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Impulsive system of ODEs with general linear boundary conditions*

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Abstract

The paper provides an operator representation for a problem which consists of a system of ordinary differential equations of the first order with impulses at fixed times and with general linear boundary conditions

$$z'(t) = A(t)z(t) + f(t, z(t)) \quad \text{for a.e. } t \in [a, b] \subset \mathbb{R},$$
$$z(t_i+) - z(t_i) = J_i(z(t_i)), \quad i = 1, \ldots, p,$$
$$\ell(z) = c_0, \quad c_0 \in \mathbb{R}^n.$$

Here $p, n \in \mathbb{N}$, $a < t_1 < \ldots < t_p < b$, $A \in \mathbb{L}^1([a, b]; \mathbb{R}^{n\times n})$, $f \in \mathrm{Car}([a, b] \times \mathbb{R}^n; \mathbb{R}^n)$, $J_i \in \mathbb{C}(\mathbb{R}^n; \mathbb{R}^n)$, $i = 1, \ldots, p$, and ℓ is a linear bounded operator on the space of left-continuous regulated functions on interval $[a, b]$. The operator ℓ is expressed by means of the Kurzweil-Stieltjes integral and covers all linear boundary conditions for solutions of the above system subject to impulse conditions. The representation, which is based on the Green matrix to a corresponding linear homogeneous problem, leads to an existence principle for the original problem. A special case of the n-th order scalar differential equation is discussed. This approach can be also used for analogical problems with state-dependent impulses.

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1 Introduction

In the literature there is a large amount of papers investigating the solvability of impulsive boundary value problems with *impulses at fixed times*. Such problems often differ from one another only by different choices of linear boundary conditions which are mostly two-point, multipoint or integral ones. On the other hand, boundary value problems with state-dependent impulses have been studied very rarely and only with two-point boundary conditions, see [1, 2, 3, 4, 5, 6, 8, 9, 10]. The aim of our paper is to find an operator representation which yields the solvability for a quite general impulsive problem of the form

$$z'(t) = A(t)z(t) + f(t, z(t)) \quad \text{for a.e. } t \in [a, b] \subset \mathbb{R}, \tag{1}$$

$$z(t_i+) - z(t_i) = J_i(z(t_i)), \quad i = 1, \dots, p, \tag{2}$$

$$\ell(z) = c_0, \quad c_0 \in \mathbb{R}^n, \tag{3}$$

where all possible linear boundary conditions are covered by condition (3). In addition, the approach presented here can be applied to problems with *state-dependent impulses*, which will be shown in our next papers.

In what follows we use this notation. Let us denote for $p \in \mathbb{N}$

$$\mathcal{J}_0 = [a, t_1],\ \mathcal{J}_1 = (t_1, t_2],\ \mathcal{J}_2 = (t_2, t_3],\ \dots,\ \mathcal{J}_p = (t_p, b].$$

Let $m, n \in \mathbb{N}$. By $\mathbb{R}^{m\times n}$ we denote the set of all matrices of the type $m \times n$ with real valued coefficients equipped with the maximum norm

$$\|K\| = \max_{i,j\in\{1,\dots,n\}} |K_{ij}| \quad \text{for } K = (K_{ij})_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}.$$

Let A^T denote the transpose of $A \in \mathbb{R}^{m\times n}$. Let $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ be the set of all n–dimensional column vectors $c = (c_1, \dots, c_n)^T$, where $c_i \in \mathbb{R}$, $i = 1, \dots, n$, and $\mathbb{R} = \mathbb{R}^{1\times 1}$. By $\mathbb{C}(\mathbb{R}^n; \mathbb{R}^m)$ we denote the set of all mappings $x : \mathbb{R}^n \to \mathbb{R}^m$ with continuous components. By $\mathbb{L}^\infty([a, b]; \mathbb{R}^{m\times n})$, $\mathbb{L}^1([a, b]; \mathbb{R}^{m\times n})$, $\mathbb{G}_{\mathrm{L}}([a, b]; \mathbb{R}^{m\times n})$, $\mathbb{AC}([a, b]; \mathbb{R}^{m\times n})$, $\mathbb{BV}([a, b]; \mathbb{R}^{m\times n})$, we denote the sets of all mappings $x : [a, b] \to \mathbb{R}^{m\times n}$ whose components are respectively essentially bounded functions, Lebesgue integrable functions, left-continuous regulated functions, absolutely continuous functions and functions with bounded variation on the interval $[a, b]$. By $\mathbb{PC}([a, b]; \mathbb{R}^n)$ ($\mathbb{APC}([a, b]; \mathbb{R}^n)$) we mean the set of all mappings $x : [a, b] \to \mathbb{R}^n$ whose components are continuous (absolutely continuous) on the intervals $\mathcal{J}_i$ and continuously extendable to the closure of $\mathcal{J}_i$ for $i = 0, \dots, p$. By $\mathrm{Car}([a, b] \times \mathbb{R}^n; \mathbb{R}^n)$ we denote the set of all mappings $x : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ whose components are Carathéodory functions on the set $[a, b] \times \mathbb{R}^n$.

Note that a mapping $u : [a, b] \to \mathbb{R}^n$ is left-continuous regulated on $[a, b]$ if for each $t \in (a, b]$ and each $s \in [a, b)$

$$u(t) = u(t-) = \lim_{\tau\to t-} u(\tau) \in \mathbb{R}^n, \quad u(s+) = \lim_{\tau\to s+} u(\tau) \in \mathbb{R}^n.$$

$\mathbb{G}_{\mathrm{L}}([a, b]; \mathbb{R}^n)$ is a linear space and equipped with the sup-norm $\|\cdot\|_\infty$ it is a Banach space (see [7], Theorem 3.6). In particular, we set

$$\|u\|_\infty = \max_{i\in\{1,\dots,n\}} \left(\sup_{t\in[a,b]} |u_i(t)| \right) \quad \text{for } u = (u_1, \dots, u_n)^T \in \mathbb{G}_{\mathrm{L}}([a, b]; \mathbb{R}^n).$$

Finally, by χ_M we denote the characteristic function of the set $M \subset \mathbb{R}$.

We investigate system (1) and impulse conditions (2) under the following assumptions:

$$\left.\begin{array}{l} A \in \mathbb{L}^1([a,b];\mathbb{R}^{n\times n}), \quad f \in \mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R}^n), \\ J_i \in \mathbb{C}(\mathbb{R}^n;\mathbb{R}^n), \quad a < t_1 < \ldots < t_p < b, \quad n, p \in \mathbb{N}. \end{array}\right\} \tag{4}$$

Definition 1 A mapping $z \in \mathbb{APC}([a,b];\mathbb{R}^n)$ is a solution of problem (1), (2), if

- z satisfies the differential equation (1) for a.e. $t \in [a,b]$,
- z satisfies the impulse conditions (2).

Remark 2 Let $\mathcal{S}$ be the set of all solutions of problem (1), (2). If $z \in \mathcal{S}$, then z is left-continuous on $[a,b]$. In order to introduce various linear boundary conditions for mappings belonging to $\mathcal{S}$ we need to find a suitable linear space containing the set $\mathcal{S}$. Clearly $\mathcal{S} \subset \mathbb{PC}([a,b];\mathbb{R}^n) \subset \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$. Therefore we could take the Banach space $\mathbb{PC}([a,b];\mathbb{R}^n)$ (*cf.* Remark 12). But we choose a more general space – the space $\mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$. The reason is to obtain a general tool, which can be also applied to problems with *state-dependent* impulsive conditions. Solutions of such problems are left-continuous and can have discontinuities anywhere in the interval (a,b).

Assume that $\ell : \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator. Then condition (3) is a general linear boundary condition for each $z \in \mathcal{S}$.

Definition 3 A mapping $z \in \mathbb{APC}([a,b];\mathbb{R}^n)$ is a solution of problem (1)–(3) if z is a solution of problem (1), (2) and fulfils (3).

We are able to construct a form of ℓ. In the scalar case, it is known (*cf.* [11], Theorem 3.8) that every linear bounded functional φ on $\mathbb{G}_\mathrm{L}([a,b];\mathbb{R})$ is uniquely determined by a couple $(k,v) \in \mathbb{R} \times \mathbb{BV}([a,b];\mathbb{R})$ such that

$$\varphi(x) = kx(a) + (\mathrm{KS})\int_a^b v(t)\,\mathrm{d}[x(t)], \quad x \in \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}), \tag{5}$$

where $(\mathrm{KS})\int_a^b$ is the Kurzweil-Stieltjes integral, whose definition and properties can be found in [13] (see Perron-Stieltjes integral based on the work of J. Kurzweil). Lemma 4 deals with a general $n \in \mathbb{N}$ and provides a form of the operator ℓ from (3).

Lemma 4 *([12], Lemma 1.8) A mapping $\ell : \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator if and only if there exist $K \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{BV}([a,b];\mathbb{R}^{n\times n})$ such that*

$$\ell(z) = Kz(a) + (\mathrm{KS})\int_a^b V(t)\,\mathrm{d}[z(t)], \quad z \in \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n). \tag{6}$$

Proof. Let $z = (z_1,\ldots,z_n)^T \in \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$ and $\ell = (\ell_1,\ldots,\ell_n)^T$. Then

$$\ell(z) = \sum_{i=1}^n \left(\sum_{j=1}^n \ell_i(z_j e_j)\right) e_i, \tag{7}$$

where e_j is the j–th element of the standard basis in $\mathbb{R}^n$. Let $i,j \in \{1,\dots,n\}$. It is easy to prove that for the linear bounded operator ℓ the mapping $\varphi_{ij} : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}) \to \mathbb{R}$ defined by

$$\varphi_{ij}(x) = \ell_i(xe_j), \quad x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}),$$

is a linear bounded functional on $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R})$. By (5), this is equivalent with the fact that there exist $k_{ij} \in \mathbb{R}$ and $v_{ij} \in \mathbb{BV}([a,b];\mathbb{R})$ such that

$$\varphi_{ij}(x) = k_{ij}x(a) + (\mathrm{KS})\int_a^b v_{ij}(t)\,\mathrm{d}[x(t)], \quad x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}).$$

This, together with (7), gives

$$\ell(z) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(k_{ij}z_j(a) + (\mathrm{KS})\int_a^b v_{ij}(t)\,\mathrm{d}[z_j(t)] \right) \right) e_i.$$

If we denote

$$K = (k_{ij})_{i,j=1}^{n}, \quad V(t) = (v_{ij}(t))_{i,j=1}^{n},$$

we get (6). $\square$

Lemma 5 *Let $\Phi : [a,b] \to \mathbb{R}^{n\times n}$, $\tau \in [a,b)$ and $Q \in \mathbb{R}^{n\times n}$. Then*

$$(\mathrm{KS})\int_a^b \Phi(t)\,\mathrm{d}\left[\chi_{(\tau,b]}(t)Q\right] = \Phi(\tau)Q.$$

Let $g \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$, $\tau \in (a,b]$. Then

$$(\mathrm{KS})\int_a^b \chi_{[a,\tau)}(t)\,\mathrm{d}[g(t)] = g(\tau) - g(a).$$

Proof. It is known (*cf.* [11], Proposition 2.3) that for any $f : [a,b] \to \mathbb{R}$ and $\tau \in (a,b)$ the formula

$$(\mathrm{KS})\int_a^b f(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)] = f(\tau) \tag{8}$$

is valid. Let $\Phi(t) = (\Phi_{ij}(t))_{i,j=1}^{n}$, $Q = (Q_{ij})_{i,j=1}^{n}$. From (8) we have

$$\sum_{j=1}^{n} (\mathrm{KS})\int_a^b \Phi_{ij}(t)\,\mathrm{d}\left[\chi_{(\tau,b]}(t)Q_{jk}\right] = \sum_{j=1}^{n} \Phi_{ij}(\tau)Q_{jk}$$

for $i,k = 1,\dots,n$. The second formula follows from its scalar case ([11], Proposition 2.3) and the fact, that g is left-continuous at $t = \tau$. $\square$

2 Operator representation of problem (1)–(3)

In this section we assume that $A \in \mathbb{L}^1([a,b];\mathbb{R}^{n\times n})$,

$$\ell \text{ is given by (6), where } K \in \mathbb{R}^{n\times n}, \quad V \in \mathbb{BV}([a,b];\mathbb{R}^{n\times n}). \tag{9}$$

For further investigation we will need a linear homogeneous problem corresponding to problem (1)–(3) which has the form

$$z'(t) = A(t)z(t), \tag{10}$$

$$\ell(z) = 0, \tag{11}$$

because putting $J_i = 0$ in (2), we get $z(t_i+) = z(t_i)$ for $i = 1, \dots, p$, and the impulse condition disappears. We will also use the non-homogeneous equation

$$z'(t) = A(t)z(t) + q(t) \tag{12}$$

for $q \in \mathbb{L}^1([a,b];\mathbb{R}^n)$.

Finally, we will consider the constant impulse conditions

$$z(t_i+) - z(t_i) = I_i \in \mathbb{R}^n, \quad i = 1, \dots, p. \tag{13}$$

A solution of problem (12), (11) is a mapping $z \in \mathbb{AC}([a,b];\mathbb{R}^n)$ satisfying equation (12) for a.e. $t \in [a,b]$ and fulfilling condition (11).

Remark 6 In what follows we denote by Y a fundamental matrix of equation (10). By $\ell(\Phi)$ we mean the matrix with columns $\ell(\Phi_1)$, …, $\ell(\Phi_n)$ if $\Phi \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^{n\times n})$ has columns Φ_1, …, Φ_n.

Definition 7 A mapping $G : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ is the Green matrix of problem (10), (11), if:

(a) $G(\cdot,\tau)$ is continuous on $[a,\tau]$, $(\tau,b]$ for each $\tau \in [a,b]$,

(b) $G(t,\cdot) \in \mathbb{BV}([a,b];\mathbb{R}^{n\times n})$ for each $t \in [a,b]$;

(c) for any $q \in \mathbb{L}^1([a,b];\mathbb{R}^n)$ the function

$$x(t) = \int_a^b G(t,\tau)q(\tau)\,\mathrm{d}\tau, \quad t \in [a,b] \tag{14}$$

is a unique solution of (12), (11).

Lemma 8 *Assume* (9). *Problem* (12), (11) *has a unique solution if and only if*

$$\det \ell(Y) \neq 0. \tag{15}$$

If (15) *is valid, then there exists a Green matrix of problem* (10), (11) *which is in the form*

$$G(t,\tau) = Y(t)H(\tau) + \chi_{(\tau,b]}(t)Y(t)Y^{-1}(\tau), \quad t,\tau \in [a,b], \tag{16}$$

where H is defined by

$$H(\tau) = -\left[\ell(Y)\right]^{-1}\left(\int_\tau^b V(s)A(s)Y(s)\,\mathrm{d}s \cdot Y^{-1}(\tau) + V(\tau)\right), \quad \tau \in [a,b] \tag{17}$$

and it has the following properties:

(i) *G is bounded on $[a,b]\times[a,b]$,*

(ii) *$G(\cdot,\tau)$ is absolutely continuous on $[a,\tau]$ and $(\tau,b]$ for each $\tau\in[a,b]$ and its columns satisfy the differential equation* (10) *a.e. on $[a,b]$,*

(iii) *$G(\tau+,\tau)-G(\tau,\tau)=E$ for each $\tau\in[a,b)$,*

(iv) *$G(\cdot,\tau)\in\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^{n\times n})$ for each $\tau\in[a,b]$ and*

$$\ell(G(\cdot,\tau))=0 \quad \textit{for each } \tau\in[a,b).$$

Proof. Step 1. The general solution $x_0\in\mathbb{AC}([a,b];\mathbb{R}^n)$ of (10) is written as $x_0(t)=Y(t)c$ for $t\in[a,b]$, where $c\in\mathbb{R}^n$. By (11) we get

$$\ell(x_0)=\ell(Yc)=\ell(Y)\cdot c=0,$$

which yields that problem (10), (11) has only the trivial solution if and only if (15) is satisfied. Since $Y\in\mathbb{AC}([a,b];\mathbb{R}^{n\times n})$, we get from (6)

$$\ell(Y)=KY(a)+(\mathrm{KS})\int_a^b V(t)\,\mathrm{d}[Y(t)]=KY(a)+\int_a^b V(t)Y'(t)\,\mathrm{d}t.$$

Therefore (10) implies

$$\ell(Y)=KY(a)+\int_a^b V(t)A(t)Y(t)\,\mathrm{d}t.$$

The general solution $x\in\mathbb{AC}([a,b];\mathbb{R}^n)$ of equation (12) has the form

$$x(t)=Y(t)c+r(t), \quad t\in[a,b], \tag{18}$$

where

$$r(t)=Y(t)\int_a^t Y^{-1}(s)q(s)\,\mathrm{d}s\in\mathbb{AC}([a,b];\mathbb{R}^n). \tag{19}$$

Substituing (18) to (11) we get the equation

$$\ell(Y)c+\ell(r)=0. \tag{20}$$

A unique solution $c\in\mathbb{R}^n$ of equation (20) exists if and only if (15) holds.
Step 2. Let (15) be satisfied. Then from (20) we have

$$c=-[\ell(Y)]^{-1}\ell(r). \tag{21}$$

By virtue of (6) and (19),

$$\ell(r)=Kr(a)+(\mathrm{KS})\int_a^b V(t)\,\mathrm{d}[r(t)]=\int_a^b V(t)r'(t)\,\mathrm{d}t,$$

hence

$$\ell(r)=\int_a^b V(t)Y'(t)\int_a^t Y^{-1}(s)q(s)\,\mathrm{d}s\,\mathrm{d}t+\int_a^b V(t)q(t)\,\mathrm{d}t.$$

Using the integration *per partes* in the first integral, we derive

$$\ell(r) = \int_a^b \left(\int_t^b V(s)A(s)Y(s)\,\mathrm{d}s \cdot Y^{-1}(t) + V(t) \right) q(t)\,\mathrm{d}t. \tag{22}$$

Substituing c from (21) into (18) we have by (22)

$$\begin{aligned} x(t) &= Y(t)\left(-[\ell(Y)]^{-1}\ell(r)\right) + r(t) \\ &= Y(t)\left(-[\ell(Y)]^{-1} \cdot \int_a^b \left(\int_\tau^b V(s)A(s)Y(s)\,\mathrm{d}s\; Y^{-1}(\tau) + V(\tau) \right) q(\tau)\,\mathrm{d}\tau \right) + r(t), \end{aligned}$$

for $t \in [a, b]$. Hence we get a unique solution x of problem (12), (11) in the form

$$\begin{aligned} x(t) = Y(t) &\left(-[\ell(Y)]^{-1} \int_a^b \left(\int_\tau^b V(s)A(s)Y(s)\,\mathrm{d}s\; Y^{-1}(\tau) + V(\tau) \right) q(\tau)\,\mathrm{d}\tau \right) \\ &+ Y(t)\int_a^t Y^{-1}(\tau)q(\tau)\,\mathrm{d}\tau, \end{aligned}$$

which can be written as (14) with G defined by (16). This yields (a), (b) and (c) of Definition 7.
Step 3. Let G be the Green matrix given by (16) and (17). The properties (i) and (ii) follow directly from (9) and Remark 6. From (16) we have

$$G(\tau+, \tau) - G(\tau, \tau) = Y(\tau)H(\tau) + Y(\tau)Y^{-1}(\tau) - Y(\tau)H(\tau) = E$$

for each $\tau \in [a, b)$, which is the property (iii). Let us prove the property (iv). Clearly, (i) and (ii) imply $G(\cdot, \tau) \in \mathbb{G}_{\mathrm{L}}([a, b]; \mathbb{R}^{n\times n})$ for each $\tau \in [a, b]$. Let $\tau \in [a, b)$. From the linearity of the operator ℓ we get

$$\ell(G(\cdot, \tau)) = \ell(Y)H(\tau) + \ell(\chi_{(\tau,b]}Y)Y^{-1}(\tau). \tag{23}$$

In view of (17) and (25), the first summand in (23) is transformed into

$$\ell(Y)H(\tau) = -\left(R(\tau)Y^{-1}(\tau) + V(\tau)\right), \tag{24}$$

where

$$R(\tau) = \int_\tau^b V(s)A(s)Y(s)\,\mathrm{d}s, \quad \tau \in [a, b]. \tag{25}$$

Treating the second term in (23) we obtain

$$\begin{aligned} \ell(\chi_{(\tau,b]}Y) &= (\mathrm{KS})\!\int_a^b V(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)Y(t)] \\ &= (\mathrm{KS})\!\int_a^b V(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)(Y(t) - Y(\tau))] + (\mathrm{KS})\!\int_a^b V(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)Y(\tau)]. \end{aligned}$$

Since $\chi_{(\tau,b]}(Y - Y(\tau))$ is absolutely continuous on $[a, b]$ and it vanishes on $[a, \tau]$, we get

$$(\mathrm{KS})\!\int_a^b V(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)(Y(t) - Y(\tau))] = (\mathrm{KS})\!\int_\tau^b V(t)\,\mathrm{d}[Y(t) - Y(\tau)] = R(\tau),$$

where R is defined by (25). According to Lemma 5, we have

$$(\mathrm{KS})\int_a^b V(t)\,\mathrm{d}[\chi_{(\tau,b]}(t)Y(\tau)] = V(\tau)Y(\tau).$$

Therefore,

$$\ell(\chi_{(\tau,b]}Y) = R(\tau) + V(\tau)Y(\tau).$$

Using this equality, (23) and (24) we get

$$\ell(G(\cdot,\tau)) = -\left(R(\tau)Y^{-1}(\tau) + V(\tau)\right) + \left(R(\tau) + V(\tau)Y(\tau)\right)Y^{-1}(\tau) = 0.$$

$\square$

Remark 9 Let us note that the Green matrix of problem (10), (11) is not determined uniquely. According to the continuity of $G(\cdot,\tau)$ on intervals $[a,\tau]$, $(\tau,b]$ for $\tau\in[a,b]$ we can see that each Green matrix is in form (16), with H determined uniquely up to a set of measure zero.

Lemma 10 *Assume that* (9) *and* (15) *hold. Then the linear impulsive boundary value problem* (12), (13), (3) *has a unique solution* z *which has the form*

$$z(t) = \int_a^b G(t,s)q(s)\,\mathrm{d}s + \sum_{i=1}^{p} G(t,t_i)I_i + Y(t)\left[\ell(Y)\right]^{-1}c_0, \quad t\in[a,b], \tag{26}$$

where Y *is a fundamental matrix of equation* (10) *and* G *takes form* (16) *with* H *of* (17).

Proof. From Lemma 8 and (c) of Definition 7, it follows that the function

$$x(t) = \int_a^b G(t,s)q(s)\,\mathrm{d}s, \quad t\in[a,b],$$

is a unique solution of problem (12),(11). Since x is continuous, it satisfies (13) with $I_i\equiv 0$ for $i=1,\ldots,p$. From (ii) in Lemma 8 we obtain that the function

$$y(t) = \sum_{i=1}^{p} G(t,t_i)I_i, \quad t\in[a,b],$$

satisfies (10) for a.e. $t\in[a,b]$, and due to (iv) in Lemma 8, y satisfies (11). Moreover, the properties (ii) and (iii) in Lemma 8 yields

$$y(t_j+) - y(t_j) = \sum_{i=1}^{p}\left[G(t_j+,t_i) - G(t_j,t_i)\right]I_i = I_j$$

for $j=1,\ldots,p$, i.e. y satisfies (13). From the fact that Y is a fundamental matrix of equation (10) the function

$$u(t) = Y(t)\left[\ell(Y)\right]^{-1}c_0, \quad t\in[a,b],$$

satisfies (10) for a.e. $t \in [a,b]$ and since u is absolutely continuous it satisfies (13) with $I_i \equiv 0$, $i = 1, \dots, p$. Moreover

$$\ell(u) = \ell(Y)\,[\ell(Y)]^{-1}\,c_0 = c_0,$$

i.e. u satisfies (3). Using superposition principle we see that the function z in (26) is a solution of problem (12),(13),(3). Uniqueness follows from the fact that if $\tilde{z}$ is a solution of problem (12), (13), (3) different from z, then $w = z - \tilde{z}$ is a nontrivial solution of problem (10), (11), contrary to (15). □

Now, due to Lemma 10, we are able to construct an operator representation of the nonlinear impulsive boundary value problem (1)–(3).

Theorem 11 *Let assumptions* (4), (9) *and* (15) *be satisfied and let* G *be given by* (16) *with* H *of* (17). *Then* $z \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ *is a fixed point of an operator* $\mathcal{F} : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ *defined by*

$$(\mathcal{F}z)(t) = \int_a^b G(t,s) f(s,z(s))\,\mathrm{d}s + \sum_{i=1}^{p} G(t,t_i) J_i(z(t_i)) + Y(t)\,[\ell(Y)]^{-1}\,c_0,$$

for $t \in [a,b]$, *if and only if* z *is a solution of problem* (1)–(3). *Moreover, the operator* $\mathcal{F}$ *is completely continuous.*

Proof. The first assertion follows directly from Lemma 10. Let us sketch the proof of complete continuity of $\mathcal{F}$. In a standard way using Arzelà–Ascoli theorem, there can be proved that an operator $\widehat{\mathcal{F}} : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{C}([a,b];\mathbb{R}^n)$ defined by

$$(\widehat{\mathcal{F}}z)(t) = \int_a^b G(t,s) f(s,z(s))\,\mathrm{d}s, \quad t \in [a,b],$$

is completely continuous. An image of an operator $\widetilde{\mathcal{F}} : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{PC}([a,b];\mathbb{R}^n)$ defined by

$$(\widetilde{\mathcal{F}}z)(t) = \sum_{i=1}^{p} G(t,t_i) J_i(z(t_i)), \quad t \in [a,b],$$

is a subset of a p–dimensional subspace in $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$. Finally, an operator $\overline{\mathcal{F}} : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{C}([a,b];\mathbb{R}^n)$ defined by

$$(\overline{\mathcal{F}}z)(t) = Y(t)\,[\ell(Y)]^{-1}\,c_0, \quad t \in [a,b],$$

is a constant mapping, therefore it is completely continuous, too. □

Remark 12 Let us note, that the operator $\mathcal{F}$ of Theorem 11 maps into $\mathbb{PC}([a,b];\mathbb{R}^n)$. According to the well–known fact that $\mathbb{PC}([a,b];\mathbb{R}^n)$ forms a Banach space, it is sufficient to consider the operator $\mathcal{F}$ on this space, only. The reason for choosing the space $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ in Theorem 11 has been explained in Remark 2.

Remark 13 The boundary condition (3) with ℓ of (9) is the most general linear condition for a function from $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$. Let us mention some common conditions and show that they are covered by ℓ:

- *Two-point boundary conditions:* Let $M, N \in \mathbb{R}^{n\times n}$ and consider

$$\ell(x) = Mx(a) + Nx(b), \quad x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n).$$

  Then ℓ has the form (6) where

$$K = M + N, \quad V(t) = N, \quad t \in [a,b].$$

  Indeed, for $x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ we have

$$\begin{aligned}\ell(x) &= (M+N)x(a) + (\mathrm{KS})\int_a^b N \,\mathrm{d}[x(t)] = (M+N)x(a) + N\,(x(b) - x(a)) \\ &= Mx(a) + Nx(a) + Nx(b) - Nx(a) = Mx(a) + Nx(b).\end{aligned}$$

- *Multi-point boundary conditions:* Let $\xi_1, \dots, \xi_m \in (a,b)$, $A_1, \dots, A_m \in \mathbb{R}^{n\times n}$ and consider

$$\ell(x) = x(b) - \sum_{i=1}^m A_i x(\xi_i), \quad x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n).$$

  Then ℓ has the form (6) where

$$K = I - \sum_{i=1}^m A_i, \quad V(t) = I - \sum_{i=1}^m A_i \chi_{[a,\xi_i)}(t), \quad t \in [a,b].$$

  Indeed, for $x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ we have

$$\begin{aligned}\ell(x) &= \left(I - \sum_{i=1}^m A_i\right) x(a) + (\mathrm{KS})\int_a^b \left(I - \sum_{i=1}^m A_i \chi_{[a,\xi_i)}(t)\right) \mathrm{d}[x(t)] \\ &= \left(I - \sum_{i=1}^m A_i\right) x(a) + x(b) - x(a) - \sum_{i=1}^m A_i\,(x(\xi_i) - x(a)) \\ &= x(b) - \sum_{i=1}^m A_i x(\xi_i).\end{aligned}$$

- *Integral conditions:* Let $H \in \mathbb{L}^1([a,b];\mathbb{R}^{n\times n})$ and consider

$$\ell(x) = x(b) - \int_a^b H(t)x(t)\,\mathrm{d}t, \quad x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n).$$

  Then ℓ has the form (6) where

$$K = I - \int_a^b H(s)\,\mathrm{d}s, \quad V(t) = I - \int_t^b H(s)\,\mathrm{d}s, \quad t \in [a,b].$$

Indeed, for $x \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$

$$\begin{aligned}\ell(x) &= \left(I - \int_a^b H(s)\,\mathrm{d}s\right) x(a) + (\mathrm{KS})\int_a^b \left(I - \int_t^b H(s)\,\mathrm{d}s\right) \mathrm{d}[x(t)] \\ &= \left(I - \int_a^b H(s)\,\mathrm{d}s\right) x(a) - (\mathrm{KS})\int_a^b \mathrm{d}\left[I - \int_t^b H(s)\,\mathrm{d}s\right] x(t) \\ &+ x(b) - \left(I - \int_a^b H(s)\,\mathrm{d}s\right) x(a) = x(b) - \int_a^b H(s)x(s)\,\mathrm{d}s.\end{aligned}$$

3 Application to n-th order differential equations

The results of Section 2 can be applied directly to the n-th order differential equation

$$\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h(t, u(t), \dots, u^{(n-1)}(t)), \tag{27}$$

subject to the impulse conditions

$$u^{(j-1)}(t_i+) - u^{(j-1)}(t_i) = J_{ij}(u(t_i), \dots, u^{(n-1)}(t_i)), \quad i = 1, \dots, p,\ j = 1, \dots, n, \tag{28}$$

and the boundary conditions

$$\ell_j(u, u', \dots, u^{(n-1)}) = c_{j0}, \quad j = 1, \dots, n. \tag{29}$$

Here we assume that

$$\left.\begin{array}{l} p, n \in \mathbb{N}, \quad a < t_1 < \ldots < t_p < b, \\ \dfrac{a_j}{a_n} \in \mathbb{L}^1([a,b];\mathbb{R}), \quad j = 0, \dots, n-1, \quad \dfrac{h(t,x)}{a_n(t)} \in \mathrm{Car}([a,b] \times \mathbb{R}^n;\mathbb{R}), \\ c_{j0} \in \mathbb{R}, J_{ij} \in \mathbb{C}(\mathbb{R}^n;\mathbb{R}), \quad i = 1, \dots, p, \quad j = 1, \dots, n, \\ \ell_j : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{R} \text{ is a linear bounded functional, } j = 1, \dots, n. \end{array}\right\} \tag{30}$$

First, we introduce a function space in which solutions of the stated problem will be considered. According to Remark 12 we restrict considerations in this section onto the space $\mathbb{PC}([a,b];\mathbb{R})$ which is more convenient for equation (27).
If $n > 1$, then by $\mathbb{PC}^{n-1}([a,b];\mathbb{R})$ ($\mathbb{APC}^{n-1}([a,b];\mathbb{R})$) we mean a set of all functions $u \in \mathbb{PC}([a,b];\mathbb{R})$ such that there exist continuous (absolutely continuous) derivatives $u', \dots, u^{(n-1)}$ on the interior of $\mathcal{J}_i$ and they are continuously extendable onto the closure of $\mathcal{J}_i$ for $i = 0, \dots, p$. For $u \in \mathbb{PC}^{n-1}([a,b];\mathbb{R})$ we define

$$u^{(k)}(a) = u^{(k)}(a+), \quad u^{(k)}(t_i) = u^{(k)}(t_i-) \quad \text{for } k = 1, \dots, n-1,\ i = 1, \dots, p,$$

i.e. $u^{(k)} \in \mathbb{PC}([a,b];\mathbb{R})$ for $k = 1, \dots, n-1$. For $n = 1$ we put $\mathbb{PC}^0([a,b];\mathbb{R}) = \mathbb{PC}([a,b];\mathbb{R})$ and $\mathbb{APC}^0([a,b];\mathbb{R}) = \mathbb{APC}([a,b];\mathbb{R})$.

Definition 14 A function $u \in \mathbb{APC}^{n-1}([a,b];\mathbb{R})$ is a solution of problem (27)–(29) if

- u satisfies the differential equation (27) for a.e. $t \in [a,b]$,
- u satisfies the impulse conditions (28) and boundary conditions (29).

Problem (27)–(29) can be transformed into problem (1)–(3) with

$$\left.\begin{aligned} &A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_0(t)}{a_n(t)} & -\frac{a_1(t)}{a_n(t)} & -\frac{a_2(t)}{a_n(t)} & \dots & -\frac{a_{n-1}(t)}{a_n(t)} \end{pmatrix}, \\ &f(t,x) = \left(0,0,\dots,0,\frac{h(t,x)}{a_n(t)}\right)^T, \quad t \in [a,b],\ x \in \mathbb{R}^n, \\ &J_i = (J_{i1},\dots,J_{in})^T, \quad i = 1,\dots,p, \\ &\ell = (\ell_1,\dots,\ell_n)^T, \quad c_0 = (c_{10},\dots,c_{n0})^T, \end{aligned}\right\} \tag{31}$$

via the classical transformation

$$z(t) = (u(t), u'(t), \dots, u^{(n-1)}(t))^T, \quad t \in [a,b]. \tag{32}$$

The assumptions (30) imply that (4) is satisfied for A, f, J_i defined in (31).

Remark 15 A function u is a solution of problem (27)–(29) if and only if z defined by (32) is a solution of (1)–(3), where data are given by (31). Since $z_1 = u$, it follows that the solution z is uniquely determined by its first component z_1.

Let us take some fundamental system of the corresponding homogeneous equation to (27), i.e. linearly independent solutions of the equation

$$\sum_{j=0}^{n} a_j(t) u^{(j)}(t) = 0, \tag{33}$$

and denote them by

$$u_{[1]}, \dots, u_{[n]}.$$

Further, denote by w the *row* vector

$$w(t) = (u_{[1]}(t), \dots, u_{[n]}(t)), \quad t \in [a,b], \tag{34}$$

and by W the Wronski matrix to equation (27)

$$W(t) = \begin{pmatrix} u_{[1]}(t) & \dots & u_{[n]}(t) \\ u'_{[1]}(t) & \dots & u'_{[n]}(t) \\ \dots & \dots & \dots \\ u_{[1]}^{(n-1)}(t) & \dots & u_{[n]}^{(n-1)}(t) \end{pmatrix}, \quad t \in [a,b]. \tag{35}$$

Since W is a fundamental matrix of system (10) with A from (31), we can use Lemma 8. Therefore, if ℓ defined by (31) with a representation by (9) is such that

$$\det \ell(W) \neq 0, \tag{36}$$

we get the Green matrix G of problem (10), (11) with A from (31). Here G has the form

$$G(t,\tau) = W(t)H(\tau) + \chi_{(\tau,b]}(t)W(t)W^{-1}(\tau), \quad t,\tau \in [a,b], \tag{37}$$

where H is defined by

$$H(\tau) = -\left[\ell(W)\right]^{-1}\left(\int_\tau^b V(s)A(s)W(s)\,\mathrm{d}s \cdot W^{-1}(\tau) + V(\tau)\right), \quad \tau \in [a,b]. \tag{38}$$

Denote

$$G = (G_{ij})_{i,j=1}^n, \quad g_j(t,\tau) = G_{1j}(t,\tau), \quad t,\tau \in [a,b], \quad j = 1,\dots,n. \tag{39}$$

Choose $\tau \in [a,b]$. Due to (35) and (37) we get

$$G_{ij}(t,\tau) = \frac{\partial^{i-1} g_j}{\partial t^{i-1}}(t,\tau), \quad t \in (a,b),\ t \neq \tau,\ i,j = 1,\dots,n.$$

In order to get needed properties of functions g_j (*cf.* Corollary 16) we extend the definition of derivatives of functions $g_j(\cdot,\tau)$ to be continuous from the left at $t = \tau$. It suffices to put

$$\frac{\partial^{i-1} g_j}{\partial t^{i-1}}(t,\tau) = G_{ij}(t,\tau), \quad t,\tau \in [a,b],\ i,j = 1,\dots,n. \tag{40}$$

With this notation, the next result is a consequence of Lemma 8.

Corollary 16 *Assume* (9) *and* (36). *Then functions $g_j = g_j(t,\tau)$, $j = 1,\dots,n$, defined by* (39) *(having derivatives in the sense of* (40)*) have the following properties:*

(i) $g_j, \dfrac{\partial g_j}{\partial t}, \dots, \dfrac{\partial^{n-1} g_j}{\partial t^{n-1}}$, $j = 1,\dots,n$, *are bounded on* $[a,b] \times [a,b]$,

(ii) $g_j(\cdot,\tau)$, $j = 1,\dots,n$, *are absolutely continuous on* $[a,\tau]$, $(\tau,b]$ *and they satisfy* (33) *a.e. on* $[a,b]$ *for each* $\tau \in [a,b]$,

(iii) *for each* $\tau \in [a,b)$

$$\frac{\partial^{i-1} g_j}{\partial t^{i-1}}(\tau+,\tau) - \frac{\partial^{i-1} g_j}{\partial t^{i-1}}(\tau,\tau) = \delta_{ij}, \quad i,j = 1,\dots,n,$$

(iv) $g_j(\cdot,\tau), \dfrac{\partial g_j}{\partial t}(\cdot,\tau), \dots, \dfrac{\partial^{n-1} g_j}{\partial t^{n-1}}(\cdot,\tau) \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R})$ *and*

$$\ell_i\left(g_j(\cdot,\tau), \frac{\partial g_j}{\partial t}(\cdot,\tau), \dots, \frac{\partial^{n-1} g_j}{\partial t^{n-1}}(\cdot,\tau)\right) = 0$$

for $i,j = 1,\dots,n$, $\tau \in [a,b)$.

We are ready to give an operator representation to problem (27)–(29).

Theorem 17 *Let* (30), (9) *and* (36) *be satisfied and* w, W *and* g_j, $j = 1, \ldots, n$, *be given in* (34), (35) *and* (39), *respectively. Then* $u \in \mathbb{PC}^{n-1}([a,b];\mathbb{R})$ *is a fixed point of an operator* $\mathcal{H} : \mathbb{PC}^{n-1}([a,b];\mathbb{R}) \to \mathbb{PC}^{n-1}([a,b];\mathbb{R})$ *defined by*

$$(\mathcal{H}u)(t) = \int_a^b \frac{g_n(t,s)}{a_n(s)} h(s, u(s), \ldots, u^{(n-1)}(s))\,\mathrm{d}s + \sum_{j=1}^{n}\sum_{i=1}^{p} g_j(t,t_i) J_{ij}(u(t_i), \ldots, u^{(n-1)}(t_i)) + w(t)\,[\ell(W)]^{-1}\,c_0,$$

$t \in [a,b]$, *if and only if* u *is a solution of problem* (27)–(29). *Moreover, the operator* $\mathcal{H}$ *is completely continuous.*

Proof. As it was mentioned in Remark 15, problem (27)–(29) can be transformed into problem (1)–(3) with (31). By (30) and Lemma 8, there exists a Green matrix G of problem (10), (11) with (31), which is in the form (37) and (38).
Let $u \in \mathbb{PC}^{n-1}([a,b];\mathbb{R})$ be a solution of (27)–(29). From Remark 15 we deduce that this is equivalent to the fact that $z \in \mathbb{PC}([a,b];\mathbb{R}^n)$ defined by (32) is a solution of problem (1)–(3) with (31). This is equivalent to the fact that z is a fixed point of the operator $\mathcal{F}$ from Theorem 11 which can be written here as

$$(\mathcal{F}z)(t) = \int_a^b G(t,s) f(s, z(s))\,\mathrm{d}s + \sum_{i=1}^{p} G(t,t_i) J_i(z(t_i)) + W(t)\,[\ell(W)]^{-1}\,c_0,$$

$t \in [a,b]$, $z \in \mathbb{PC}([a,b];\mathbb{R}^n)$, due to Remark 12. Since z is uniquely determined by its first component $z_1 = u$, we see, that $\mathcal{F}z = z$ is equivalent to $(\mathcal{F}z)_1 = z_1$, which means

$$u(t) = z_1(t) = (\mathcal{F}z)_1(t) = \int_a^b G_{1n}(t,s)\frac{h(s,z(s))}{a_n(s)}\,\mathrm{d}s + \sum_{j=1}^{n}\sum_{i=1}^{p} G_{1j}(t,t_i) J_{ij}(z(t_i)) + w(t)\,[\ell(W)]^{-1}\,c_0 = (\mathcal{H}u)(t),$$

for each $t \in [a,b]$, taking account of (31), (32) and (39). The complete continuity of $\mathcal{H}$ can be obtained from the complete continuity of $\mathcal{F}$. □

A similar result for a linear equation with two-point boundary conditions can be found in [14].

4 Fredholm-type existence theorems

Theorems 11 and 17 combined with the Schauder fixed point theorem imply the validity of existence theorems of the Fredholm type for problem (1)–(3) (Theorem 18) and for problem (27)–(29) (Theorem 19), respectively. Such theorems guarantee the solvability of a nonlinear problem provided a corresponding linear homogeneous problem has only the trivial solution and data functions in the nonlinear problem are bounded.

Theorem 18 *Let assumptions* (4)*,* (9) *and* (15) *be satisfied and let there exist* $h \in \mathbb{L}^1([a,b];\mathbb{R})$ *and* $c \in (0,\infty)$ *such that*

$$\|f(t,x)\| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}^n,$$

$$\|J_i(x)\| \le c, \quad x \in \mathbb{R}^n, \quad i = 1,\dots,p.$$

Then problem (1)–(3) *is solvable.*

Theorem 19 *Let assumptions* (30)*,* (9) *and* (36) *be satisfied and let there exists* $h \in \mathbb{L}^1([a,b];\mathbb{R})$ *and* $c \in (0,\infty)$ *such that*

$$|f(t,x)| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}^n,$$

$$|J_{ij}(x)| \le c, \quad x \in \mathbb{R}^n, \quad i = 1,\dots,p, \quad j = 1,\dots,n.$$

Then problem (27)–(29) *is solvable.*

Remark 20 Let us mention that the Fredholm-type theorems are not valid for the case with state–dependent impulses. This fact was shown in [10].

Remark 21 Combining the presented Fredholm–type theorems together with the method of a priori estimates for concrete boundary conditions we can obtain existence results for corresponding problems with unbounded data.

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