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Existence Results for Nondensely Defined Semilinear Functional Differential Inclusions in Fréchet Spaces

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Abstract

In this paper, a recent Frigon nonlinear alternative for contractive multivalued maps in Fréchet spaces, combined with semigroup theory, is used to investigate the existence of integral solutions for first order semilinear functional differential inclusions. An application to a control problem is studied. We assume that the linear part of the differential inclusion is a nondensely defined operator and satisfies the Hille-Yosida condition.

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1 Introduction

This paper is concerned with an application of a recent Frigon nonlinear alternative for contractive multivalued maps in Fréchet spaces [20] to obtain the existence of integral solutions of some classes of initial value problems for first order semilinear functional differential inclusions. In Section 3, we will consider the first order semilinear functional differential inclusion of the form,

$$y'(t) - Ay(t) \in F(t, y_t), \quad a.e.\ t \in J = [0, \infty), \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

where $r > 0$, $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ is a multivalued map with compacts value ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $\phi \in C([-r, 0], E)$, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, and E is real Banach space with norm $|\cdot|$. For any continuous function y defined on $[-r,\infty)$ and any $t\in[0,\infty)$, we denote by y_t the element of $C([-r,0],\overline{D(A)})$ defined by $y_t(\theta)=y(t+\theta),\quad \theta\in[-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t. in Section 4, we are concerned with the existence of integral solutions of the above problem subject to a control parameter. More precisely we consider the control problem,

$$y'(t)-Ay(t)\in F(t,y_t)+(Bu)(t),\quad a.e.\ t\in[0,\infty), \tag{3}$$

$$y(t)=\phi(t),\quad t\in[-r,0], \tag{4}$$

where F,A,ϕ are as in (1)-(2), B is a bounded linear operator from $\overline{D(A)}$ into $\overline{D(A)}$ and the control parameter $u(\cdot)$ belongs to $L^2(J,U)$, a space of admissible controls, and U is a Banach space.

In the case where F is either a single or a multivalued map, and A is a densely defined linear operator generating a C_0-semigroup of bounded linear operators, the problems (1)–(2) and (3)–(4) have been investigated on compact intervals in, for instance, the monographs by Ahmed [1], Hale and Lunel [21], Wu [31], Hu and Papageorgiou [22], Kamenskii, Obukhovskii and Zecca [24], and in the papers of Benchohra, Ntouyas [9, 10, 11] for the controllability of differential inclusions with different conditions; see also the monograph of Benchohra, Ntouyas, Górniewicz [13] and the papers of Balachandran and Manimegolai [6], Benchohra *et al* [7], Benchohra and Ntouyas [8], and Li and Xue [23]and the references cited therein. On infinite intervals, and still when A is a densely defined linear operator generating C_0-semigroup families of linear bounded operators and F is a single map, the problems (1)–(2), (3)–(4) were studied by Arara, Benchohra and Ouahab [2] by means of the nonlinear alternative for contraction maps in Fréchet spaces due to Frigon and Granas [19]. Other recent results, on the controllablity question for problem (3)–(4) and other classes of equations, can be found, for instance, in the survey paper by Balachandran and Dauer [5] and in the references cited therein.

Recently, the existence of integral solutions on compact intervals for the problem (1)–(2) with periodic boundary conditions in Banach space was considered by Ezzinbi and Liu [18]. For more details on nondensely defined operators and the concept of integrated semigroup we refer to the monograph [1] and to the papers [3, 12, 16, 27, 30].

Our goal here is to give existence results for the above problems. These results extend some ones existing in the previous literature in the case of densely defined linear operators.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C([-r,0],\overline{D(A)})$ is the Banach space of all continuous functions from $[-r,0]$ into $\overline{D(A)}$ with the norm

$$\|\phi\| := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

$B(E)$ is the Banach space of all linear bounded operator from $\overline{D(A)} \subset E$ into $\overline{D(A)}$ with norm

$$\|N\|_{B(E)} := \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : [0,\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [32]).
$L^1([0,\infty),\overline{D(A)})$ denotes the Banach space of functions $y : [0,\infty) \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)|dt.$$

Definition 2.1 *[3]. We say that a family $\{S(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is an integrated semigroup family if:*

(1) $S(0) = 0$;

(2) $t \to S(t)$ is strongly continuous;

(3) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \ge 0$.

Definition 2.2 *[25]. An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) and there exists a strongly continuous exponentially bounded family $(S(t))_{t\ge 0}$ of bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t}S(t)dt$ exists for all λ with $\lambda > \omega$.*

Lemma 2.1 *[3] Let A be the generator of an integrated semigroup $(S(t))_{t\ge 0}$. Then for all $x \in E$ and $t \ge 0$,*

$$\int_0^t S(s)xds \in D(A) \quad \textit{and} \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Definition 2.3 *We say that a linear operator A satisfies the "Hille-Yosida condition" if there exist $M \ge 0$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and*

$$\sup\{(\lambda-\omega)^n|(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\} \le M.$$

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [3], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0- semigroup on $\overline{D(A)}$. Here and hereafter, we assume that

(H1) A satisfies the Hille-Yosida condition.

Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. Then we have the following from [3] and [25].

Theorem 2.1 *Let $f : [0,T] \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0,T] \to E$ such that*

(i) $\int_0^t y(s)ds \in D(A), \quad t \in [0,T],$

(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s)ds, \quad t \in [0,T],$

(iii) $|y(t)| \leq Me^{\omega t}(|y_0| + \int_0^t e^{-\omega s}|f(s)|ds), \quad t \in [0,T].$

Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad t \geq 0. \tag{5}$$

Let $B_\lambda = \lambda R(\lambda, A)$, where $R(\lambda, A) := (\lambda I - A)^{-1}$, then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. As a consequence, if y satisfies (5), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds, \quad t \geq 0.$$

For properties from semigroup theory, we refer the interested reader to the books of Ahmed [1], Engel and Nagel [17] and Pazy [28].

Given a space X and metrics $d_\alpha, \alpha \in \bigwedge$ on X, define $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$. We denote by $D_\alpha,\ \alpha \in \bigwedge$, the Hausdorff pseudometric induced by d_α; that is, for $A, B \in \mathcal{P}(X)$,

$$\begin{aligned} D_\alpha(A,B) &= \inf\Big\{\varepsilon > 0 : \forall x \in A,\ \forall y \in B,\ \exists \bar{x} \in A,\ \bar{y} \in B \text{ such that} \\ &\qquad\qquad d_\alpha(x,\bar{y}) < \varepsilon, d_\alpha(\bar{x},y) < \varepsilon\Big\} \end{aligned}$$

with $\inf\emptyset = \infty$. In the particular case where X is a complete locally convex space, we say that a subset $A \subset X$ is bounded if $D_\alpha(\{0\}, A) < \infty$ for every $\alpha \in \bigwedge$.

Definition 2.4 *A multi-valued map $F : X \to \mathcal{P}(E)$ is called an admissible contraction with constant $\{k_\alpha\}_{\alpha\in\bigwedge}$ if for each $\alpha \in \bigwedge$ there exists $k_\alpha \in (0,1)$ such that*

i) $D_\alpha(F(x), F(y)) \le k_\alpha d_\alpha(x,y)$ *for all* $x, y \in X$.

ii) for every $x \in X$ *and every* $\varepsilon \in (0,\infty)^{\bigwedge}$, *there exists* $y \in F(x)$ *such that*

$$d_\alpha(x,y) \le d_\alpha(x, F(x)) + \varepsilon_\alpha \ for\ every\, \alpha \in \bigwedge.$$

Lemma 2.2 *(Nonlinear Alternative, [20]). Let E be a Fréchet space and U an open negihborhood of the origin in E, and let $N : \overline{U} \to \mathcal{P}(E)$ be an admissible multi-valued contraction. Assume that N is bounded. Then one of the following statements holds:*

(C1) N has a fixed point;

(C2) there exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.

For applications of Lemma 2.2 we consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b),\ d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space see [26]. In what follows, we will assume that the function $F : [0,\infty) \times C([-r,0],E) \to \mathcal{P}(E)$ is an L^1_{loc}- Carathéodory function, i.e.

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in C([-r,0],E)$;

(ii) $x \longmapsto F(t,x)$ is continuous for almost all $t \in [0,\infty)$;

(iii) For each $q > 0$, there exists $h_q \in L^1_{loc}([0,\infty), \mathbb{R}_+)$ such that

$$\|F(t,x)\| \le h_q(t) \quad \text{for all} \quad \|x\| \le q \quad \text{and for almost all} \quad t \in [0,\infty).$$

3 Functional Differential Inclusions

The main result of this section concerns the IVP (1)-(2). Before stating and proving this one, we give first the definition of its mild solution.

Definition 3.1 *A function $y \in C([-r,\infty),\overline{D}(A))$ is said to be an integral solution of (1)–(2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e $t \in [0,\infty)$ and*

$$y(t) = S'(t)\phi(0) + A\int_0^t y(s)ds + \int_0^t v(s)ds,$$

$\int_0^t y(s)ds \in \overline{D(A)}$, *for* $t \in [0,\infty)$, *and* $y(t) = \phi(t)$, $t \in [-r,0]$.

We assume hereafter the following hypotheses:

(H2) There exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1_{loc}([0,\infty), \mathbb{R}_+)$ such that

$$\|F(t,x)\| \leq p(t)\psi(\|x\|) \text{ for a.e. } t \in [0,\infty) \text{ and each } x \in C([-r,0],\overline{D(A)})$$

with

$$\int_1^\infty \frac{ds}{s+\psi(s)} = \infty.$$

(H3) for all $R > 0$ there exists $l_R \in L^1_{loc}([-r,\infty), \mathbb{R}_+)$ such that

$$H_d(F(t,x),F(t,\overline{x})) \leq l_R(t)\|x-\overline{x}\| \text{ for all } x,\ \overline{u} \in C([-r,0],E) \text{ with } \|x\|,\|\overline{x}\| \leq R,$$

and

$$d(0,F(t,0)) \leq l_R(t) \text{ for a.e. } t \in J.$$

For each $n \in \mathbb{N}$ we define in $C([-r,\infty),\overline{D(A)})$ the semi-norms by

$$\|y\|_n = \sup\{e^{-(\omega t+\tau L_n(t))}|y(t)| : t \leq n\},$$

where $L_n(t) = \int_0^t Ml_n(s)ds$. Then $C([-r,\infty),\overline{D(A)})$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}$. In what follows we will choose τ sufficiently large.

Theorem 3.1 *Suppose that hypotheses (H1)-(H3) are satisfied. Then problem (1)–(2) has at least one integral solution on $[-r,\infty)$.*

Proof Transform the problem (1)–(2) into a fixed point problem. Consider the operator $N : C([-r,\infty),\overline{D(A)}) \to \mathcal{P}(C([-r,\infty),\overline{D(A)}))$ defined by,

$$N(y) = \left\{ h \in C([-r,\infty),\overline{D(A)}) \mid h(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds, & t \in [0,\infty), \end{cases} \right\}$$

where

$$v \in S_{F,y} = \{u \in L^1(J,E) \mid u \in F(t,y_t)\, a.e\ t \in J\}.$$

Clearly, the fixed points of the operator N are integral solutions of the problem (1)–(2). Let y be a possible solution of the problem (1)–(2). Given $n \in \mathbb{N}$ and $t \leq n$, then $y \in N(y)$, and there exists $v \in S_{F,y}$ such that, for each $t \in [0, \infty)$, we have

$$y(t) = S'(t)\phi(0) + A \int_0^t y(s)ds + \int_0^t v(s)ds.$$

Then

$$|y(t)| \leq Me^{\omega t} \left[|\phi(0)| + \int_0^t e^{-\omega s} p(s)\psi(\|y_s\|)ds \right].$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq n.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, n]$, by the previous inequality we have for $t \in [0, n]$

$$\begin{aligned} e^{-\omega t}\mu(t) &\leq M[\|\phi\| + \int_0^t e^{-\omega s} p(s)\psi(\mu(s))ds] \\ &= M\|\phi\| + M \int_0^t e^{-\omega s} p(s)\psi(\mu(s)))ds. \end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq e^{\omega t} v(t) \text{ for all } \quad t \in [0, n],$$

and

$$v(0) = M\|\phi\|, \qquad v'(t) = e^{-\omega t} M p(t)\psi(\mu(t)).$$

Using the increasing character of ψ we get

$$\psi(\mu(t)) \leq \psi(e^{\omega t} v(t)), \quad t \in [0, n],$$

$$v'(t) \leq e^{-\omega t} M p(t)\psi(e^{\omega t} v(t)), \quad a.e. \ t \in [0, n].$$

Since

$$(e^{\omega t} v(t))' = \omega e^{\omega t} v(t) + v'(t) e^{\omega t}, \ a.e. \ t \in [0, n],$$

then for a.e. $t \in [0, n]$ we have

$$(e^{\omega t} v(t))' \leq m(t)[e^{\omega t} v(t) + \psi(e^{\omega t} v(t))] \quad \text{where } m(t) = \max(\omega, Mp(t)).$$

Thus

$$\int_{v(0)}^{e^{\omega t} v(t)} \frac{du}{u + \psi(u)} \leq \int_0^n m(s)ds < \infty.$$

Consequently, from (H2) there exists a constant d_n such that $e^{\omega t}v(t) \leq d_n,\ t \in [0,n]$, and hence $\|y\|_n \leq \max(\|\phi\|, d_n) := M_n$. Set

$$U = \{y \in C([-r,\infty), E) : \sup\{|y(t)| : t \leq n\} < M_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, U is a open subset of $C([-r,\infty), E)$. We shall show that $N : \overline{U} \to \mathcal{P}(C([-r,\infty), \overline{D(A)}))$ is a contraction and admissible operator.

First, we prove that N is a contraction; that is, there exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_n \ \text{ for each } \ y, \overline{y} \in C([-r,\infty), \overline{D(A)}).$$

Let $y, \overline{y} \in C([-r,\infty), \overline{D(A)})$ and $h \in N(y)$. Then there exists $v(t) \in F(t, y_t)$ such that for each $t \in [0,n]$

$$y(t) = S'(t)\phi(0) + A\int_0^t y(s)ds + \int_0^t v(s)ds.$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \leq l(t)\|y_t - \overline{y}_t\|.$$

Hence there is $w \in F(t, \overline{y}_t)$ such that

$$|v(t) - w| \leq l(t)\|y_t - \overline{y}_t\|, \quad t \in J.$$

Consider $U_* : [0,n] \to \mathcal{P}(E)$, given by

$$U_*(t) = \{w \in E : |v(t) - w| \leq l(t)\|y_t - \overline{y}_t\|\}.$$

Since the multi-valued operator $V_*(t) = U_*(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [14]), there exists a function $\overline{v}(t)$, which is a measurable selection for V_*. So, $\overline{v}(t) \in F(t, \overline{y}_t)$ and

$$|v(t) - \overline{v}(t)| \leq l(t)\|y_t - \overline{y}_t\|, \ \text{ for each } \ t \in [0,n].$$

Let us define for each $t \in [0,n]$

$$\overline{h}(t) = S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)\overline{v}(s)ds, \ t \in [0,n].$$

Then

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\leq \left|\tfrac{d}{dt}\int_0^t S(t-s)[v(s) - \overline{v}(s)]ds\right| \\
&\leq Me^{\omega t}\int_0^t l_n(s)e^{-\omega s}e^{\tau L_n(s)}e^{-\tau L_n(s)}\|y_s - \overline{y}_s\|ds \\
&\leq \tfrac{M}{\tau}e^{\omega t}\int_0^t (e^{\tau L_n(s)})'ds\|y - \overline{y}\|_n \\
&\leq \tfrac{1}{\tau}e^{(\omega t + \tau L_n(t))}\|y - \overline{y}\|_n.
\end{aligned}$$

Therefore,

$$\|h-\overline{h}\|_n \le \frac{1}{\tau}\|y-\overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y})) \le \frac{1}{\tau}\|y-\overline{y}\|_n.$$

So, N is a contraction. Let $y \in C([-r,\infty),\overline{D(A)})$. Consider $N : C([-r,n],\overline{D(A)}) \to \mathcal{P}_{cl}(C([-r,n],\overline{D(A)}))$, given by

$$N(y) = \left\{ h \in C([-r,n],\overline{D(A)}) : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \in [-r,0], \\ \\ S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds, & t \in [0,n], \end{array} \right. \right\}$$

where $v \in S^n_{F,y} = \{h \in L^1([0,n],\overline{D(A)}) : v \in F(t,y_t)\ a.e. t \in [0,n]\}$. From (H1)-(H3) and since F is a multi-valued map with compact values, we can prove that for every $y \in C([-r,n],\overline{D(A)})$, $N(y) \in \mathcal{P}_{cp}(C([-r,n],\overline{D(A)}))$, and there exists $y_* \in C([-r,n],E)$ such that $y_* \in N(y_*)$. Let $h \in C([-r,n],\overline{D(A)})$, $\overline{y} \in \overline{U}$ and $\varepsilon > 0$. Assume that $y_* \in N(\overline{y})$, then we have

$$\begin{array}{rcl} \|\overline{y}(t)-y_*(t)\| & \le & \|\overline{y}(t)-h(t)\| + \|y_*(t)-h(t)\| \\ \\ & \le & \|\overline{y}-N\overline{y}\|_n e^{-\omega t-\tau L_n(t)} + \|y_*(t)-h(t)\|. \end{array}$$

Since h is arbitrary we may suppose that $h \in B(y_*,\varepsilon) = \{h \in C([-r,n],\overline{D(A)}) : \|h-y_*\|_n \le \varepsilon\}$. Therefore,

$$\|\overline{y}-y_*\|_n \le \|\overline{y}-N\overline{y}\|_n + \varepsilon.$$

If $y_* \notin N(\overline{y})$, then $\|y_*-N(\overline{y})\| \neq 0$. Since $N(\overline{y})$ is compact, there exists $x \in N(\overline{y})$ such that $\|y_*-N(\overline{y})\| = \|y_*-x\|$. Then we have

$$\begin{array}{rcl} \|\overline{y}(t)-x(t)| & \le & \|\overline{y}(t)-h(t)\| + \|x(t)-h(t)\| \\ \\ & \le & \|\overline{y}-N\overline{y}\|_n e^{-\omega t-\tau L_n(t)} + \|x(t)-h(t)\|. \end{array}$$

Thus,

$$\|\overline{y}-x\|_n \le \|\overline{y}-N\overline{y}\|_n + \varepsilon.$$

So, N is an admissible operator contraction. By Lemma 2.2, N has a fixed point y, which is a integral solution to (1)–(2).

4 Controllability Functional Differential Inclusions

In this section we are concerned with the existence of integral solutions for problem (3)–(4).

Definition 4.1 *A function $y \in C([-r,\infty), \overline{D(A)})$ is said to be an integral solution of (3)–(4) if there exists $v \in S_{F,y}$ such that y is the solution of the integral equation*

$$y(t) = S'(t)\phi(0) + A\int_0^t y(s)ds + \int_0^t v(s)ds + \int_0^t (Bu)(s)ds,$$

$$\int_0^t y(s)ds \in D(A), \quad t \in [0,\infty) \quad \textit{and} \quad y(t) = \phi(t), \quad t \in [-r, 0].$$

From the definition it follows that $y(t) \in \overline{D(A)}$, $t \geq 0$. Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)v(s)ds + \frac{d}{dt}\int_0^t S(t-s)(Bu)(s)ds, \quad t \geq 0. \qquad (6)$$

Let $B_\lambda = \lambda R(\lambda, A)$. Then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. As a consequence, if y satisfies (6), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda[v(s) + (Bu)(s)]ds, \quad t \geq 0.$$

Definition 4.2 *The system (3)–(4) is said to be infinite controllable if for any continuous function ϕ on $[-r, 0]$ and any $x_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0,n], U)$ such that the integral solution y of (3) satisfies $y(n) = x_1$.*

Let us introduce the following hypotheses:

(A1) For each $n > 0$ the linear operator $W : L^2([0,n], U) \to E$ defined by

$$Wu = \int_0^n S'(n-s)Bu(s)ds,$$

has an invertible operator W^{-1} which takes values in $L^2([0,n], U)\backslash KerW$ and there exist positive constants $M_1,\ M_2$ such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$,

Remark 4.1 *The question of the existence of the operator W and its inverse is discussed in the paper by Quinn and Carmichael [29].*

Theorem 4.1 *Assume that hypotheses (H1)-(H3) and (A1) hold. Then the problem (3)–(4) is controllable.*

Proof. Using hypothesis (A3) for each $y(\cdot)$ and $v \in S_{F,y}$, and for each $n \in \mathbb{N}$, define the control

$$u_y^n(t) = W^{-1}\left[x_1 - S'(n)\phi(0) - \lim_{\lambda \to +\infty} \int_0^n S'(n-s)B_\lambda v(s)ds\right](t).$$

Consider the operator $N_1 : C([-r,\infty), \overline{D(A)}) \longrightarrow \mathcal{P}(C([-r,\infty), \overline{D(A)}))$ defined by:

$$N_1(y)\left\{h \in C([-r,\infty), \overline{D(A)})/h(t) = \left\{\begin{array}{ll} \phi(t), & t \in [-r,0], \\ S'(t)\phi(0) + \frac{d}{dt}\displaystyle\int_0^t S(t-s)v(s)ds & \\ +\frac{d}{dt}\displaystyle\int_0^t S(t-s)Bu_y^n(s)ds, & t \in [0,\infty), \end{array}\right.\right\}$$

where $v \in S_{F,y}$. It is clear that the fixed points of N_1 are integral solutions to problem (3)–(4).

Let $y \in C([-r,\infty), \overline{D(A)})$ be a possible solution of the problem (3)–(4). Then there exists $v \in S_{F,y}$ such that for each $t \in [0,\infty)$,

$$y(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds + \frac{d}{dt}\int_0^t S(t-s)Bu_y^n(s)ds.$$

This implies by (H2) and (A1) that, for each $t \in [0,n]$, we have

$$\begin{aligned} |y(t)| &\leq Me^{\omega t}\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds + Me^{\omega t}\int_0^t e^{-\omega s}|(Bu_y^n)(s)|ds \\ &\leq Me^{\omega t}\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds \\ &\quad +Me^{\omega t}M_1M_2n\left(|x_1| + Me^{\omega n}\|\phi\| + Me^{\omega n}\int_0^n e^{-\omega s}p(s)\psi(\|y_s\|)ds\right). \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq n.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,n]$, by the previous inequality, we have for $t \in [0,n]$,

$$\begin{aligned} e^{-\omega t}\mu(t) &\leq M\|\phi\| + M\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)ds \\ &\quad +MM_1M_2n\left(|x_1| + Me^{\omega n}\|\phi\| + Me^{\omega n}\int_0^n e^{-\omega s}p(s)\psi(\|y_s\|)ds\right). \end{aligned}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq e^{\omega t}v(t) \text{ for all } \quad t \in [0,n],$$

$$v(0) = M\|\phi\| + nMM_1M_2\left(|x_1| + Me^{\omega n}\|\phi\| + Me^{\omega n}\int_0^n e^{-\omega s}p(s)\psi(\mu(s))ds\right),$$

and

$$v'(t) = Me^{-\omega t}p(t)\psi(\mu(t)), \;\; a.e. \; t \in [0,n].$$

Using the increasing character of ψ we get

$$v'(t) \le Me^{-\omega t}p(t)\psi(e^{\omega t}v(t)) \;\; \text{a.e.} \quad t \in [0,n].$$

Then for each $t \in [0,n]$ we have

$$\begin{aligned}(e^{\omega t}v(t))' &= \omega e^{\omega t}v(t) + v'(t)e^{\omega t}\\ &\le \omega e^{\omega t}v(t) + Mp(t)\psi(e^{\omega t}v(t))\\ &\le m_1(t)[e^{\omega t}v(t) + \psi(e^{\omega t}v(t))], \;\; t \in [0,n].\end{aligned}$$

Thus

$$\int_{v(0)}^{e^{\omega t}v(t)} \frac{du}{u + \psi(u)} \le \int_0^n m_1(s)ds < +\infty.$$

Consequently, by (H2), there exists a constant $\overline{d}_n$ such that $e^{\omega t}v(t) \le \overline{d}_n, \; t \in [0,n]$, and hence $\|y\|_n \le \max(\|\phi\|, \overline{d}_n) := K_n$. Set

$$U_1 = \{y \in C([-r,\infty), \overline{D(A)}) : \sup\{|y(t)| : t \le n\} < K_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, we define in $C([0,\infty), \overline{D(A)})$ the semi-norms by

$$\|y\|_n = \sup\{e^{-(\omega t + \tau \overline{L}_n(t))}|y(t)| : t \le n\},$$

where $\overline{L}_n(t) = \int_0^t \bar{l}_n(s)ds$ and $\bar{l}_n(t) = \max(Ml_n(t), M_1M^2M_2e^{\omega n}\|l_n\|_{L^1([0,n])})$.

We shall show that the operator N_1 is a contraction and admissible operator.

First, we prove that N_1 is contraction. Indeed, consider $y, \overline{y} \in C([-r,\infty), \overline{D(A)})$. Thus for each $t \in [0,n]$ and $n \in \mathbb{N}$, and $h \in N(y)$, there exists $v(t) \in F(t, y_t)$ such that for each $t \in [0,n]$,

$$h(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds + \frac{d}{dt}\int_0^t S(t-s)Bu_y^n(s)ds.$$

From (H2) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \le l(t)\|y_t - \overline{y}_t\|.$$

Hence there is a $w \in F(t, \overline{y}_t)$ such that

$$|v(t) - w| \le l(t)\|y_t - \overline{y}_t\|, \quad t \in J.$$

Consider $U_*: [0,n] \to \mathcal{P}(E)$, given by

$$U_*(t) = \{w \in E : |v(t) - w| \leq l(t)\|y_t - \overline{y}_t\|\}.$$

Since the multi-valued operator $V_*(t) = U_*(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition III.4 in [14]), there exists a function $\overline{v}(t)$, which is a measurable selection for V_*. So, $\overline{v}(t) \in F(t, \overline{y}_t)$, and

$$|v(t) - \overline{v}(t)| \leq l(t)\|y_t - \overline{y}_t\|, \quad \text{for each} \quad t \in [0,n].$$

Let us define for each $t \in [0,n]$,

$$\overline{h}(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)\overline{v}(ss)ds + \frac{d}{dt}\int_0^t S(t-s)Bu^n_{\overline{y}}(s)ds.$$

Then

$$\begin{aligned}
|h(t) - \overline{h}(t)| &= \Big|\tfrac{d}{dt}\int_0^t S'(t-s)[(Bu^n_y)(s) - (Bu^n_{\overline{y}})(s)]ds \\
&\quad + \tfrac{d}{dt}\int_0^t S'(t-s)[v(s) - \overline{v}(s)]ds\Big| \\
&\leq Me^{\omega t}\int_0^t e^{-\omega s}\|B\||u^n_y(s) - u^n_{\overline{y}}(s)|ds \\
&\quad + Me^{\omega t}\int_0^t l_n(s)e^{-\omega s}\|y_s - \overline{y}_s\|ds \\
&\leq e^{\omega t}\int_0^t l_n(s)\|y_s - \overline{y}_s\|ds + Me^{\omega t}M_1\int_0^t e^{-\omega s}|W^{-1}[x_1 - S'(n)\phi(0) \\
&\quad - \lim_{\lambda\to+\infty}\int_0^n S'(n-s)B_\lambda f(r, y_r)drds] - W^{-1}[x_1 - S'(n)\phi(0) \\
&\quad - \lim_{\lambda\to+\infty}\int_0^n S'(n-s)B_\lambda f(r, \overline{y}_r)drds| \\
&\leq e^{\omega t}\int_0^t e^{-\omega s}l_n(s)\|y_s - \overline{y}_s\|ds \\
&\quad + M_1M^2M_2e^{\omega n}e^{\omega t}\int_0^t e^{-\omega s}\int_0^n |f(r, y_r) - f(r, \overline{y}_r)|drds \\
&\leq e^{\omega t}\int_0^t l_n(s)e^{-\omega s}\|y_s - \overline{y}_s\|ds \\
&\quad + M_1M^2M_2ne^{\omega n}e^{\omega t}\int_0^t e^{-\omega s}\int_0^t l_n(s)\|y_s - \overline{y}_s\|ds \\
&\leq 2e^{\omega t}\int_0^t e^{-\omega s}\overline{l}_n(s)\|y_s - \overline{y}_s\|ds \\
&\leq 2e^{\omega t}\int_0^t l_n(s)e^{\tau\overline{L}_n(s)}e^{-(\omega s + \tau\overline{L}_n(s))}\|y_s - \overline{y}_s\|ds \\
&\leq 2e^{\omega t}\int_0^t (e^{\tau\overline{L}_n(s)})'ds\|y - \overline{y}\|_n \\
&\leq \frac{2}{\tau}e^{(\omega t + \tau\overline{L}_n(t))}\|y - \overline{y}\|_n.
\end{aligned}$$

Therefore,

$$\|h-\overline{h}\|_n \leq \frac{2}{\tau}\|y-\overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N_1(y), N_1(\overline{y})) \leq \frac{2}{\tau}\|y-\overline{y}\|_n.$$

So, N_1 is a contraction, and as in Theorem 3.1, we can prove that N_1 is an admissible multivalued map. From the choice of U_1 there is no $y \in \partial U_1$ such that $y \in \lambda N_1(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative [20] we deduce that N_1 has at least one fixed point which is a integral solution to (3)–(4).

5 An Example

As an application of our results we consider the following partial neutral functional differential inclusion,

$$\frac{\partial z(t,x)}{\partial t} - \Delta z(t,x) \in Q(t, z(t-r,x)),\ t \in [0,\infty),\ 0 \leq x \leq \pi, \tag{7}$$

$$z(t,0) = z(t,\pi),\ t \in [0,\infty), \tag{8}$$

$$z(t,x) = \phi(t,x), \quad -r \leq t \leq 0, \quad 0 \leq x \leq \pi, \tag{9}$$

where $r > 0$, $\phi \in C([-r,0]\times[0,\pi], \mathbb{R})$, $Q : [0,\infty)\times[0,\pi] \to \mathcal{P}(\mathbb{R})$, is a multivalued map with compacts values, and there exist constants $k_p > 0$ such that

$$H_d(Q(t,x), Q(t,y)) \leq k_q\|x-y\| \text{ for all } x,y \in [0,\pi], \quad t \in [0,\infty),$$

and

$$H_d(0, Q(t,0)) \leq k_q.$$

Consider $E = C([0,\pi], \mathbb{R})$ the Banach space of continuous functions on $[0,\pi]$ with values in $\mathbb{R}$. Define the linear operator A in E by $Az = \Delta z$, in

$$D(A) = \{z \in C([0,\pi], \mathbb{R}) : z(0) = z(\pi) = 0,\ \Delta z \in C([0,\pi], \mathbb{R})\},$$

where Δ is the Laplacian operator in the sense of distributions on $[0,\pi]$. Now we have

$$\overline{D(A)} = C_0([0,\pi], \mathbb{R}) = \{z \in C([0,\pi], \mathbb{R}) : z(0) = z(\pi) = 0\}.$$

It is well known from [16] that Δ satisfies the following properties:

(i) $(0,\infty) \subset \rho(\Delta)$,

(ii) $\|R(\lambda, \Delta)\| \le \dfrac{1}{\lambda}$, for some $\lambda > 0$.

It follows that Δ satisfies (H1) and hence it generates an integrated semigroup $(S(t))_t$, $t \ge 0$ and that $|S'(t)| \le e^{-\mu t}$, for $t \ge 0$ and some constant $\mu > 0$. Let

$$F(t, w_t)(x) = Q(t, w(t - x)), \ 0 \le x \le \pi.$$

Then problem (7)-(9) takes the abstract form (1)-(2). We can easily see that all hypotheses of Theorem 3.1 are satisfied. Hence from Theorem 3.1 the problem (7)-(9) has at last on integral solution on $[-r, \infty)$.

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