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ALMOST AUTOMORPHIC MILD SOLUTIONS TO SOME SEMILINEAR ABSTRACT DIFFERENTIAL EQUATIONS WITH DEVIATED ARGUMENT IN FRÉCHET SPACES

CIPRIAN G. GAL

Abstract. In this paper we consider the semilinear differential equation with deviated argument in a Fréchet space $x'(t) = Ax(t) + f(t, x(t), x[\alpha(x(t), t)])$, $t \in \mathbb{R}$, where A is the infinitesimal (bounded) generator of a C_0-semigroup satisfying some conditions of exponential stability. Under suitable conditions on the functions f and α we prove the existence and uniqueness of an almost automorphic mild solution to the equation.

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1. Introduction

In a very recent paper [2], the existence and uniqueness of almost automorphic mild solutions with values in Banach spaces, for the differential equation

$$x'(t) = Ax(t) + f(t, x(t), x[\alpha(x(t), t)]),\ t \in \mathbb{R}, \tag{1.1}$$

is proved, where A is the infinitesimal (bounded) generator of a C_0-semigroup of operators $(T(t))_{t\geq 0}$ on a Banach space, satisfying some exponential-type conditions of stability and f and α satisfy suitable conditions.

The goal of the present note is to prove the existence and uniqueness of almost automorphic mild solutions for the differential equation (1.1), but in the more general setting of Fréchet spaces. We now give the framework which is necessary to study (1.1) in locally convex spaces. We recall the following:

Definition 1.1 A linear space $(X, +, \cdot)$ over $\mathbb{R}$ is called Fréchet space if X is a metrizable, complete, locally convex space.

Remark 1.1. It is a known fact that the Fréchet spaces are characterized by the existence of a countable, sufficient and increasing family of seminorms $(p_i)_{i\in\mathbb{N}}$ (that is $p_i(x) = 0, \forall i \in \mathbb{N}$ implies $x = 0_X$ and $p_i(x) \leq p_{i+1}(x), \forall x \in X, i \in \mathbb{N}$), which define the pseudonorm

$$|x|_X = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{p_i(x)}{1 + p_i(x)}, x \in X. \tag{1.2}$$

The metric $d(x,y) = |x-y|_X$ is invariant with respect to translations and generates a complete (by sequences) topology equivalent to that of locally convex space. Moreover, d has the properties : $d(x,y) = 0$ iff $x = y$, $d(x,y) = d(y,x)$, $d(x,y) \le d(x,z)+d(z,y)$, $d(x+u, y+u) = d(x,y)$ for all $x,y,z \in X$. Also, we note that since $\frac{p_i(x)}{1+p_i(x)} \le 1$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1$, it follows that $|x|_X \le 1, \forall x \in X$.

Furthermore, d has the following properties:

Theorem 1.2(see e.g. [3])

(i) $d(cx, cy) \le d(x,y)$ *for* $|c| \le 1$;

(ii) $d(x+u, y+v) \le d(x,y) + d(u,v)$;

(iii) $d(kx, ky) \le d(rx, ry)$ *if* $k, r \in \mathbb{R}, 0 < k \le r$;

(iv) $d(kx, ky) \le kd(x,y), \forall k \in \mathbb{N}, k \ge 2$;

(v) $d(cx, cy) \le (|c|+1)d(x,y), \forall c \in \mathbb{R}$.

Remark 1.2. Everywhere in the rest of this paper, $(X, (p_i)_{i\in\mathbb{N}}, d)$ will be a Fréchet space with $(p_i)_{i\in\mathbb{N}}$ and d as in the Remark 1.1 following Definition 1.1.

The concept of almost automorphy is a generalization of periodicity. It has been introduced in the literature by S. Bochner in relation to some aspect of differential geometry. There are many important contributions that have been made to the theory of almost automorphic functions with values in Banach spaces. We refer the reader to the book [8] and the references therein. Moreover, in [3] , the authors develop the theory of almost automorphic functions with values in Fréchet spaces and apply it to abstract differential equations of the form (1.1) in the special case when the semilinear term f (in (1.1)) depends only on the first two arguments t and $x(t)$. Our goal is to generalize (at least, partially) such results when f is as in (1.1) and α is an almost automorphic function that satisfies suitable conditions.

We start with the following Bochner-kind definition.

Definition 1.3 (see e.g. [3]) We say that a continuous function $f : \mathbb{R} \to X$ is almost automorphic, if every sequence of real numbers $(r_n)_n$ contains a subsequence $(s_n)_n$ such that for each $t \in \mathbb{R}$, there exists $g(t) \in X$ with the property

$$\lim_{n\to+\infty} d(g(t), f(t+s_n)) = \lim_{n\to+\infty} d(g(t-s_n), f(t)) = 0. \tag{1.3}$$

(The above convergence on $\mathbb{R}$ is pointwise).

The set of all almost automorphic functions with values in X is denoted by $AA(X)$.

2. Basic Result

First let us recall some known concepts and results in locally convex (Fréchet) spaces.

Theorem 2.1 *(see e.g. [4, p. 128]) Let* $(X, (p_i)_{i\in J_1}), (Y, (q_j)_{j\in J_2})$ *be two locally convex spaces, where* $(p_i)_i$ *and* $(q_j)_j$ *are the corresponding families of semi-norms. A* *linear operator $A : X \to Y$ is continuous on X if and only if for any $j \in J_2$, there exists $i \in J_1$ and a constant $M_j > 0$, such that*

$$q_j(A(x)) \le M_j p_i(x), \forall x \in x. \tag{2.1}$$

The space of all linear and continuous operators from X to Y is denoted by $B(X,Y)$. If $X = Y$, then $B(X,Y)$ will be denoted by $B(X)$.

Remark 2.1. For $A \in B(X)$, let us denote

$$||A||_{i,j} = \sup\{p_j(A(x)); x \in X, p_i(x) \le 1\}.$$

Then it is well-known that $A \in B(X)$ if and only if for every j there exists i (depending on j) such that $||A||_{i,j} < +\infty$.

Definition 2.2 (see e.g. [6],[9]) Let $(X,(p_j)_{j\in J})$ be a locally convex space. A family $T = (T(t))_{t\ge 0}$ with $T(t) \in B(X), \forall t \ge 0$ is called C_0-semigroup on X if :

(i) $T(0) = I$ (the identity operator on X) ;

(ii) $T(t+s) = T(t)T(s), \forall t, s \ge 0$ (here the product means composition) ;

(iii) For all $j \in J$, $x \in X$ and $t_0 \in \mathbb{R}_+$, we have $\lim_{t\to t_0} p_j[T(t)(x) - T(t_0)(x)] = 0$.

(iv) The operator A is called the (infinitesimal) (possibly unbounded) generator of the C_0-semigroup T on X, if for every $j \in J$ we have

$$\lim_{t\to 0^+} p_j[A(x) - \frac{T(t)(x) - x}{t}] = 0, \tag{2.2}$$

for all $x \in X$.

Remark 2.2. In a similar manner, we can define a C_0-group on X by replacing $\mathbb{R}_+$ with $\mathbb{R}$.

Definition 2.3 (see e.g. [8, p. 99, Definition 7.1.1]) Let $(X,(p_j)_{j\in J})$ be a complete, Hausdorff locally convex space. A family $F = (A_i)_{i\in\Gamma}$, $A_i \in B(X), \forall i$, is called equicontinuous, if for any $j_1 \in J$ there exists $j_2 \in J$ such that

$$p_{j_1}[A_i(x)] \le p_{j_2}(x), \forall x \in X, i \in \Gamma. \tag{2.3}$$

According to e.g. [8, p. 100-103, Theorems 7.1.2, 7.1.3, 7.1.5, 7.1.6], we can state the following:

Theorem 2.4 *Let $(X,(p_j)_{j\in J})$ be a complete, Hausdorff locally convex space and $A \in B(X)$ such that the countable family $\{A^k; k = 1, 2, ...,\}$ is equicontinuous. For $x \in X$ and $t \ge 0$, let us define $S_m(t,x) = \sum_{k=0}^{m} \frac{t^k}{k!} A^k(x)$. It follows :*

(i) For each $x \in X$ and $t \ge 0$, the sequence $S_m(t,x), m \in \mathbb{N}$ is convergent in X, that is, there exists an element in X denoted by $e^{tA}(x)$, such that

$$\lim_{m\to+\infty} p_j(e^{tA}(x) - S_m(t,x)) = 0, \forall j \in J \tag{2.4}$$

and we write $e^{tA}(x) = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k(x)$;

(ii) For any fixed $t \geq 0$, we have $e^{tA} \in B(X)$;

(iii) $e^{(t+s)A} = e^{tA}e^{sA}, \forall t, s \geq 0$;

(iv) For every $j \in J$, we have

$$\lim_{t \to 0^+} p_j[A(x) - \frac{e^{tA}(x) - x}{t}] = 0, \tag{2.5}$$

for all $x \in X$;

(iv) $\frac{d}{dt}[e^{(t-a)A}(x)] = A[e^{(t-a)A}(x)]$, for every $t \geq a, a \in \mathbb{R}$ and the function $e^{(t-a)A}(x\,(a)) : \mathbb{R} \to X$ is the unique solution of the problem $x'(t) = A[x(t)]$, for every $t \geq a$, $a \in \mathbb{R}$.

If $(X, (p_i)_{i \in \mathbb{N}}, d)$ is a Fréchet space, then let us recall that for $f : \mathbb{R} \to X$, the derivative of f at $x \in \mathbb{R}$ denoted by $f'(x) \in X$, is defined by the relation

$$lim_{h \to 0} d(f'(x), \frac{f(x+h) - f(x)}{h}) = 0. \tag{2.6}$$

It easily follows that this is equivalent to

$$lim_{h \to 0} p_i[f'(x) - \frac{f(x+h) - f(x)}{h}] = 0, \forall i \in \mathbb{N}.$$

For $A \in B(X)$, denote by $(T(t))_{t \geq 0}$ a C_0-semigroup of operators on X generated by A (according to Definition 2.2).

Now, let us consider the following abstract differential equation with deviated argument in the Fréchet space $(X, (p_i)_{i \in \mathbb{N}}, d)$,

$$x'(t) = Ax(t) + f(t, x(t), x[\alpha(x(t), t)]), \; t \in \mathbb{R}. \tag{2.7}$$

It is easy to prove (see [3, proof of Theorem 3.5]) that if $x(t)$ is a mild solution of the differential equation (2.7), then it has the form

$$x(t) = T(t-a)[x\,(a)] + \int_a^t T(t-s)[f(s, x(s), x(\alpha(x(s), s)))]ds,$$

for every $a \in \mathbb{R}$, every $t \geq a$ and we refer to any continuous $x \in C\,(\mathbb{R}, X)$ satisfying the above relation as a mild solution of the above problem . Obviously, because of the absence, in general, of its differentiability, a mild solution is not a strong solution of the problem.

This section is concerned with existence and uniqueness of almost automorphic mild solutions of the differential equation (2.7) with deviated argument. The almost automorphic property of the deviation function $\alpha(s, t)$ with respect to t and a Lipschitz condition in s, uniformly with respect to t, permits us to generalize some of the results found in the literature for the semilinear ordinary differential equations with deviated arguments in Fréchet spaces.

The main result is the following:

Theorem 2.5 *Let $(X, (p_i)_{i\in\mathbb{N}}, d)$ be a Fréchet space and let us assume that $A \in B\,(X)$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X which satisfies the condition : for any $j \in \mathbb{N}$ there exist $K_j > 0, \omega_j < 0$, such that*

$$||T(t)||_{\beta(j),j} \leq K_j e^{\omega_j t}, \forall t \geq 0, \tag{2.8}$$

where $\beta : \mathbb{N} \to \mathbb{N}$ is an application satisfying the condition $\beta[\beta(j)] = \beta(j), \forall j \in \mathbb{N}$. Also, assume that $f(t, x, y)$ is almost automorphic in t for each $x, y \in X$, and that $f : \mathbb{R} \times X \times X \to X$ satisfies the Lipschitz-type conditions uniformly in t of the form

$$p_j[f(t,x,u)-f(t,y,v)] \leq C_j[p_j(x-y)+p_j(u-v)], \forall x,y,u,v \in X,\ t \in \mathbb{R},\ j \in \mathbb{N}, \tag{2.9}$$

$\alpha : X \times \mathbb{R} \to \mathbb{R}$ is almost automorphic in $t \in \mathbb{R}$ for each $x \in X$ and satisfies the conditions

$$p_j[\alpha(u,t) - \alpha(v,t)] \leq S_j p_j(u - v), \forall u, v \in X,\ t \in \mathbb{R}, \tag{2.10}$$

and that $||A||_{k,k} < +\infty, \forall k \in \mathbb{N}$.

Denoting $M_j = \sup\{p_j[f(t, x, y)]; t \in \mathbb{R}, x, y \in X\} < +\infty$, for all $j \in \mathbb{N}$ and

$$L_j = M_j + M_{\beta(j)} \frac{K_j}{|\omega_j|} ||A||_{\beta(j),\beta(j)}, \tag{2.11}$$

for all $j \in \mathbb{N}$, under the conditions

$$C_{\beta_j} \left(S_{\beta(j)} L_{\beta(j)} + 2\right) \frac{K_j}{|\omega_j|} < 1, \forall j \in \mathbb{N}, \tag{2.12}$$

then the equation

$$x'(t) = A[x(t)] + f[t, x(t), x(\alpha(x(t), t))],\ t \in \mathbb{R},$$

has a unique almost automorphic mild solution in the Fréchet space

$$AA_{(L_j)_j}(X) = \{\Phi \in AA(X);\ p_j[\Phi(u) - \Phi(v)] \leq L_j|u - v|, \forall u, v \in \mathbb{R},\ j \in \mathbb{N}\}.$$

Proof. Let $x(t)$ be a mild solution of (2.7). It is continuous and satisfies the integral equation

$$x(t) = T(t - a)[x(a)] + \int_a^t T(t - s)[f(s, x(s), x(\alpha(x(s), s)))]ds, \forall a \in \mathbb{R}, \forall t \geq a.$$

Since by [3, Theorem 2.14], $AA(X)$ is a Fréchet space with respect to the countable family of seminorms $q_j(f) = \sup\{p_j(f(t)); t \in \mathbb{R}\}$, $j \in \mathbb{N}$, it is easy to show that $AA_{(L_j)_j}(X)$ is closed under the convergence with respect to the family of seminorms $(q_j)_j$. It follows that $AA_{(L_j)_j}(X)$ is also a Fréchet space with respect to the same family of seminorms.

Consider now $\int_a^t T(t-s)[f(s,x(s),x(\alpha(x(s),s)))]ds$ and the nonlinear operator $G : AA_{(L_j)_j}(X) \to AA(X)$ given by

$$(G\Phi)(t) := \int_{-\infty}^{t} T(t-s)[f(s,\Phi(s),\Phi(\alpha(\Phi(s),s)))]ds.$$

First we show that $G\Phi \in AA_{(L_j)_j}(X)$ for $\Phi \in AA_{(L_j)_j}(X)$. Denote

$$F(s) = f(s,\Phi(s),\Phi[\alpha(\Phi(s),s)]),\ s \in \mathbb{R},$$

with $\Phi \in AA_{(L_j)j}(X)$.

Since $\Phi \in AA(X)$, by the hypothesis on α and by Theorem 2.8, (iv) (see also [3]), it follows that $\alpha(\Phi(s),s) : \mathbb{R} \to \mathbb{R}$ is in $AA(\mathbb{R})$. Since Φ also is continuous, by Theorem 2.4, (viii), we get that $\Phi[\alpha(\Phi(\cdot),\cdot)] \in AA(X)$. Denoting $\gamma(s) = \Phi[\alpha(\Phi(s),s)]$, we have $\gamma \in AA(X)$ and since $f(t,u,v)$ is almost automorphic in t for each u and v, and Lipschitz in u and v, by similar reasonings with those in the proof of Theorem 2.8, (iv) in [3], we immediately get $\gamma \in AA(X)$, $F(s) = f(s,\Phi(s),\beta(s)) \in AA(X)$, $s \in \mathbb{R}$. Because $F(s) \in AA(X)$, then it is bounded in norm so that $M_j = \sup_{s\in\mathbb{R}} p_j[F(s)]$ exist and are finite for all $j \in \mathbb{N}$. Moreover, as in the proof of Theorem 4.1 in [3], we immediately obtain that $G\Phi \in AA(X)$. Thus, the map G is well defined.

As in e.g. [5], we obtain

$$(G\Phi)'(t) = F(t) + \int_{-\infty}^{t} T(t-s)A[F(s)]ds,$$

which implies

$$p_j[(G\Phi)'(t)] \le p_j[F(t)] + \int_{-\infty}^{t} p_j(T(t-s)A[F(s)])ds \le$$

$$M_j + \int_{-\infty}^{t} ||T(t-s)||_{\beta(j),j} \cdot p_{\beta(j)}(A[F(s)])ds \le$$

$$M_j + \int_{-\infty}^{t} K_j e^{\omega_j(t-s)} ||A||_{\beta(j),\beta(j)} p_{\beta(j)}(F(s))ds \le$$

$$M_j + K_j M_{\beta(f)} \frac{||A||_{\beta(j),\beta(j)}}{|\omega_j|} = L_j.$$

Then by the mean value theorem in locally convex spaces (see e.g [7, p. 15, Proposition 3]), we obtain

$$p_j[(G\Phi)(u) - (G\Phi)(v)] \le L_j|u-v|, \forall u,v \in \mathbb{R},$$

that is $G\Phi \in AA_{(L_j)_j}(X)$ for $\Phi \in AA_{(L_j)_j}(X)$.

Finally, it remains to check that G is a contraction. Let $\Phi_1, \Phi_2 \in AA_{(L_j)_j}(X)$. We calculate

$$q_j[G\Phi_1 - G\Phi_2]$$

$$= \sup_{t\in\mathbb{R}} p_j[\int_{-\infty}^{t} T(t-s)[f(s,\Phi_1(s),\Phi_1(\alpha(\Phi_1(s),s))) - f(s,\Phi_2(s),\Phi_2(\alpha(\Phi_2(s),s)))]ds] \leq$$

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} p_j(T(t-s)[f(s,\Phi_1(s),\Phi_1(\alpha(\Phi_1(s),s))) - f(s,\Phi_2(s),\Phi_2(\alpha(\Phi_2(s),s)))])ds \leq$$

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} ||T(t-s)||_{\beta(j),j}\cdot$$

$$p_{\beta(j)}[f(s,\Phi_1(s),\Phi_1(\alpha(\Phi_1(s),s))) - f(s,\Phi_2(s),\Phi_2(\alpha(\Phi_2(s),s)))]ds \leq$$

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} ||T(t-s)||_{\beta(j),j} C_{\beta(j)}[q_{\beta(j)}(\Phi_1-\Phi_2) + p_{\beta(j)}(\Phi_1(\alpha(\Phi_1(s),s)) - \Phi_1(\alpha(\Phi_2(s),s)))+$$

$$p_{\beta(j)}(\Phi_1(\alpha(\Phi_2(s),s)) - \Phi_2(\alpha(\Phi_2(s),s)))ds \leq$$

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} ||T(t-s)||_{\beta(j),j} C_{\beta(j)}[2q_{\beta(j)}(\Phi_1-\Phi_2) + L_{\beta(j)} p_{\beta(j)}(\alpha(\Phi_1(s),s) - \alpha(\Phi_2(s),s))]ds \leq$$

$$C_{\beta(j)} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} K_j e^{\omega_j(t-s)} C_{\beta(j)}[2q_{\beta(j)}(\Phi_1-\Phi_2) + L_{\beta(j)} S_{\beta(j)} q_{\beta(j)}(\Phi_1-\Phi_2)]ds \leq$$

$$C_{\beta(j)}[S_{\beta(j)}L_{\beta(j)} + 2]\frac{K_j}{|\omega_j|} q_{\beta(j)}(\Phi_1-\Phi_2) < q_{\beta(j)}(\Phi_1-\Phi_2),$$

because of the assumption (2.12). It follows from [1, p. 92, Theorem 1] that there exists a unique $u \in AA_{(L_j)_j}(X)$ such that $Gu = u$, that is,

$$u(t) = \int_{-\infty}^{t} T(t-s)[f(s,u(s),u(\alpha(u(s),s)))]ds.$$

Reasoning now exactly as in the proof of Theorem 4.2 in [3], the proof is complete. We omit the details.

In conclusion, in the present note, we have obtained an existence and uniqueness result concerning almost automorphic mild solutions for differential equations of the form (1.1) in locally convex (*Fréchet)* spaces and in Banach spaces [2], but only when the generator A is a bounded operator. The next natural step is to extend such results to the case when A is a (possibly) unbounded operator in Banach and *Fréchet* spaces. Such problems will be investigated in future articles.

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Morgan State University, Department of Mathematical Sciences, Baltimore, MD, 21212, U.S.A.

E-mail address: cgal@morgan.edu