p+1}{2}} f(p,q)$,
2. examine the monotonicity of $f(p,\cdot)$
3. and prove that $f(p,1) < 0$ for all $p \in (-1,0)$,

which will make us able to describe the sets of (p,q) where f is positive, zero or negative.

1. Let $p \in (-1,1)$. Since $\lim_{q\to\frac{p+1}{2}} g(p,q) = \infty$, [15, Lemma 3.4] can be used. We need only the first term of the asymptotic expansion of $I_p(y)$ for $y \to \infty$ to calculate

$$\lim_{q\to\frac{p+1}{2}} \frac{f(p,q)}{g^{\frac{1-p}{2}}(p,q)} = \frac{-7p-1}{3\sqrt{p+1}},$$

thus $\lim_{q\to\frac{p+1}{2}} f(p,q)$ is equal to ∞ for $p < -\frac{1}{7}$, and $-\infty$ for $p > -\frac{1}{7}$.

Now assume that $p = -\frac{1}{7}$, and set $r := 2q - p - 1$. Approximating $I_p(y)$ with its two-term asymptotic expansion for $y \to \infty$, we obtain that

$$f\left(-\frac{1}{7},q\right) = \underbrace{\left(\frac{7r+36}{3\sqrt{7(7r+6)}} - \frac{2\sqrt{6}}{\sqrt{7}}\right)}_{=O(r)} \underbrace{g^{4/7}\left(-\frac{1}{7},q\right)}_{=O\left(\frac{1}{r^{2/3}}\right)} - \frac{8\sqrt{6}}{7\sqrt{7}}B_{-1/7} + o(1)$$

$$\longrightarrow -\frac{8\sqrt{6}}{7\sqrt{7}}B_{-1/7} < 0$$

for $r \to 0+$.

To sum up,

$$\lim_{q\to\frac{p+1}{2}} f(p,q)\begin{cases} >0 & \text{if } p\in\left(-1,-\frac{1}{7}\right),\\ <0 & \text{if } p\in\left[-\frac{1}{7},1\right).\end{cases}$$

2. Let $p\in(-1,1)$ again. One can calculate that

$$\begin{aligned}\frac{\partial f}{\partial q}(p,q) &= \frac{q{+}p{-}2}{3q\sqrt{2q}}\frac{1}{\sqrt{g^{p+1}(p,q)}}g(p,q)+\frac{(1{-}p)(q{-}p{+}2)\sqrt{2q{-}p{-}1}}{6q}\frac{\partial g}{\partial q}(p,q)\\ &\quad+(p{-}1)\sqrt{\frac{p{+}1}{g^{p+1}(p,q){-}1}\frac{\partial g}{\partial q}(p,q)}\\ &= \frac{\sqrt{2q{-}p{-}1}}{6q^2}\left((p{+}q{-}2)g(p,q)+q(p{-}1)(p{+}5q{-}2)\frac{\partial g}{\partial q}(p,q)\right)\end{aligned}$$

and

$$\frac{\partial g}{\partial q}(p,q) = -\frac{g(p,q)}{q(2q-p-1)},$$

consequently,

$$\frac{\partial f}{\partial q}(p,q) = \underbrace{\left(q^2-2pq-p^2+2p\right)}_{=:\xi(p,q)}\underbrace{\frac{g(p,q)}{3q^2\sqrt{2q-p-1}}}_{>0}.$$

It is easy to see that

$$\xi(p,q)=0 \iff p\le 0 \text{ and } q=p\pm\sqrt{2p(p-1)},$$

while $p-\sqrt{2p(p-1)}<\frac{p+1}{2}$ for all $p\le 1$, and $p+\sqrt{2p(p-1)}>\frac{p+1}{2}$ only if $p<-\frac{1}{7}$.

So we conclude that

- if $p\in[-\frac{1}{7},1)$, then $f(p,\cdot)$ increases on $(\frac{p+1}{2},\infty)$,
- if $p\in(-1,-\frac{1}{7})$, then $f(p,\cdot)$ decreases on $(\frac{p+1}{2},p+\sqrt{2p(p-1)}]$ and increases on $[p+\sqrt{2p(p-1)},\infty)$.

3. In this step we prove that $f(p,1)<0$ for all $p\in(-1,0)$, or equivalently,

$$I_p\left(\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}}\right) > \frac{\sqrt{2}(3-p)}{3(1-p)}\left(\frac{2}{1-p}\right)^{\frac{1-p}{2(p+1)}}, \qquad p\in(-1,0). \tag{16}$$

Our method is to gradually derive simpler and simpler sufficient conditions for (16), the last of which will be proved directly.

(a) Since $p\mapsto I_p(y)$ decreases on $\mathbb{R}$ for all $y>1$ according to [15, Theorem 3.5], a sufficient condition for (16) can be obtained replacing I_p on its left-hand side with I_0 (see also (7)). After squaring, this new inequality reads

$$\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}}-1>\frac{1}{18}\left(\frac{2}{1-p}+1\right)^2\left(\frac{2}{1-p}\right)^{\frac{1-p}{p+1}}, \qquad p\in(-1,0).$$

Denoting $\frac{2}{1-p} =: x$, it simplifies to

$$\sqrt{x \cdot x^{\frac{1}{x-1}}} - 1 > \frac{(x+1)^2 x^{\frac{1}{x-1}}}{18}, \qquad x \in (1,2).$$

It is convenient to introduce the notation $\omega(x) := x^{1/(x-1)}$, by means of which the last inequality transforms to

$$-\frac{x^2 - 7x + 1}{9}\omega(x) - \frac{(x+1)^4}{324}\omega^2(x) > 1, \qquad x \in (1,2). \tag{17}$$

(b) Let us prove that

$$\frac{x(4-x)}{8}\omega(x) > 1, \qquad x \in (1,2). \tag{18}$$

Equivalently, it can be written as

$$\zeta(x) := x \ln x + (x-1)\big(\ln(4-x) - \ln 8\big), \qquad x \in (1,2).$$

We have

$$\zeta''(x) = \frac{2x^2 - 15x + 16}{x(x-4)^2},$$

and one can see that ζ'' is positive on $[1, x_0)$ and negative on $(x_0, 2]$, while $x_0 = (15 - \sqrt{97})/4$. Consequently, $\zeta' > \zeta'(1) = \ln \frac{3e}{8} > 0$ on $(1, x_0]$, and since $\zeta(1) = 0$, the positivity of ζ on $(1, x_0]$ follows. Therefore, the concavity of ζ on $[x_0, 2]$ with $\zeta(2) = 0$ ensures its positivity on $[x_0, 2)$, and (18) is verified.

Replacing the right-hand side of (17) with the left-hand side of (18), we obtain a sufficient condition for (17), which can be simplified to

$$\omega(x) < \frac{9\big(x^2 + 20x - 8\big)}{2(x+1)^4}, \qquad x \in (1,2). \tag{19}$$

(c) Our next auxiliary inequality is

$$\frac{6}{x+1} < \frac{9\big(x^2 + 20x - 8\big)}{2(x+1)^4}, \qquad x \in (1,2),$$

which is equivalent to

$$P(x) := 4x^3 + 9x^2 - 48x + 28 < 0, \qquad x \in (1,2),$$

and which can be proved realising that $P(1) = -7 < 0$, $P(2) = 0$ and $P'' > 0$ on $(1,2)$. It provides a sufficient condition for (19) in the form of

$$\omega(x) < \frac{6}{x+1}, \qquad x \in (1,2),$$

or equivalently,

$$\eta(x) := \ln x + (x-1)\big(\ln(x+1) - \ln 6\big) < 0, \qquad x \in (1,2),$$

which is a true inequality, since $\eta(1) = \eta(2) = 0$ and

$$\eta''(x) = \frac{(x-1)\big(x^2 + 3x + 1\big)}{x^2(x+1)^2} > 0, \qquad x \in (1,2).$$

Define

$$q_1(p) := \begin{cases} p + \sqrt{2p(p-1)} & \text{if } p \in \left(-1, -\frac{1}{7}\right), \\ \dfrac{p+1}{2} & \text{if } p \in \left[-\frac{1}{7}, 1\right). \end{cases}$$

As a consequence of *1.*, *2.* and *3.*, $\lim_{q \to q_1(p)} f(p,q) < 0$ for all $p \in (-1,1)$. Taking (15) and the increase of $f(p,\cdot)$ on $(q_1(p), \infty)$ into account as well, we obtain that

$$\forall p \in (-1,1)\colon \quad \exists!\ \widehat{q}(p) \in \big(q_1(p), \infty\big)\colon \quad f\big(p, \widehat{q}(p)\big) = 0.$$

Clearly, if $p \in (-1,1)$, $q > \widehat{q}(p)$, then $f(p,q) > 0$ and consequently, $L_1 + L_2 > 2L(M)$ near M. On the other hand, if $p \in (-1,1)$, $q \in [q_1(p), \widehat{q}(p))$, then $f(p,q) < 0$, and $L_1 + L_2 < 2L(M)$ near M. Furthermore, $\widehat{q} > 1$ on $(-1,0)$ due to *3.*, while the continuous differentiability of $\widehat{q}$ follows from the implicit function theorem and the continuous differentiability of f (see Theorem 2.5).

Similarly, since $f(p, q_1(p)) < 0$ for $p \in (-1, -\frac{1}{7})$, *1.* and *2.* imply that

$$\forall p \in \left(-1, -\tfrac{1}{7}\right)\colon \quad \exists!\ \overline{q}(p) \in \left(\tfrac{p+1}{2}, q_1(p)\right)\colon \quad f\big(p, \overline{q}(p)\big) = 0.$$

Again, $f(p,q)$ is positive for $p \in (-1, -\frac{1}{7})$, $q \in (\frac{p+1}{2}, \overline{q}(p))$, and negative for $p \in (-1, -\frac{1}{7})$, $q \in (\overline{q}(p), q_1(p)]$, making clear the behaviour of $L_1 + L_2$ near M for these values of p and q, and obviously, $\overline{q}$ is continuously differentiable. $\square$

The next lemma describes the basic properties of $\widehat{q}$.

3.8 Lemma. *The limit*

$$\lim_{p \to -1} \widehat{q}(p) =: \widehat{q}(-1) \in (1, \infty),$$

exists and it is the only solution of the equation

$$\varphi(q) := \frac{\sqrt{2}(q+3)}{3\sqrt{q}} \mathrm{e}^{\frac{1}{2q}} - 2I_{-1}\left(\mathrm{e}^{\frac{1}{2q}}\right) = 0 \tag{20}$$

in $[1, \infty)$. *Furthermore,* $\widehat{q} > 1$ *on* $(-1,0)$, $\widehat{q}(-\frac{1}{2}) = \frac{3}{2}$, $\widehat{q}(0) = 1$, $\widehat{q} < 1$ *on* $(0,1)$, *and* $\lim_{p \to 1} \widehat{q}(p) = 1$.

Proof: It is a part of Lemma 3.7 that $\widehat{q} > 1$ on $(-1,0)$. We also know from it that $L_1 + L_2 \neq 2L(M)$ near M for $p = 0$, $q \in (0,\infty) \setminus \{\widehat{q}(0)\}$, which, in view of Theorem 3.2 (i), yields $\widehat{q}(0) = 1$. It remains to

1. prove the existence and properties of $\lim_{p \to -1} \widehat{q}(p)$,
2. compute $\widehat{q}(-\frac{1}{2})$
3. and prove that $\widehat{q} < 1$ on $(0,1)$.

We will obtain $\lim_{p \to 1} \widehat{q}(p)$ as a direct consequence of *3.* and $\widehat{q}(p) > \frac{p+1}{2}$.

1. [15, Theorem 3.5] and some simple calculations yield that $\lim_{p \to -1} f(p,q) = \varphi(q)$ for any $q > 0$ (see Lemma 3.7 for the definition of f).

Clearly, $\lim_{q\to\infty}\varphi(q)=\infty$. Since $I_{-1}(\mathrm{e}^{1/2q})=O(\sqrt{q})\mathrm{e}^{1/2q}$ for $q\to 0$ due to [15, Lemma 3.6],

$$\varphi(q)=\frac{\sqrt{2}}{\sqrt{q}}\mathrm{e}^{\frac{1}{2q}}\big(1+O(q)\big)\ \longrightarrow\ \infty, \qquad q\longrightarrow 0.$$

It is not hard to derive that

$$\varphi'(q)=\frac{(q-1)(q+3)}{3q^2\sqrt{2q}}\mathrm{e}^{\frac{1}{2q}}, \qquad q>0,$$

which implies that φ is decreasing on $(0,1]$ and increasing on $[1,\infty)$. Furthermore,

$$\varphi(1)=\frac{4}{3}\sqrt{2\mathrm{e}}-2I_{-1}(\sqrt{\mathrm{e}})<\frac{4}{3}\sqrt{2\mathrm{e}}-2I_0(\sqrt{\mathrm{e}})=4\left(\frac{\sqrt{2\mathrm{e}}}{3}-\sqrt{\sqrt{\mathrm{e}}-1}\right)<0$$

(see [15, Theorem 3.5] and (7)).

So one can see that $\varphi|_{(1,\infty)}$ has a unique zero, which will be denoted by q_0. Since $\varphi=\lim_{p\to-1}f(p,\cdot)$, and it increases on $(1,\infty)$, we have that for arbitrary $\varepsilon\in(0,q_0-1)$ there exists $\delta>0$ such that

$$\forall p\in(-1,-1+\delta)\colon \quad f(p,q_0-\varepsilon)<0<f(p,q_0+\varepsilon)$$

and therefore,

$$\forall p\in(-1,-1+\delta)\colon \quad q_0-\varepsilon<\widehat{q}(p)<q_0+\varepsilon,$$

following from the increase of $f(p,\cdot)$ on $(1,\infty)$ (see step *2.* of the proof of Lemma 3.7). Consequently, $\lim_{p\to-1}\widehat{q}(p)=q_0$.

2. One can calculate that

$$f\left(-\frac{1}{2},q\right)=\frac{4\sqrt{2}q(2q+5)}{3(4q-1)^{3/2}}-\frac{3}{2}I_{-1/2}\left(\left(\frac{4q}{4q-1}\right)^2\right)=\frac{2\sqrt{2}\big(4q^2-8q+3\big)}{3(4q-1)^{3/2}}$$

for $q>\frac{1}{4}$, which vanishes only for $q=\frac{1}{2}$ and $q=\frac{3}{2}$, meaning that $\widehat{q}(-\frac{1}{2})=\frac{3}{2}$.

3. Now we prove that $f(p,1)>0$ for all $p\in(0,1)$, guaranteeing that $\widehat{q}<1$ on $(0,1)$. It is equivalent to

$$I_p\left(\left(\frac{2}{1-p}\right)^{\frac{1}{p+1}}\right)<\frac{\sqrt{2}(3-p)}{3(1-p)}\left(\frac{2}{1-p}\right)^{\frac{1-p}{2(p+1)}}, \qquad p\in(0,1), \tag{21}$$

which will be gradually simplified, similarly to step *3.* of the proof of Lemma 3.7.

(a) The first sufficient condition for (21) is

$$-\frac{x^2-7x+1}{9}\omega(x)-\frac{(x+1)^4}{324}\omega^2(x)<1, \qquad x>2 \tag{22}$$

(again, $\omega(x)=x^{1/(x-1)}$), which can be derived in a way completely analogous to the corresponding part of the proof of Lemma 3.7.

(b) The opposite inequality of (18) does not hold for all $x > 2$. Instead,

$$\frac{\omega(x)}{2} < 1, \qquad x > 2 \tag{23}$$

will be used, which is equivalent to

$$\kappa(x) := (x-1)\ln 2 - \ln x > 0, \qquad x > 2,$$

and the validity of which follows from the facts that $\kappa(2) = 0$ and

$$\kappa'(x) = \ln 2 - \frac{1}{x} > \ln 2 - \frac{1}{2} > 0, \qquad x > 2.$$

Due to (23), 1 can be replaced with $\omega(x)/2$ on the right-hand side of (22), yielding a sufficient condition for (22), which can be rewritten as

$$\omega(x) > -\frac{18\bigl(2x^2 - 14x + 11\bigr)}{(x+1)^4}, \qquad x > 2.$$

(c) The final simplification will be done by virtue of the inequality

$$\frac{6}{x+1} > -\frac{18\bigl(2x^2 - 14x + 11\bigr)}{(x+1)^4}, \qquad x > 2,$$

equivalent to

$$Q(x) := x^3 + 9x^2 - 39x + 34 > 0, \qquad x > 2,$$

which holds since $Q(2) = 0$ and $Q'(x) > 9 > 0$ for $x > 2$. So now the only assertion to prove is

$$\omega(x) > \frac{6}{x+1}, \qquad x > 2.$$

And to do so, we just have to recall part (c) of step *3.* of the proof of Lemma 3.7, and to realise that $\eta(x) > 0$ for $x > 2$ because $\eta'(2) = \frac{5}{6} - \ln 2 > 0$ and $\eta'' > 0$ on $(2, \infty)$. □

As suggested by numerical calculations, $\widehat{q}(-1) \approx 2.151$, and $\widehat{q}$ seems to be convex, having $\min \widehat{q} \approx 0.822 \approx \widehat{q}(0.495)$. It can be proved that setting $\widehat{q}(1) := 1$, $\widehat{q}'(1) = \frac{1}{2}$ holds.

Recall that the line $q = -p$ forms the border between those sets of (p, q) where $L_1 + L_2 < L_2(0)$ and $L_1 + L_2 > L_2(0)$ near 0 (see Lemma 3.4). According to Lemma 3.7, the graph of $\overline{q}$ plays a similar role in the behaviour of $L_1 + L_2$ near M. Therefore, if we are interested in the behaviour of $L_1 + L_2$ on $(0, M)$, we have to know the mutual position of these to curves.

3.9 Lemma. *The limit*

$$\lim_{p \to -1} \overline{q}(p) =: \overline{q}(-1) \in (0, 1),$$

exists and it is the only solution of the equation (20) *in* $(0, 1]$. *Furthermore,* $\overline{q}(p) < -p$ *for* $p \in (-1, -\frac{1}{2})$*,* $\overline{q}(-\frac{1}{2}) = \frac{1}{2}$*,* $\overline{q}(p) > -p$ *for* $p \in (-\frac{1}{2}, -\frac{1}{7})$ *and* $\lim_{p \to -1/7} \overline{q}(p) = \frac{3}{7}$.

Proof: The existence and properties of $\lim_{p\to -1} \overline{q}(p)$ can be validated the same way as it is done in step *1.* of the proof of Lemma 3.8 for $\lim_{p\to -1} \widehat{q}(p)$. And it is clear from step *2.* of the same proof and from the definition of $\overline{q}$ (or from Theorem 3.2 (ii)) that $\overline{q}(-\frac{1}{2}) = \frac{1}{2}$. Further, since $\frac{p+1}{2} < \overline{q}(p) < p + \sqrt{2p(p-1)}$ for $p \in (-1, -\frac{1}{7})$ (see Lemma 3.7), the value of $\lim_{p\to -1/7} \overline{q}(p)$ is evident.

It remains to determine the sign of $\overline{q}(p) + p$ for $p \in (-1, -\frac{1}{3})$. (For $p \in [-\frac{1}{3}, -\frac{1}{7})$ we obviously have $-p \le \frac{p+1}{2} < \overline{q}(p)$.) Let

$$\begin{aligned} \varGamma(p) &:= g(p,-p) = \left(\frac{2p}{3p+1}\right)^{\frac{1}{p+1}}, \\ \varPhi(p) &:= \frac{f(p,-p)}{(p-1)\sqrt{p+1}} = \frac{I_p(\varGamma(p))}{\sqrt{p+1}} - \frac{2\sqrt{2}}{3\sqrt{-p(p+1)}}\varGamma^{\frac{1-p}{2}}(p), \end{aligned} \qquad p \in \left(-1, -\tfrac{1}{3}\right).$$

We prove soon that

1. $\varPhi$ decreases on $[-\frac{3}{7}, -\frac{1}{3})$,
2. $\varPhi < 0$ on $(-\frac{1}{2}, -\frac{3}{7}]$
3. and $\varPhi > 0$ on $(-1, -\frac{1}{2})$.

It will mean that $f(p,-p)$ is positive for $p \in (-\frac{1}{2}, -\frac{1}{3})$ and negative for $p \in (-1, -\frac{1}{2})$. Since for all $p \in (-1, -\frac{1}{3})$: $-p \in (\frac{p+1}{2}, p + \sqrt{2p(p-1)})$, $f(p, \cdot)$ decreases on $(\frac{p+1}{2}, p + \sqrt{2p(p-1)})$ (see step *2.* of the proof of Lemma 3.7) and $f(p, \overline{q}(p)) = 0$, the assertion of the lemma regarding the relationship between $\overline{q}(p)$ and $-p$ will follow.

1. Let $p \in (-1, -\frac{1}{3})$. We have

$$\varGamma'(p) = \left(\frac{1}{p(3p+1)} - \frac{1}{p+1}\ln\frac{2p}{3p+1}\right)\frac{\varGamma(p)}{p+1}$$

and

$$\begin{aligned} \left(\varGamma^{\frac{1-p}{2}}(p)\right)' &= \left(\sqrt{\frac{3p+1}{2p}}\varGamma(p)\right)' \\ &= \left(\frac{1-p}{2p(3p+1)} - \frac{1}{p+1}\ln\frac{2p}{3p+1}\right)\sqrt{\frac{3p+1}{2p}}\frac{\varGamma(p)}{p+1}. \end{aligned}$$

Thanks to Theorem 2.5, $\varPhi$ is differentiable, and

$$\varPhi'(p) = \underbrace{J_p(\varGamma(p))}_{<0} - \underbrace{\left(\frac{2p+1}{3p+1} + \frac{3p+2}{3(p+1)}\ln\frac{2p}{3p+1}\right)}_{=:H(p)}\underbrace{\sqrt{\frac{-3p-1}{p+1}}\frac{\varGamma(p)}{p(p+1)}}_{<0}.$$

Numerical calculations indicate that $\varPhi$ is decreasing. If we could prove it, the proof would be complete (since we know that $\varPhi(-\frac{1}{2}) = 0$). The non-positivity of H is a sufficient condition for it.

Instead of H, we will investigate h, defined as

$$\begin{aligned} h(p) :&= \frac{3(p+1)}{3p+2}H(p) \\ &= \frac{3(p+1)(2p+1)}{(3p+1)(3p+2)} + \ln\frac{2p}{3p+1}, \end{aligned} \qquad p \in \left(-1, -\tfrac{1}{3}\right) \setminus \left\{\tfrac{2}{3}\right\},$$

because it has a simpler derivative:

$$h'(p) = \frac{15p^2 + 15p + 4}{p(3p+1)^2(3p+2)^2} < 0.$$

Since $\lim_{p\to -1} h(p) = 0$, $h < 0$ on $(-1, -\frac{2}{3})$. One can also derive that $\lim_{p\to -2/3+} h(p) = \infty$ and $\lim_{p\to -1/3-} h(p) = -\infty$. Consequently, $h > 0$ on $(-\frac{2}{3}, p_0)$ and $h < 0$ on $(p_0, -\frac{1}{3})$ for some $p_0 \in (-\frac{2}{3}, -\frac{1}{3})$. It means that the sufficient condition for the decrease of Φ is met only for $p \in (p_0, -\frac{1}{3})$. Since $h(-\frac{3}{7}) = \ln 3 - \frac{6}{5} < 0$, we have $p_0 < -\frac{3}{7}$. (According to numerical calculations, $p_0 \approx -0.434$.)

2. The proof of $\Phi < 0$ on $(-\frac{1}{2}, -\frac{3}{7}]$ is based on the method of gradual simplification from step *3.* of the proof of Lemma 3.7.

(a) Let

$$\widetilde{\Phi}(p) := \frac{I_{-1/2}(\Gamma(p))}{\sqrt{p+1}} - \frac{2\sqrt{2}}{3\sqrt{-p(p+1)}}\Gamma^{\frac{1-p}{2}}(p), \qquad p \in \left(-1, -\tfrac{1}{3}\right).$$

Due to [15, Theorem 3.5], $\widetilde{\Phi}(p) < 0$ is a sufficient condition for $\Phi(p) < 0$ for $p \in (-\frac{1}{2}, -\frac{3}{7}]$. (Naturally, the same holds even for $p \in (-\frac{1}{2}, -\frac{1}{3})$, but numerical calculations suggest that $\widetilde{\Phi} < 0$ on $(-\frac{1}{2}, p_1)$ and $\widetilde{\Phi} > 0$ on $(p_1, -\frac{1}{3})$ with $p_1 \approx -0.338$. This explains why we have executed step *1.*) Using (8), the condition we want to verify can be rewritten as

$$\left(\left(\frac{2p}{3p+1}\right)^{\frac{1}{2(p+1)}} - 1\right)\left(\left(\frac{2p}{3p+1}\right)^{\frac{1}{2(p+1)}} + 2\right)^2 < -\frac{1}{p}\left(\frac{2p}{3p+1}\right)^{\frac{1-p}{p+1}}, \qquad p \in \left(-\tfrac{1}{2}, -\tfrac{3}{7}\right],$$

or equivalently as

$$\left(x^{\frac{3x-2}{4(x-1)}} - 1\right)\left(x^{\frac{3x-2}{4(x-1)}} + 2\right)^2 < \frac{3x-2}{x^2}x^{\frac{3x-2}{x-1}}, \qquad x \in (2,3],$$

where $x := \frac{2p}{3p+1}$. After introducing

$$\tau(x) := x^{\frac{3x-2}{4(x-1)}}, \qquad x > 1,$$

we can rearrange it into the form

$$3\tau^2(x) + \tau^3(x) + \frac{2-3x}{x^2}\tau^4(x) < 4, \qquad x \in (2,3]. \tag{24}$$

(b) Now the inequality

$$\frac{2\tau^3(x)}{x^2} < 4, \qquad x \in (2,3] \tag{25}$$

will be used. Its validity follows from its equivalent form

$$\zeta(x) := (x+2)\ln x - (x-1)4\ln 2 < 0, \qquad x \in (2,3],$$

after realising that $\zeta(2) = 0$, $\zeta(3) = \ln \frac{243}{256} < 0$ and $\zeta''(x) = \frac{x-2}{x^2} > 0$ for $x \in (2,3)$. So the right-hand side of (24) can be replaced by the left-hand side of (25), yielding a sufficient condition for (24), which can be simplified to

$$\frac{x^2-2}{x^2}\tau(x) + \frac{2-3x}{x^2}\tau^2(x) < -3, \qquad x \in (2,3]. \tag{26}$$

(c) Let us now prove that

$$-\frac{2x+5}{3x}\tau(x) < -3, \qquad x \in (2,3]. \tag{27}$$

The given inequality can be rearranged into

$$\eta(x) := (2-x)\ln x + 4(x-1)\big(\ln(2x+5) - \ln 9\big) > 0, \qquad x \in (2,3].$$

One can derive that

$$\eta''(x) = \frac{P(x)}{x^2(2x+5)^2}$$

with

$$P(x) = 12x^3 + 68x^2 - 65x - 50.$$

Apparently, $P(x) > 75 > 0$ for $x \in (2,3)$ and consequently, η is strictly convex on $(2,3]$. And since $\eta(2) = 0$ and $\eta'(2) = \frac{8}{9} - \ln 2 > 0$, we have that $\eta > 0$ on $(2,3]$.

Thanks to (27), a sufficient condition for (26) follows, namely

$$\tau(x) > \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (2,3].$$

(d) It is easy to see that

$$\frac{3x+4}{5} > \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (2,3]$$

because it is equivalent to

$$Q(x) := 2x^2 - 7x + 6 > 0, \qquad x \in (2,3],$$

while $\frac{3}{2}$ and 2 are the roots of Q. So proving

$$\tau(x) > \frac{3x+4}{5}, \qquad x \in (2,3], \tag{28}$$

will finish step *2*. Let us express (28) in the form

$$\kappa(x) := (3x-2)\ln x + 4(1-x)\big(\ln(3x+4) - \ln 5\big) > 0, \qquad x \in (2,3].$$

We have

$$\kappa''(x) = -\frac{S(x)}{x^2(3x+4)^2},$$

where

$$S(x) = 9x^3 + 42x^2 - 96x - 32.$$

Since $S(2) = 16 > 0$ and $S'(x) > 180 > 0$ for $x > 2$, κ is strictly concave on $(2,3]$, which together with $\kappa(2) = 0$ and $\kappa(3) = \ln \frac{3^7 5^8}{13^8} > 0$ yields that $\kappa > 0$ indeed on $(2,3]$.

3. Parts (a), (b) and (c) of step *2.* are applicable for the proof of the positivity of Φ on $(-1, -\frac{1}{2})$ with minor changes.

(a) It suffices to prove that $\widetilde{\Phi} > 0$ on $(-1, -\frac{1}{2})$, which is equivalent to

$$3\tau^2(x) + \tau^3(x) + \frac{2-3x}{x^2}\tau^4(x) > 4, \qquad x \in (1,2). \tag{29}$$

(b) Since $\zeta(1) = \zeta(2) = 0$ and $\zeta''(x) < 0$ for $x \in (1,2)$, $\zeta > 0$ on $(1,2)$, yielding a sufficient condition for (29) in the form

$$\frac{x^2-2}{x^2}\tau(x) + \frac{2-3x}{x^2}\tau^2(x) > -3, \qquad x \in (1,2). \tag{30}$$

(c) We have $P(1) = -35 < 0$, $P(2) = 188 > 0$ and $P'(x) > 107 > 0$ for $x > 1$. Consequently, P has a unique root x_0 in $(1,2)$, and η is strictly concave on $(1, x_0]$ and strictly convex on $[x_0, 2)$. However, $\eta(1) = \eta(2) = 0$, and $\eta'(1) = 1 + 4\ln\frac{4}{9} < 0$, which ensure that $\eta < 0$ on $(1,2)$, and

$$\tau(x) < \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (1,2)$$

is a sufficient condition for (30).

(d) As we have seen, $Q(\frac{3}{2}) = 0$ and therefore, we cannot proceed as in part (d) of step *2.* Instead, let us prove that

$$\frac{8x+4}{x+8} < \frac{5x^2+5x-6}{3(3x-2)}, \qquad x \in (1,2).$$

The desired inequality is equivalent to

$$T(x) := 5x^3 - 27x^2 + 46x - 24 > 0, \qquad x \in (1,2).$$

Let us notice that $T'' < 0$ on $(1, \frac{9}{5})$ and $T'' > 0$ on $(\frac{9}{5}, 2)$. And since $T(1) = T(2) = 0$ and $T'(2) = -2 < 0$, the positivity of T on $(1,2)$ follows.

Consequently, it suffices to prove that

$$\tau(x) < \frac{8x+4}{x+8}, \qquad x \in (1,2).$$

Let us reformulate it as

$$\mu(x) := 4(x-1)\big(\ln 4 + \ln(2x+1) - \ln(x+8)\big) - (3x-2)\ln x > 0, \qquad x \in (1,2).$$

After differentiating we obtain that

$$\mu''(x) = -\frac{U(x)}{x^2(x+8)^2(2x+1)^2},$$

where

$$U(x) = 12x^5 + 212x^4 - 161x^3 - 522x^2 + 736x + 128.$$

We have that $U(1) = 405 > 0$, $U'(1) = 117 > 0$, $U''(1) = 774 > 0$ and $U'''(x) > 4842 > 0$ for $x > 1$, meaning that μ is strictly concave on $(1,2)$. The last fact we have to realise is that $\mu(1) = \mu(2) = 0$. □

Numerical calculations indicate that $\overline{q}(-1) \approx 0.624$, it has a unique stationary point (≈ -0.185, while $\overline{q}(-0.185) \approx 0.421$) as well as a unique inflection point (≈ -0.400). One can prove that defining $\overline{q}(-\frac{1}{7}) := \frac{3}{7}$, $\overline{q}'(-\frac{1}{7}) = \frac{1}{2}$ holds.

Joining Lemmata 3.3, 3.4, 3.7 and 3.9 with Lemma 2.4, and using the continuity of $L_1 + L_2$, we obtain Theorem 3.10.

3.10 Theorem. *Assume that* (6) *holds with* $p < 1$ *and* $q \notin \{\widehat{q}(p), \overline{q}(p)\}$. *(Functions* $\widehat{q}$ *and* $\overline{q}$ *are defined by* (14)*.) Then there exist* $0 < l_1 \le l_2 \le l_3 \le l_4 < \infty$, $l_1 < l_4$ *such that the number of all positive non-symmetric solutions of* (1) *is*

$$|\mathcal{N}(l)| \begin{cases} \ge 2 & \text{if } l \in (l_1, l_4), \\ = 0 & \text{if } l \in (0, l_1) \cup (l_4, \infty). \end{cases}$$

(Recall that $|\mathcal{N}(l)|$ *is even.)*

Moreover:

(i) *If* $p < 0$, $q > \widehat{q}(p)$ *or* $p \le -\frac{1}{7}$, $-p < q < \overline{q}(p)$, *then* $l_3 < l_4$ *and*

$$|\mathcal{N}(l)| \ge \begin{cases} 4 & \text{if } l \in (l_3, l_4), \\ 2 & \text{if } l = l_4. \end{cases} \tag{31}$$

(ii) *If* $0 < p < 1$, $q < \widehat{q}(p)$ *or* $p < -\frac{1}{2}$, $\overline{q}(p) < q < -p$, *then* $l_1 < l_2$ *and*

$$|\mathcal{N}(l)| \ge \begin{cases} 4 & \text{if } l \in (l_1, l_2), \\ 2 & \text{if } l = l_1. \end{cases} \tag{32}$$

(See Figure 3 showing the graphs of $\widehat{q}$, $\overline{q}$ *and the sets from assertions (i) and (ii)—the green and the cyan sets. Furthermore, we have* $l_1 = \inf \frac{L_1+L_2}{2}$, $l_2 = \min\{\frac{L_2(0)}{2}, L(M)\}$, $l_3 = \max\{\frac{L_2(0)}{2}, L(M)\}$ *and* $l_4 = \sup \frac{L_1+L_2}{2}$. *See Definition 2.3 and Lemmata 2.1, 2.2 for the definition of* L_1, L_2 *and* $L(M)$, *and Lemma 3.3 for the definition of* $L_2(0)$*.)*

For $p < 1$ we have succeeded in describing the behaviour of $L_1 + L_2$ only near 0 and M, except $p = 0$, $q = 1$ and $p = -\frac{1}{2}$, $q = \frac{1}{2}$, for which $L_1 + L_2$ is constant, and except $p \in (-1, 0) \cup (0, 1)$, $q = \widehat{q}(p)$ and $p \in (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{7})$, $q = \overline{q}(p)$, for which we know only the limits of $L_1 + L_2$ at 0 and M.

However, using numerical calculations, one can observe that $L_1 + L_2$ has probably at most one relative extremum for any $p \in (-1, 1)$, $q > \frac{p+1}{2}$, $(p, q) \notin \{(0, 1), (-\frac{1}{2}, \frac{1}{2})\}$. If it is true, the behaviour of $L_1 + L_2$ on $(0, M)$ is clear for all $p \in (-1, 1)$, $q \notin \{\widehat{q}(p), \overline{q}(p)\}$, and the statement of Theorem 3.10 can be modified in the following way:

A: (31) and (32) hold with "=" instead of "$\ge$",

B: $|\mathcal{N}(l)| = 2$ for all p, q, a and $l \in (l_1, l_4)$ such that the exact value of $|\mathcal{N}(l)|$ does not follow from A,

C: $l_1 = l_2$ and $\mathcal{N}(l_1) = \emptyset$ hold for all (p, q) not dealt with in (ii),

D: $l_3 = l_4$ and $\mathcal{N}(l_4) = \emptyset$ hold for all (p, q) not dealt with in (i).

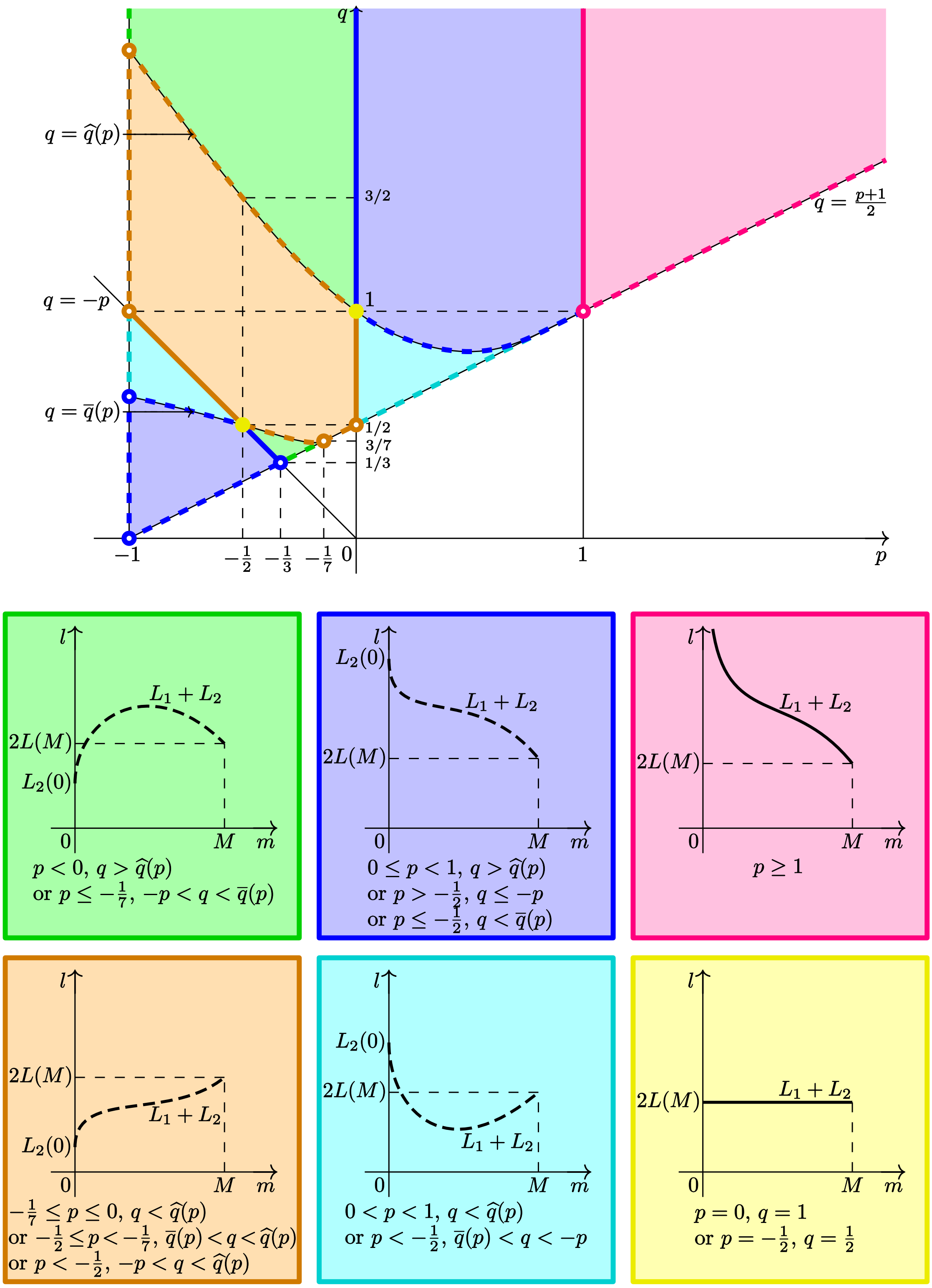

Figure 3: The behaviour of $L_1 + L_2$ for $p > -1$, $q > \frac{p+1}{2}$, $a > 0$ according to Theorem 3.2 and Lemmata 3.1, 3.3, 3.4, 3.7, 3.8 and 3.9.
The dashed graphs mean that for those values of p and q the behaviour of $L_1 + L_2$ has been examined only near 0 and M, and the graph has been plotted assuming that $L_1 + L_2$ has at most one stationary point. (This assumption is consistent with numerical calculations.)

The properties of $L_1 + L_2$ are summarised in Figure 3, which shows the graphs of $L_1 + L_2$ and the corresponding sets of (p, q). Let us notice that the graphs of $\widehat{q}$ and $\overline{q}$ in it are the output of the numerical solution of (14).

Acknowledgements

The author is grateful to Professor Marek Fila for his help. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the VEGA 1/0711/12 grant.

References

[1] Andreu F., Mazón J. M., Toledo J. and Rossi J. D., *Porous medium equation with absorption and a nonlinear boundary condition,* Nonlinear Analysis: Theory, Methods & Applications **49**(4) (2002), 541–563.

[2] Arrieta J. M. and Rodríguez-Bernal A., *Localization on the boundary of blow-up for reaction-diffusion equations with nonlinear boundary conditions,* Communications in Partial Differential Equations **29**(7–8) (2004), 1127–1148.

[3] Brighi B. and Ramaswamy M., *On some general semilinear elliptic problems with nonlinear boundary conditions,* Advances in Differential Equations **4**(3) (1999), 369–390.

[4] Cano-Casanova S., *On the positive solutions of the logistic weighted elliptic BVP with sublinear mixed boundary conditions,* Cano-Casanova, S. López-Gómez J. and Mora-Corral C. (eds.) Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, 1–15, World Scientific, Singapore 2005.

[5] Chipot M., Fila M. and Quittner P., *Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions,* Acta Mathematica Universitatis Comenianae **60**(1) (1991), 35–103.

[6] Chipot M. and Quittner P., *Equilibria, connecting orbits and a priori bounds for semilinear parabolic equations with nonlinear boundary conditions,* Journal of Dynamics and Differential Equations **16**(1) (2004), 91–138.

[7] Chipot M. and Ramaswamy M., *Semilinear elliptic problems with nonlinear boundary conditions,* Differential Equations and Dynamical Systems **6**(1–2) (1998), 51–75.

[8] Fila M. and Kawohl B., *Large time behavior of solutions to a quasilinear parabolic equation with a nonlinear boundary condition,* Advances in Mathematical Sciences and Applications **11** (2001), 113–126.

[9] Fila M., Velázquez J. J. L. and Winkler M., *Grow-up on the boundary for a semilinear parabolic problem,* Progress in Nonlinear Differential Equations and Their Applications, vol. 64: Nonlinear Elliptic and Parabolic Problems, 137–150, Birkhäuser, Basel 2005.

[10] Filo J., *Diffusivity versus absorption through the boundary,* Journal off Differential Equations **99**(2) (1992), 281–305.

[11] García-Melián J., Morales-Rodrigo C., Rossi J. D. and Suárez A., *Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary,* Annali di Matematica Pura ed Applicata **187**(3) (2008), 459–486.

[12] Leung A. W. and Zhang Q., *Reaction diffusion equations with non-linear boundary conditions, blowup and steady states,* Mathematical Methods in the Applied Sciences **21**(17) (1998), 1593–1617.

[13] López Gómez J., Márquez V. and Wolanski N., *Dynamic behavior of positive solutions to reaction-diffusion problems with nonlinear absorption through the boundary,* Revista de la Unión Matemática Argentina **38**(3–4) (1993), 196–209.

[14] Morales-Rodrigo C. and Suárez A., *Some elliptic problems with nonlinear boundary conditions,* Cano-Casanova S., López-Gómez J. and Mora-Corral C. (eds.) Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, 175–199, World Scientific, Singapore 2005.

[15] Peres S., *Solvability of a nonlinear boundary value problem,* Acta Mathematica Universitatis Comenianae **82**(1) (2013), 69–103.

[16] Peres S., *Nonsymmetric solutions of a nonlinear boundary value problem,* Czechoslovak Mathematical Journal (to appear).

[17] Quittner P., *On global existence and stationary solutions for two classes of semilinear parabolic problems,* Commentationes Mathematicae Universitatis Carolinae **34**(1) (1993), 105–124.

[18] Rodríguez-Bernal A. and Tajdine A., *Nonlinear balance for reaction-diffusion equations under nonlinear boundary conditions: dissipativity and blow-up,* Journal of Differential Equations **169**(2) (2001), 332–372.

[19] Rossi J. D., *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem,* Chipot M. and Quittner P. (eds.) Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 2, Chapter 5, 311–406, Elsevier, Amsterdam 2005.

Sámuel Peres, Department of Applied Mathematics and Statistics, Comenius University, Mlynská dolina, SK-84248 Bratislava, Slovakia, *e-mail*: `peres_samuel@hotmail.com`

(Received August 16, 2013)