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OSCILLATION AND NONOSCILLATION OF PERTURBED HIGHER ORDER EULER-TYPE DIFFERENTIAL EQUATIONS

SIMONA FIŠNAROVÁ

Abstract. Oscillatory properties of even order self-adjoint linear differential equations in the form

$$\sum_{k=0}^{n}(-1)^k\nu_k\left(\frac{y^{(k)}}{t^{2n-2k-\alpha}}\right)^{(k)}=(-1)^m\left(q_m(t)y^{(m)}\right)^{(m)},\ \nu_n:=1,$$

where $m\in\{0,1\}$, $\alpha\notin\{1,3,\dots,2n-1\}$ and $\nu_0,\dots,\nu_{n-1}$, are real constants satisfying certain conditions, are investigated. In particular, the case when $q_m(t)=\frac{\beta}{t^{2n-2m-\alpha}\ln^2 t}$ is studied.

1. Introduction

The aim of this paper is to investigate oscillatory behavior of the even order self adjoint differential equation

$$L_\nu(y)=(-1)^m\left(q_m(t)y^{(m)}\right)^{(m)},\quad m\in\{0,1,\dots,n-1\},\tag{1}$$

where q_m is a continuous function and L_ν is the Euler differential operator

$$L_\nu(y):=\sum_{k=0}^{n}(-1)^k\nu_k\left(\frac{y^{(k)}}{t^{2n-2k-\alpha}}\right)^{(k)},\quad \alpha\notin\{1,3,\dots,2n-1\},\ \nu_n:=1.$$

Moreover, we suppose that $\nu_0,\dots,\nu_{n-1}$ are real constants such that the characteristic polynomial of the Euler equation

$$L_\nu(y)=0,\tag{2}$$

i.e., the polynomial

$$P(\lambda):=\sum_{k=0}^{n}(-1)^k\nu_k\prod_{j=1}^{k}(\lambda-j+1)(\lambda-2n+j+\alpha)\tag{3}$$

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has a double root at $\frac{2n-1-\alpha}{2}$ and $2n-2$ distinct real roots. This is equivalent to the following two conditions.

$$\text{(4)} \qquad \begin{cases} \textit{The polynomial} \\ \sum_{k=0}^{n}(-1)^k \nu_k \prod_{j=1}^{k}\left(z - \frac{(2n+1-2j-\alpha)^2}{4}\right) \\ \textit{has } n-1 \textit{ distinct positive roots}, \end{cases}$$

and

$$\text{(5)} \qquad \sum_{k=0}^{n} \frac{\nu_k}{4^k} \prod_{j=1}^{k} (2n+1-2j-\alpha)^2 = 0.$$

We use the usual convention that the product $\prod_{j=1}^{k}$ equals 1 when $k < j$.

Note that the assumptions imposed on $\nu_0, \dots, \nu_{n-1}$ mean that (2) is nonoscillatory, since it has the so-called ordered system of solutions (a fundamental system of positive solutions $y_1, \dots, y_{2n}$ satisfying $y_i = o(y_{i+1})$ as $t \to \infty$, $i = 1, \dots, 2n-1$)

$$\text{(6)} \qquad \begin{aligned} &y_1 = t^{\alpha_1}, \dots, y_{n-1} = t^{\alpha_{n-1}}, y_n = t^{\alpha_0} = t^{\frac{2n-1-\alpha}{2}}, \\ &y_{n+1} = t^{\alpha_0} \ln t = t^{\frac{2n-1-\alpha}{2}} \ln t, y_{n+2} = t^{2n-1-\alpha-\alpha_{n-1}}, \dots, y_{2n} = t^{2n-1-\alpha-\alpha_1}, \end{aligned}$$

where $\alpha_1, \dots, \alpha_{n-1}, \alpha_0, \alpha_{n+1}, \dots, \alpha_{2n}$ are the roots of (3), ordered by size.

Note also that the problem of (non)oscillation of Euler differential equation (2) is treated in [15, §30, §40]. It is known that (2) is nonoscillatory if and only if its coefficients $\nu_0, \dots, \nu_{n-1}$ belong to a certain closed convex subset $\mathcal{R}_\nu$ of $\mathbb{R}^n$ which can be described using a transformation which converts (2) into an equation with constant coefficients. For example, if $n = 2$ and $\alpha = 0$, then $\mathcal{R}_\nu$ is the set of coefficients ν_0, ν_1 satisfying $\nu_0 \geq -\frac{9}{4}\nu_1 - \frac{9}{16}$ for $\nu_1 \geq -\frac{5}{2}$ and $\nu_0 \geq \frac{1}{4}(2-\nu_1)^2$ for $\nu_1 \leq -\frac{5}{2}$ and (4)–(5) mean that we consider the linear part of the boundary $\nu_0 + \frac{9}{4}\nu_1 + \frac{9}{16} = 0$, $\nu_1 > -\frac{5}{2}$. Our restriction (4)–(5) on the coefficients $\nu_0, \dots, \nu_{n-1}$ is forced by the method used in this paper, which is based on Polya's factorization of the operator L_ν. It is an open problem as how to investigate oscillatory properties of perturbed Euler operators when assumptions (4)–(5) are not satisfied; this problem is a subject of the present investigation. We refer to [15] for a more detailed treatment of the (non)oscillation of (2).

As a consequence of the presented oscillation and nonoscillation criteria, we deal with the oscillation of (1) in case $q_m(t) = \frac{\beta}{t^{2n-2m-\alpha}\ln^2 t}$, $m \in \{0, 1\}$, $\beta \in \mathbb{R}$. In particular, we show that, under the conditions (4) and (5), equation(1) is nonoscillatory if and only if $\beta \leq \tilde{\nu}_{n,\alpha}$ in case $m = 0$; it is oscillatory if $\beta > \frac{16\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$ and nonoscillatory if $\beta < \frac{4\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$ in case $m = 1$, where

$$(7) \qquad \tilde{\nu}_{n,\alpha} := \sum_{l=1}^{n} \left\{ \frac{\nu_{n+1-l}}{4^{n+1-l}} \prod_{k=l}^{n} (2k-1-\alpha)^2 \sum_{k=l}^{n} \frac{1}{(2k-1-\alpha)^2} \right\}.$$

This paper can be regarded as a continuation of some recent papers where the two-term differential equation in the form

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = p(t)y, \quad \alpha \notin \{1, 3, \ldots, 2n-1\},$$

has been investigated, see [4, 5, 6, 8, 10, 11, 12, 13, 14, 16]. Namely, we extend the results of [8] and also of [7] dealing with (1) in the case where $n = 2$ and $\alpha = 0$.

Similarly, as in the above mentioned papers, we use the methods based on the factorization of disconjugate operators, variation techniques, and the relationship between self-adjoint equations and linear Hamiltonian systems.

The paper is organized as follows. In the next sections we recall necessary definitions and some preliminary results. Our main results, the oscillation and nonoscillation criteria for (1), are contained in Section 3 and Section 4. In the last section we formulate some technical results needed in the proofs.

2. Preliminaries

Here, we present some basic results which we will apply in the next section. We will need a statement concerning factorization of formally self-adjoint differential operators. Consider the equation

$$(8) \qquad L(y) := \sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = 0, \quad r_n(t) > 0.$$

Lemma 1. ([1]) *Suppose that equation* (8) *possesses a system of positive solutions* $y_1, \ldots, y_{2n}$ *such that Wronskians* $W(y_1, \ldots, y_k) \neq 0$, $k = 1, \ldots, 2n$, *for large* t. *Then the operator* L *(given by the left-hand side of* (8)*) admits the factorization for large* t

$$L(y) = \frac{(-1)^n}{a_0(t)} \left(\frac{1}{a_1(t)} \left(\cdots \frac{r_n(t)}{a_n^2(t)} \left(\frac{1}{a_{n-1}(t)} \cdots \frac{1}{a_1(t)} \left(\frac{y}{a_0(t)} \right)' \cdots \right)' \cdots \right)' \right)',$$

where

$$a_0 = y_1, \; a_1 = \left(\frac{y_2}{y_1}\right)', \; a_i = \frac{W(y_1, \ldots, y_{i+1}) W(y_1, \ldots, y_{i-1})}{W^2(y_1, \ldots, y_i)}, \; i = 1, \ldots, n-1,$$

and $a_n = \frac{1}{a_0 \cdots a_{n-1}}$.

Using the previous result we can factor the differential operator L_ν. The proof of the following statement is almost the same as that of [8, Lemma 2.2].

Lemma 2. *Let $\alpha \neq \{1, 3, \dots, 2n-1\}$ and suppose that (4) and (5) hold. Then we have, for any sufficiently smooth function y,*

$$(9) \quad L_\nu(y) = \frac{(-1)^n}{a_0(t)} \left(\frac{1}{a_1(t)} \left(\cdots \frac{t^\alpha}{a_n^2(t)} \left(\frac{1}{a_{n-1}(t)} \cdots \frac{1}{a_1(t)} \left(\frac{y}{a_0(t)} \right)' \cdots \right)' \cdots \right)' \right)',$$

where

$$a_0(t) = t^{\alpha_0}, \; a_k(t) = t^{\alpha_k - \alpha_{k-1} - 1}, \; k = 1, \dots, n-1, \; a_n(t) = t^{(n-1) - \alpha_{n-1}},$$

with $\alpha_0 = \frac{2n-1-\alpha}{2}$ and $\alpha_1 < \cdots < \alpha_{n-1}$ the first roots (ordered by their size) of the polynomial $P(\lambda)$ given by (3).

Now we recall basic oscillatory properties of self-adjoint differential equations (8). These properties can be investigated within the scope of the oscillation theory of linear Hamiltonian systems (LHS)

$$(10) \qquad x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u,$$

where A, B, C are $n \times n$ matrices with B, C symmetric. Indeed, if y is a solution of (8) and we set

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \cdots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix};$$

then (x, u) solves (10) with A, B, C given by

$$B(t) = \operatorname{diag}\{0, \dots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \dots, r_{n-1}(t)\},$$

$$A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1, \; i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

In this case we say that the solution (x, u) of (10) *is generated* by the solution y of (8). Moreover, if $y_1, \dots, y_n$ are solutions of (8) and the columns of the matrix solution (X, U) of (10) are generated by the solutions $y_1, \dots, y_n$, we say that the solution (X, U) *is generated by the solutions* $y_1, \dots, y_n$.

Recall that two different points t_1, t_2 are said to be *conjugate* relative to system (10) if there exists a nontrivial solution (x, u) of this system such that $x(t_1) = 0 = x(t_2)$. Consequently, by the above mentioned relationship between (8) and (10), these points are conjugate relative to (8) if there exists a nontrivial solution y of this equation such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, 1, \dots, n-1$. System (10) (and hence also equation (8)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exists a pair of points

$t_1, t_2 \in [T, \infty)$ which are conjugate relative to (10) (relative to (8)), in the opposite case (10) (or (8)) is said to be *nonoscillatory.*

We say that a conjoined basis (X, U) of (10) (i.e., a matrix solution of this system with $n \times n$ matrices X, U satisfying $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank}(X^T, U^T)^T = n$) is the *principal solution* of (10) if $X(t)$ is nonsingular for large t and for any other conjoined basis $(\bar{X}, \bar{U})$ such that the (constant) matrix $X^T\bar{U} - U^T\bar{X}$ is nonsingular, $\lim_{t\to\infty} \bar{X}^{-1}(t)X(t) = 0$ holds. The last limit equals zero if and only if

$$\lim_{t\to\infty} \left(\int^t X^{-1}(s)B(s)X^{T-1}(s)\, ds \right)^{-1} = 0 \tag{11}$$

([17]). A principal solution of (10) is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix. If (X, U) is the principal solution, any conjoined basis $(\bar{X}, \bar{U})$ such that the matrix $X^T\bar{U} - U^T\bar{X}$ is nonsingular is said to be a *nonprincipal solution* of (10). Solutions $y_1, \dots, y_n$ of (8) are said to form the *principal (nonprincipal) system of solutions* if the solution (X, U) of the associated linear Hamiltonian system generated by $y_1, \dots, y_n$ is a principal (nonprincipal) solution. Note that if (8) possesses a fundamental system of positive solutions $y_1, \dots, y_{2n}$ satisfying $y_i = o(y_{i+1})$ as $t \to \infty$, $i = 1, \dots, 2n - 1$, (the so-called *ordered system* of solutions), then the "small" solutions $y_1, \dots, y_n$ form the principal system of solutions of (8).

Using the relation between (8), (10) and the so-called Roundabout Theorem for linear Hamiltonian systems (see e.g. [17]), one can easily prove the following variational lemma.

Lemma 3. ([15]) *Equation* (8) *is nonoscillatory if and only if there exists* $T \in \mathbb{R}$ *such that*

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \left[\sum_{k=0}^n r_k(t)(y^{(k)}(t))^2 \right] dt > 0$$

for any nontrivial $y \in W^{n,2}(T, \infty)$ *with a compact support in* (T, ∞)*.*

We will also need the following Wirtinger-type inequality.

Lemma 4. ([15]) *Let* $y \in W^{1,2}(T, \infty)$ *have a compact support in* (T, ∞) *and let* M *be a positive differentiable function such that* $M'(t) \neq 0$ *for* $t \in [T, \infty)$*. Then*

$$\int_T^\infty |M'(t)| y^2\, dt \le 4 \int_T^\infty \frac{M^2(t)}{|M'(t)|} y'^2\, dt.$$

The following statement can be proved using repeated integration by parts, similarly as in [6, Lemma 4].

Lemma 5. *Let $y \in W_0^{n,2}(T,\infty)$ have a compact support in (T,∞) and (4)–(5) hold. Then*

$$\int_T^\infty \left[t^\alpha \left(y^{(n)}\right)^2 + \frac{\nu_{n-1}}{t^{2-\alpha}} \left(y^{(n-1)}\right)^2 + \cdots + \frac{\nu_1}{t^{2n-2-\alpha}} (y')^2 + \frac{\nu_0}{t^{2n-\alpha}} y^2\right] dt$$

$$= \int_T^\infty \frac{t^\alpha}{a_n} \left\{ \left[\frac{1}{a_{n-1}} \left(\frac{1}{a_{n-2}} \left(\cdots \frac{1}{a_1} \left(\frac{y}{a_0} \right)' \right)' \cdots \right)' \right]' \right\}^2 dt,$$

where $a_0, \dots, a_n$ are given in Lemma 2.

We finish this section with one general oscillation criterion based on the concept of principal solutions. The proof of this statement can be found in [2]. Let us consider the equation

$$L(y) = M(y), \tag{12}$$

where

$$M(y) = \sum_{k=0}^{m} (-1)^k (\tilde{r}_k(t) y^{(k)})^{(k)}, \quad m \in \{1, \dots, n-1\}$$

and $\tilde{r}_j(t) \geq 0$ for large t.

Proposition 1. *Suppose (8) is nonoscillatory and $y_1, \dots, y_n$ is the principal system of solutions of this equation. Equation (12) is oscillatory if there exists $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ such that*

$$\limsup_{t\to\infty} \frac{\int_t^\infty [\sum_{k=0}^m \tilde{r}_k(t)(h^{(k)}(s))^2] ds}{c^T \left(\int^t X^{-1}(s) B(s) X^{T^{-1}}(s) ds\right)^{-1} c} > 1, \quad h := c_1 y_1 + \cdots + c_n y_n,$$

where (X, U) is the solution of the linear Hamiltonian system associated with (8) generated by $y_1, \dots, y_n$.

3. Oscillation and nonoscillation criteria for (1) in case $m = 0$

In this section, we deal with (1) in the case $m = 0$, i.e., with the equation

$$L_\nu(y) = q_0(t) y. \tag{13}$$

We start with a nonoscillation criterion for (13).

Theorem 1. *Suppose that (4)–(5) hold and $\tilde{\nu}_{n,\alpha}$ is given by (7). If the second order equation*

$$(tu')' + \frac{1}{4\tilde{\nu}_{n,\alpha}} t^{2n-1-\alpha} q_0(t) u = 0, \tag{14}$$

is nonoscillatory, then (13) *is also nonoscillatory.*

Proof. Let $T \in \mathbb{R}$ be such that the statement of Lemma 3 holds for (14) and let $y \in W^{n,2}(T,\infty)$ be any function with compact support in (T,∞). Using Lemma 5, Wirtinger's inequality (Lemma 4), which we apply $(n-1)$-times, and Lemma 6 from the last section, we obtain

$$\begin{aligned}
&\int_T^\infty \left[t^\alpha \left(y^{(n)}\right)^2 + \frac{\nu_{n-1}}{t^{2-\alpha}}\left(y^{(n-1)}\right)^2 + \cdots + \frac{\nu_1}{t^{2n-2-\alpha}}\left(y'\right)^2 + \frac{\nu_0}{t^{2n-\alpha}} y^2\right] dt \\
&\geq \prod_{k=1}^{n-1}\left(\frac{2n-1-\alpha}{2} - \alpha_k\right)^2 \int_T^\infty t \left[\left(\frac{y}{a_0}\right)'\right]^2 dt \\
&= 4\tilde{\nu}_{n,\alpha} \int_T^\infty t \left[\left(\frac{y}{t^{\frac{2n-1-\alpha}{2}}}\right)'\right]^2 dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\int_T^\infty \left[t^\alpha \left(y^{(n)}\right)^2 + \frac{\nu_{n-1}}{t^{2-\alpha}}\left(y^{(n-1)}\right)^2 + \cdots + \frac{\nu_1}{t^{2n-2-\alpha}}\left(y'\right)^2 + \frac{\nu_0}{t^{2n-\alpha}} y^2 - q_0(t) y^2\right] dt \\
&\geq 4\tilde{\nu}_{n,\alpha} \int_T^\infty \left\{ t \left[\left(\frac{y}{t^{\frac{2n-1-\alpha}{2}}}\right)'\right]^2 - \frac{1}{4\tilde{\nu}_{n,\alpha}} q_0(t) y^2 \right\} dt \\
&= 4\tilde{\nu}_{n,\alpha} \int_T^\infty \left\{ t \left[\left(\frac{y}{t^{\frac{2n-1-\alpha}{2}}}\right)'\right]^2 - \frac{1}{4\tilde{\nu}_{n,\alpha}} t^{2n-1-\alpha} q_0(t) \left(\frac{y}{t^{\frac{2n-1-\alpha}{2}}}\right)^2 \right\} dt > 0,
\end{aligned}$$

according to Lemma 3, since (14) is nonoscillatory (take $u = y/t^{\frac{2n-1-\alpha}{2}}$) and consequently, nonoscillation of (13) follows from this Lemma as well. ☐

Theorem 2. *Let $q_0(t) \geq 0$ for large t, $\tilde{\nu}_{n,\alpha}$ is the constant given by* (7)*, conditions* (4)*–*(5) *hold, and*

$$\int^\infty \left(q_0(t) - \frac{\tilde{\nu}_{n,\alpha}}{t^{2n-\alpha}\ln^2 t}\right) t^{2n-1-\alpha} \ln t\, dt = \infty. \tag{15}$$

Then (13) *is oscillatory.*

Proof. Let $T \in \mathbb{R}$ be arbitrary, $T < t_0 < t_1 < t_2 < t_3$ (these values will be specified later). We show that for t_2, t_3 sufficiently large, there exists a function $0 \not\equiv y \in W^{n,2}(T,\infty)$ with compact support in (T,∞) and such that

$$\begin{aligned}
&\mathcal{F}(y;T,\infty) \\
&:= \int_T^\infty \left[t^\alpha \left(y^{(n)}\right)^2 + \frac{\nu_{n-1}}{t^{2-\alpha}}\left(y^{(n-1)}\right)^2 + \cdots + \frac{\nu_1}{t^{2n-2-\alpha}}\left(y'\right)^2 + \frac{\nu_0}{t^{2n-\alpha}} y^2 - q_0(t) y^2\right] dt \leq 0
\end{aligned}$$

and then, nonoscillation of (1) will be a consequence of Lemma 3. We construct the function y as follows:

$$y(t)=\begin{cases}0, & t\le t_0,\\ f(t), & t_0\le t\le t_1,\\ h(t), & t_1\le t\le t_2,\\ g(t), & t_2\le t\le t_3,\\ 0, & t\ge t_3,\end{cases}$$

where

$$h(t)=t^{\frac{2n-1-\alpha}{2}}\sqrt{\ln t},$$

$f\in C^n[t_0,t_1]$ is any function such that

$$f^{(j)}(t_0)=0,\ f^{(j)}(t_1)=h^{(j)}(t_1),\ j=0,\dots,n-1$$

and g is the solution of (2) satisfying the boundary conditions

$$g^{(j)}(t_2)=h^{(j)}(t_2),\ g^{(j)}(t_3)=0,\ j=0,\dots,n-1. \tag{16}$$

Denote

$$K:=\int_{t_0}^{t_1}\left[t^\alpha\left(f^{(n)}\right)^2+\frac{\nu_{n-1}}{t^{2-\alpha}}\left(f^{(n-1)}\right)^2+\cdots+\frac{\nu_1}{t^{2n-2-\alpha}}\left(f'\right)^2+\frac{\nu_0}{t^{2n-\alpha}}f^2-q_0(t)f^2\right]dt.$$

By a direct computation and using Lemma 7, we have for $k=1,\dots,n$,

$$h^{(k)}(t)=t^{\frac{2n-1-\alpha}{2}-k}\left[\frac{1}{2^k}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\sqrt{\ln t}+\frac{A_k}{\sqrt{\ln t}}+\frac{B_k}{\sqrt{\ln^3 t}}+o(\ln^{-\frac{3}{2}}t)\right]$$

as $t\to\infty$, where

$$A_k = \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\frac{a_j}{2^{k-j}}\prod_{l=n-k+j+1}^{n}(2l-1-\alpha), \tag{17}$$

$$B_k = \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\frac{b_j}{2^{k-j}}\prod_{l=n-k+j+1}^{n}(2l-1-\alpha). \tag{18}$$

Consequently,

$$\begin{aligned}\left(h^{(k)}\right)^2 &= t^{2n-2k-1-\alpha}\left[\frac{\ln t}{4^k}\prod_{l=n-k+1}^{n}(2l-1-\alpha)^2+\frac{A_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\right.\\ &\quad\left.+\frac{B_k}{2^{k-1}\ln t}\prod_{l=n-k+1}^{n}(2l-1-\alpha)+\frac{A_k^2}{\ln t}+2\frac{A_kB_k}{\ln^2 t}+\frac{B_k^2}{\ln^3 t}+O\left(\ln^{-3}t\right)\right]\end{aligned}$$

as $t\to\infty$.

Since

$$\int \frac{\left(h^{(k)}\right)^2}{t^{2n-2k-\alpha}}dt \;=\; \frac{1}{4^k}\prod_{l=n-k+1}^{n}(2l-1-\alpha)^2\int\frac{\ln t}{t}dt + \frac{A_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\ln t$$
$$+\left(A_k^2+\frac{B_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\right)\int\frac{dt}{t\ln t}+o(1),$$

as $t\to\infty$, $k=0,\dots,n$ (take $A_0=B_0=0$), we obtain

$$\int_{t_1}^{t_2}\left[t^\alpha\left(h^{(n)}\right)^2+\frac{\nu_{n-1}}{t^{2-\alpha}}\left(h^{(n-1)}\right)^2+\cdots+\frac{\nu_1}{t^{2n-2-\alpha}}\left(h'\right)^2+\frac{\nu_0}{t^{2n-\alpha}}h^2\right]dt$$
$$=\left(\sum_{k=0}^{n}\frac{\nu_k}{4^k}\prod_{j=1}^{k}(2n+1-2j-\alpha)^2\right)\int_{t_1}^{t_2}\frac{\ln t}{t}dt+\tilde K_{n,\alpha}\ln t_2+\hat\nu_{n,\alpha}\int_{t_1}^{t_2}\frac{dt}{t\ln t}$$
$$+L_1+o(1)$$
$$=\tilde K_{n,\alpha}\ln t_2+\hat\nu_{n,\alpha}\int_{t_1}^{t_2}\frac{dt}{t\ln t}+L_1+o(1),$$

as $t_2\to\infty$, $L_1\in\mathbb{R}$, where we have used (5) and denoted

$$\tilde K_{n,\alpha}:=\sum_{k=1}^{n}\frac{A_k\nu_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha), \tag{19}$$

$$\hat\nu_{n,\alpha}:=\sum_{k=1}^{n}\nu_k\left[A_k^2+\frac{B_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\right]. \tag{20}$$

Concerning the interval $[t_2,t_3]$, since $q_0(t)\ge 0$ for large t, we have

$$\int_{t_2}^{t_3}\left[t^\alpha\left(g^{(n)}\right)^2+\frac{\nu_{n-1}}{t^{2-\alpha}}\left(g^{(n-1)}\right)^2+\cdots+\frac{\nu_1}{t^{2n-2-\alpha}}\left(g'\right)^2+\frac{\nu_0}{t^{2n-\alpha}}g^2-q_0(t)g^2\right]dt$$
$$\le\int_{t_2}^{t_3}\left[t^\alpha\left(g^{(n)}\right)^2+\frac{\nu_{n-1}}{t^{2-\alpha}}\left(g^{(n-1)}\right)^2+\cdots+\frac{\nu_1}{t^{2n-2-\alpha}}\left(g'\right)^2+\frac{\nu_0}{t^{2n-\alpha}}g^2\right]dt.$$

Next we use the relationship between equation (2) and corresponding LHS (10). Since g is a solution of (2), we have

$$x=\begin{pmatrix} g\\ g'\\ \vdots\\ g^{(n-1)}\end{pmatrix},\quad u=\begin{pmatrix}(-1)^{n-1}(t^\alpha g^{(n)})^{(n-1)}+\cdots+\nu_1 t^{\alpha-2n+2}g'\\ \vdots\\ -(t^\alpha g^{(n)})'+\nu_{n-1}t^{\alpha-2}g^{(n-1)}\\ t^\alpha g^{(n)}\end{pmatrix}$$

and

$$B(t)=\operatorname{diag}\{0,\dots,0,t^{-\alpha}\},\quad C(t)=\operatorname{diag}\{\nu_0 t^{\alpha-2n},\dots,\nu_{n-1}t^{\alpha-2}\},$$

$$A=A_{i,j}=\begin{cases}1, & \text{if } j=i+1,\ i=1,\dots,n-1,\\ 0, & \text{elsewhere.}\end{cases}$$

Then, using conditions (16),

$$\begin{aligned}
&\int_{t_2}^{t_3}\left[t^{\alpha}\left(g^{(n)}\right)^2+\frac{\nu_{n-1}}{t^{2-\alpha}}\left(g^{(n-1)}\right)^2+\cdots+\frac{\nu_1}{t^{2n-2-\alpha}}\left(g'\right)^2+\frac{\nu_0}{t^{2n-\alpha}}g^2\right]dt\\
&=\int_{t_2}^{t_3}[u^T(t)B(t)u(t)+x^T(t)C(t)x(t)]dt\\
&=\int_{t_2}^{t_3}[u^T(t)(x'(t)-Ax(t))+x^T(t)C(t)x(t)]dt\\
&=u^T(t)x(t)|_{t_2}^{t_3}+\int_{t_2}^{t_3}x^T(t)[-u'(t)-A^Tu(t)+C(t)x(t)]dt\\
&=-u^T(t_2)x(t_2).
\end{aligned}$$

Further, let (X,U) be the principal solution of the LHS associated with (2). Then $(\tilde X,\tilde U)$ defined by

$$\begin{aligned}
\bar X(t) &= X(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds,\\
\bar U(t) &= U(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds-X^{T-1}(t)
\end{aligned}$$

is also a conjoined basis of this LHS, and according to (16), if we let

$$\tilde h=(h,h',\dots,h^{(n-1)})^T,$$

we obtain

$$\begin{aligned}
x(t) &= X(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\\
&\quad\times\left(\int_{t_2}^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1}X^{-1}(t_2)\tilde h(t_2),\\
u(t) &= \left(U(t)\int_t^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds-X^{T-1}(t)\right)\\
&\quad\times\left(\int_{t_2}^{t_3}X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1}X^{-1}(t_2)\tilde h(t_2),
\end{aligned}$$

(see e.g. [1]) and hence

$$\begin{aligned} -u^T(t_2)x(t_2) &= \tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \\ &\quad -\tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2). \end{aligned}$$

Using the fact that the principal solution (X,U) is generated by

$$y_1 = t^{\alpha_1}, \dots, y_{n-1} = t^{\alpha_{n-1}}, y_n = t^{\frac{2n-1-\alpha}{2}},$$

where $\alpha_1, \dots, \alpha_{n-1}, \alpha_0 = \frac{2n-1-\alpha}{2}$ are the first roots (ordered by size) of (3), by a direct computation (similarly to that in [8, Theorem 3.2]), we get

$$\tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2) = \hat{K}_{n,\alpha}\ln t_2 + L_2 + o(1), \quad \text{as } t_2 \to \infty,$$

where L_2 is a real constant and

$$\hat{K}_{n,\alpha} := \sum_{k=1}^{n}\left\{\frac{\nu_k}{2^{2k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha)^2\sum_{l=n-k+1}^{n}\frac{1}{2l-1-\alpha}\right\}. \tag{21}$$

If we summarize all the above computations, we obtain

$$\begin{aligned} \mathcal{F}(y;T,\infty) &\le K + \tilde{K}_{n,\alpha}\ln t_2 + \hat{\nu}_{n,\alpha}\int_{t_1}^{t_2}\frac{dt}{t\ln t} + L_1 + o(1) - \int_{t_1}^{t_2} q_0(t)h^2(t)dt \\ &\quad +\tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \\ &\quad -\hat{K}_{n,\alpha}\ln t_2 - L_2 - o(1), \quad \text{as } t_2 \to \infty. \end{aligned}$$

It follows from Lemma 9 that $\tilde{K}_{n,\alpha} = \hat{K}_{n,\alpha}$ and $\tilde{\nu}_{n,\alpha} = \hat{\nu}_{n,\alpha}$, and according to (15), it is possible to choose $t_2 > t_1$ so large that

$$\hat{\nu}_{n,\alpha}\int_{t_1}^{t_2}\frac{dt}{t\ln t} - \int_{t_1}^{t_2} q_0(t)h^2(t)dt \le -(K + L_1 - L_2 + 2)$$

and that the sum of all the terms o(1) is less than 1. Moreover, since (X,U) is the principal solution, we can choose $t_3 > t_2$ such that

$$\tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}(s)B(s)X^{T-1}(s)ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) \le 1.$$

All together means that

$$\mathcal{F}(y;T,\infty) \le K - (K + L_1 - L_2 + 2) + L_1 + 1 + 1 - L_2 = 0,$$

and hence (1) is oscillatory. $\square$

Corollary 1. *Suppose that* (4) *and* (5) *hold. The equation*

$$L_\nu(y) = \frac{\beta}{t^{2n-\alpha}\ln^2 t} y \tag{22}$$

is nonoscillatory if and only if $\beta \le \tilde{\nu}_{n,\alpha}$.

Proof. If $\beta > \tilde{\nu}_{n,\alpha}$, then

$$\int^\infty \left(q_0(t) - \frac{\tilde{\nu}_{n,\alpha}}{t^{2n-\alpha}\ln^2 t} \right) t^{2n-1-\alpha} \ln t dt = \int^\infty \frac{\beta - \tilde{\nu}_{n,\alpha}}{t \ln t} dt = \infty$$

and (22) is oscillatory according to Theorem 2.

On the other hand, since the second order equation

$$(tu')' + \frac{\mu}{t \ln^2 t} u = 0$$

is nonoscillatory for $\mu \le \frac{1}{4}$, we have nonoscillation of

$$(tu')' + \frac{1}{4\tilde{\nu}_{n,\alpha}} t^{2n-1-\alpha} \frac{\beta}{t^{2n-\alpha}\ln^2 t} u = 0$$

for $\frac{\beta}{4\tilde{\nu}_{n,\alpha}} \le \frac{1}{4}$, i.e., for $\beta \le \tilde{\nu}_{n,\alpha}$. The nonoscillation of (22) for these ν follows from Theorem 1. ☐

4. Oscillation and nonoscillation criteria for (1) in case $m = 1$

In this section we turn our attention to the equation

$$L_\nu(y) = -(q_1(t)y')'. \tag{23}$$

Theorem 3. *Suppose that* (4) *and* (5) *hold and* $4\tilde{\nu}_{n,\alpha} > t^{2n-2-\alpha} q_1(t)$ *for large* t. *If the second order equation*

$$\left[t \left(1 - \frac{1}{4\tilde{\nu}_{n,\alpha}} t^{2n-2-\alpha} q_1(t) \right) u' \right]' - \frac{2n-1-\alpha}{8\tilde{\nu}_{n,\alpha}} t^{\frac{2n-1-\alpha}{2}} \left(q_1(t) t^{\frac{2n-3-\alpha}{2}} \right)' u = 0 \tag{24}$$

is nonoscillatory, then equation (23) *is also nonoscillatory.*

Proof. Let $T \in \mathbb{R}$ be such that the statement of Lemma 3 holds for (24) and let $y \in W^{n,2}(T,\infty)$ with compact support in (T,∞) be arbitrary. We show that the quadratic functional associated with (23) is positive for any nontrivial $y \in W^{n,2}(T,\infty)$ with compact support in (T,∞) by using the same argument as in the proof of

Theorem 1 and the transformation of this functional by the substitution $y = t^{\frac{2n-1-\alpha}{2}} u$ (applied to the term $\int_T^\infty q_1(t)(y')^2 dt$) as follows

$$\int_T^\infty \left[t^\alpha \left(y^{(n)}\right)^2 + \frac{\nu_{n-1}}{t^{2-\alpha}} \left(y^{(n-1)}\right)^2 + \cdots + \frac{\nu_1}{t^{2n-2-\alpha}} (y')^2 + \frac{\nu_0}{t^{2n-\alpha}} y^2 - q_1(t)(y')^2 \right] dt$$

$$\geq 4\tilde{\nu}_{n,\alpha} \int_T^\infty t \left[\left(\frac{y}{t^{\frac{2n-1-\alpha}{2}}} \right)' \right]^2 dt - \int_T^\infty q_1(t)(y')^2 dt$$

$$= 4\tilde{\nu}_{n,\alpha} \int_T^\infty t(u')^2 dt$$

$$- \int_T^\infty \left[q_1(t) t^{2n-1-\alpha} (u')^2 - \frac{2n-1-\alpha}{2} t^{\frac{2n-1-\alpha}{2}} \left(q_1(t) t^{\frac{2n-3-\alpha}{2}} \right)' u^2 \right] dt$$

$$= 4\tilde{\nu}_{n,\alpha} \int_T^\infty \left\{ \left(t - \frac{1}{4\tilde{\nu}_{n,\alpha}} q_1(t) t^{2n-1-\alpha} \right) (u')^2 \right.$$

$$\left. + \frac{1}{4\tilde{\nu}_{n,\alpha}} \frac{2n-1-\alpha}{2} t^{\frac{2n-1-\alpha}{2}} \left(q_1(t) t^{\frac{2n-3-\alpha}{2}} \right)' u^2 \right\} dt > 0.$$

□

The following oscillation criterion is based on Proposition 1 and we prove it similarly to [10, Theorem 4.1].

Theorem 4. *Let $q_1(t) \geq 0$ for large t, (4), (5) hold and*

$$\limsup_{t\to\infty} \ln t \int_t^\infty q_1(s) s^{2n-3-\alpha} ds > \frac{16\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}.$$

Then (23) is oscillatory.

Proof. First, recall that equation (2) is nonoscillatory and its ordered system of solutions is

$$\begin{aligned} &y_1 = t^{\alpha_1}, \dots, y_{n-1} = t^{\alpha_{n-1}}, y_n = t^{\alpha_0} = t^{\frac{2n-1-\alpha}{2}}, \\ &\tilde{y}_1 = t^{\alpha_0} \ln t = t^{\frac{2n-1-\alpha}{2}} \ln t, \tilde{y}_2 = t^{2n-1-\alpha-\alpha_{n-1}}, \dots, \tilde{y}_n = t^{2n-1-\alpha-\alpha_1}. \end{aligned} \tag{25}$$

Let (X, U) denote the principal solution of LHS associated with (2) generated by $y_1, \dots, y_n$ and let $(\tilde{X}, \tilde{U})$ be the solution of this LHS generated by $\tilde{y}_1, \dots, \tilde{y}_n$. According to Proposition 1, which we apply to equation (23) by taking $c = (0, \dots, 0, 1)^T$, we have that (23) is oscillatory if

$$\limsup_{t\to\infty} \frac{\frac{(2n-1-\alpha)^2}{4} \int_t^\infty q_1(s) s^{2n-3-\alpha} ds}{\left(\int^t X^{-1}(s) B(s) X^{T^{-1}}(s) ds \right)^{-1}_{n,n}} > 1.$$

It remains to show that $\left(\int^t X^{-1}(s)B(s)X^{T^{-1}}(s)ds\right)^{-1}_{n,n} \sim \frac{4\tilde{\nu}_{n,\alpha}}{\ln t}$ as $t \to \infty$. (Here by the symbol $f(t) \sim g(t)$ we mean $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.) Since

$$\begin{aligned}
\left(X^{-1}(t)\tilde{X}(t)\right)' &= -X^{-1}(t)\left[A(t)X(t) + B(t)U(t)\right]X^{-1}(t)\tilde{X}(t) \\
&\quad +X^{-1}(t)\left[A(t)\tilde{X}(t) + B(t)\tilde{U}(t)\right] \\
&= X^{-1}(t)B(t)X^{T^{-1}}(t)X^T(t)\tilde{U}(t) - X^{-1}(t)B(t)X^{T^{-1}}(t)U^T(t)\tilde{X}(t) \\
&= X^{-1}(t)B(t)X^{T^{-1}}(t)L,
\end{aligned}$$

where $L := X^T(t)\tilde{U}(t) - U^T(t)\tilde{X}(t)$, we have

$$\left(\int^t X^{-1}(s)B(s)X^{T^{-1}}(s)ds\right)^{-1} = L\tilde{X}^{-1}(t)X(t).$$

It follows from Lemma 10 below that

$$\left(\int^t X^{-1}(s)B(s)X^{T^{-1}}(s)ds\right)^{-1}_{n,n} = \left(L\tilde{X}^{-1}(t)X(t)\right)_{n,n}$$

$$= \sum_{j=1}^n L_{n,j}\left(\tilde{X}^{-1}(t)X(t)\right)_{j,n} = \sum_{j=1}^n L_{n,j}\frac{\tilde{W}_{j,n}(t)}{\tilde{W}(t)},$$

where $\tilde{W}_{j,n}(t) := W(\tilde{y}_1,\dots,\tilde{y}_{j-1},y_n,\tilde{y}_{j+1},\dots,\tilde{y}_n)$ and $\tilde{W}(t) := W(\tilde{y}_1,\dots,\tilde{y}_n)$. (Here $L_{i,j}$, $i,j = 1,\dots,n$ denote the entries of L.) Consequently,

$$\sum_{j=1}^n L_{n,j}\frac{\tilde{W}_{j,n}(t)}{\tilde{W}(t)} \sim L_{n,l}\frac{\tilde{W}_{l,n}(t)}{\tilde{W}(t)}, \qquad l := \min\left\{j \in \{1,\dots,n\}, L_{n,j} \neq 0\right\},$$

as $t \to \infty$, since $\lim_{t\to\infty}\frac{\tilde{W}_{k,n}(t)}{\tilde{W}_{l,n}(t)} = 0$ if $l < k$, by Lemma 11 below. By a direct computation (see Lemma 13), we obtain

$$L_{n,1} = x^T_{[n]}\tilde{u}_{[1]} - u^T_{[n]}\tilde{x}_{[1]} = 4\tilde{\nu}_{n,\alpha},$$

where $x_{[n]}$, $u_{[n]}$ denote the n-th column of X, U respectively and $\tilde{x}_{[1]}$, $\tilde{u}_{[1]}$ denote the first column of $\tilde{X}$, $\tilde{U}$ respectively. It means that we take $l = 1$, since $\tilde{\nu}_{n,\alpha}$ is positive (see Lemma 6). Next, we compute the above mentioned wronskians. Using Lemma

12, we have

$$\begin{aligned}\tilde{W}_{1,n}(t) &= W\left(t^{\frac{2n-1-\alpha}{2}}, t^{2n-1-\alpha-\alpha_{n-1}}, \ldots, t^{2n-1-\alpha-\alpha_1}\right)\\ &= t^{\frac{n(2n-1-\alpha)}{2}} W\left(1, t^{\frac{2n-1-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-1-\alpha}{2}-\alpha_1}\right)\\ &= \prod_{k=1}^{n-1}(\alpha_0-\alpha_k)\, t^{\frac{n(2n-1-\alpha)}{2}} W\left(t^{\frac{2n-3-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-3-\alpha}{2}-\alpha_1}\right),\end{aligned}$$

and similarly

$$\begin{aligned}\tilde{W}(t) &= W\left(t^{\frac{2n-1-\alpha}{2}}\ln t, t^{2n-1-\alpha-\alpha_{n-1}}, \ldots, t^{2n-1-\alpha-\alpha_1}\right)\\ &= t^{\frac{n(2n-1-\alpha)}{2}} W\left(\ln t, t^{\frac{2n-1-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-1-\alpha}{2}-\alpha_1}\right)\\ &= t^{\frac{n(2n-1-\alpha)}{2}} \left\{\ln t \prod_{k=1}^{n-1}(\alpha_0-\alpha_k)\, W\left(t^{\frac{2n-3-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-3-\alpha}{2}-\alpha_1}\right)\right.\\ &\quad -t^{\frac{2n-1-\alpha}{2}-\alpha_{n-1}} \prod_{k=1}^{n-2}(\alpha_0-\alpha_k)\, W\left(t^{-1}, t^{\frac{2n-3-\alpha}{2}-\alpha_{n-2}}, \ldots, t^{\frac{2n-3-\alpha}{2}-\alpha_1}\right)\\ &\quad +\\ &\quad \vdots\\ &\quad \left.+(-1)^{n+1} t^{\frac{2n-1-\alpha}{2}-\alpha_1} \prod_{k=2}^{n-1}(\alpha_0-\alpha_k)\, W\left(t^{-1}, t^{\frac{2n-3-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-3-\alpha}{2}-\alpha_2}\right)\right\}\\ &\sim \prod_{k=1}^{n-1}(\alpha_0-\alpha_k)\, t^{\frac{n(2n-1-\alpha)}{2}} W\left(t^{\frac{2n-3-\alpha}{2}-\alpha_{n-1}}, \ldots, t^{\frac{2n-3-\alpha}{2}-\alpha_1}\right)\ln t,\end{aligned}$$

as $t \to \infty$, by Lemma 11. Finally,

$$\frac{\tilde{W}_{1,n}(t)}{\tilde{W}(t)} \sim \frac{1}{\ln t},$$

as $t \to \infty$ and the proof is completed. □

Remark 1. By Hille's nonoscillation criterion, the equation

$$(r(t)x')' + c(t)x = 0,$$

where $c(t) \geq 0$ for large t and $\int^\infty r^{-1}(t)dt = \infty$, is nonoscillatory if

$$\lim_{t\to\infty}\left(\int^t r^{-1}(s)ds\right)\left(\int_t^\infty c(s)ds\right) < \frac{1}{4}.$$

If we apply this criterion to (24), we obtain that (24) is nonoscillatory if

$$\int^{\infty} t^{-1}\left(4\tilde{\nu}_{n,\alpha}-t^{2n-2-\alpha}q_1(t)\right)^{-1}ds=\infty,\quad (2n-1-\alpha)\left(q_1(t)t^{\frac{2n-3-\alpha}{2}}\right)'\le 0 \text{ for large } t$$

and

$$\lim_{t\to\infty}\left(\int^{t} s^{-1}\left(4\tilde{\nu}_{n,\alpha}-s^{2n-2-\alpha}q_1(s)\right)^{-1}ds\right) \times\left(\int_t^{\infty} s^{\frac{2n-1-\alpha}{2}}\left|\left(q_1(s)s^{\frac{2n-3-\alpha}{2}}\right)'\right|ds\right)<\frac{1}{2|2n-1-\alpha|}.$$

Corollary 2. *Let* (4) *and* (5) *hold and set* $q_1(t)=\frac{\beta}{t^{2n-2-\alpha}\ln^2 t}$. *Then* (23) *is oscillatory if* $\beta>\frac{16\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$ *and nonoscillatory if* $\beta<\frac{4\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$.

Proof. If $\beta>\frac{16\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$, then $\limsup_{t\to\infty}\ln t\int_t^{\infty}q_1(s)s^{2n-3-\alpha}ds=\beta>\frac{16\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$ and (23) is oscillatory according to Theorem 4. If $0\le\beta<\frac{4\tilde{\nu}_{n,\alpha}}{(2n-1-\alpha)^2}$, the nonoscillation of (23) is a consequence of Theorem 3 in view of Remark 1. Indeed, one can verify, by a direct computation, that all the assumptions are satisfied and that

$$\int_t^{\infty} s^{\frac{2n-1-\alpha}{2}}\left|\left(q_1(s)s^{\frac{2n-3-\alpha}{2}}\right)'\right|ds=\begin{cases}\frac{|2n-1-\alpha|}{2}\frac{\beta}{\ln t}+\frac{\beta}{\ln^2 t}, & \text{if } 2n-1-\alpha>0,\\ \frac{|2n-1-\alpha|}{2}\frac{\beta}{\ln t}-\frac{\beta}{\ln^2 t}, & \text{if } 2n-1-\alpha<0\end{cases}$$

and

$$\int^{t} s^{-1}\left(4\tilde{\nu}_{n,\alpha}-s^{2n-2-\alpha}q_1(s)\right)^{-1}ds=\frac{1}{4\tilde{\nu}_{n,\alpha}}\ln t+o(\ln t).$$

Hence,

$$\lim_{t\to\infty}\left(\int^{t} s^{-1}\left(4\tilde{\nu}_{n,\alpha}-s^{2n-2-\alpha}q_1(s)\right)^{-1}ds\right)\left(\int_t^{\infty} s^{\frac{2n-1-\alpha}{2}}\left|\left(q_1(s)s^{\frac{2n-3-\alpha}{2}}\right)'\right|ds\right) =\frac{|2n-1-\alpha|}{8\tilde{\nu}_{n,\alpha}}\beta<\frac{1}{2|2n-1-\alpha|}.$$

If $\beta<0$, then nonoscillation of (23) follows from comparing this equation with the nonoscillatory equation (2). □

We can apply Proposition 1 to equation (1) for arbitrary $m\in\{1,\dots,n-1\}$ as well. If we choose $c=(0,\dots,0,1)^T$ and noting that

$$\left(y_n^{(m)}\right)^2=\frac{1}{4^m}(2n-1-\alpha)^2(2n-3-\alpha)^2\cdots(2n+1-2m-\alpha)^2t^{2n-1-2m-\alpha},$$

where $y_n=t^{\frac{2n-1-\alpha}{2}}$ is the solution of (2), we get the following statement.

Theorem 5. *Suppose* (4) *and* (5) *hold,* $q_m(t) \ge 0$ *for large* t*, and*

$$\limsup_{t\to\infty} \ln t \int_t^\infty q(s) s^{2n-1-2m-\alpha} ds > 4^{m+1} \tilde{\nu}_{n,\alpha} \prod_{j=1}^m \frac{1}{(2n+1-2j-\alpha)^2}.$$

Then (1) *is oscillatory.*

Corollary 3. *Let* (4) *and* (5) *hold and set* $q_m(t) = \frac{\beta}{t^{2n-2m-\alpha} \ln^2 t}$*. Then* (1) *is oscillatory if* $\beta > 4^{m+1} \tilde{\nu}_{n,\alpha} \prod_{j=1}^m \frac{1}{(2n+1-2j-\alpha)^2}$*.*

5. Technical results

Lemma 6. *Let* α_k*,* $k = 1, \dots, n-1$*, be the first* $n-1$ *roots (ordered by size) of the polynomial* (3) *and* $\alpha_0 = \frac{2n-1-\alpha}{2}$*. Then*

$$4\tilde{\nu}_{n,\alpha} = \prod_{k=1}^{n-1} \left(\frac{2n-1-\alpha}{2} - \alpha_k \right)^2,$$

where $\tilde{\nu}_{n,\alpha}$ *is given by* (7)*.*

Proof. The substitution $\mu = \lambda - \frac{2n-1-\alpha}{2}$ converts the polynomial $P(\lambda)$ (given by (3)) into the polynomial

$$Q(\mu) = \sum_{k=0}^n (-1)^k \nu_k \prod_{j=1}^k \left(\mu^2 - \frac{(2n+1-2j-\alpha)^2}{4} \right),$$

whose roots are $\mu = 0$ (double), $\mu = \pm\beta_k$, where $\beta_k = \frac{2n-1-\alpha}{2} - \alpha_k$, $k = 1, \dots, n-1$. This means, that

$$Q(\mu) = (-1)^n \mu^2 (\mu^2 - \beta_1^2) \cdots (\mu^2 - \beta_{n-1}^2).$$

Comparing the coefficients of μ^2 in both expressions for $Q(\mu)$, we obtain the assertion of this Lemma. □

Lemma 7. ([8]) *For arbitrary* $k \in \mathbb{N}$*,*

$$\left(\sqrt{\ln t}\right)^{(k)} = \frac{(-1)^{k-1}}{t^k} \left(\frac{a_k}{\sqrt{\ln t}} + \frac{b_k}{\sqrt{\ln^3 t}} + o\left(\ln^{-\frac{3}{2}} t\right) \right),$$

where a_k*,* b_k *are given by the recursion*

$$a_1 = \frac{1}{2},\ a_{k+1} = k a_k; \quad b_1 = 0,\ b_{k+1} = k b_k + \frac{a_k}{2},$$

or explicitly by

$$a_k = a_1 \prod_{j=1}^{k-1} j = \frac{1}{2}(k-1)!, \quad b_k = \frac{(k-1)!}{4} \sum_{j=1}^{k-1} \frac{1}{j},\ k \ge 2. \tag{26}$$

Lemma 8. ([8]) *Let a_j, b_j be given by* (26) *and set*

$$A_{k,\bar\alpha} := \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\frac{a_j}{2^{k-j}}\prod_{l=j+1}^{k}(2l-1-\bar\alpha)$$

and

$$B_{k,\bar\alpha} := \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\frac{b_j}{2^{k-j}}\prod_{l=j+1}^{k}(2l-1-\bar\alpha).$$

Then, for arbitrary $k \in \mathbb{N}$,

$$A_{k,\bar\alpha} = \frac{1}{2^k}\prod_{l=1}^{k}(2l-1-\bar\alpha)\sum_{l=1}^{k}\frac{1}{2l-1-\bar\alpha} \tag{27}$$

and

$$\frac{1}{4^k}\prod_{l=1}^{k}(2l-1-\bar\alpha)^2\sum_{l=1}^{k}\frac{1}{(2l-1-\bar\alpha)^2} = A_{k,\bar\alpha}^2 + \frac{B_{k,\bar\alpha}}{2^{k-1}}\prod_{l=1}^{k}(2l-1-\bar\alpha). \tag{28}$$

Lemma 9. *Let $\tilde K_{n,\alpha}$, $\hat K_{n,\alpha}$ $\tilde\nu_{n,\alpha}$ and $\hat\nu_{n,\alpha}$ be given by* (19), (21), (7) *and* (20)*. Then*

$$\tilde K_{n,\alpha} = \hat K_{n,\alpha}$$

and

$$\tilde\nu_{n,\alpha} = \hat\nu_{n,\alpha}.$$

Proof. To prove the first equality, it suffices to show that

$$A_k = \frac{1}{2^k}\prod_{l=n-k+1}^{n}(2l-1-\alpha)\sum_{l=n-k+1}^{n}\frac{1}{2l-1-\alpha}, \quad k = 1,\dots,n.$$

This follows from the definition of A_k by (17) and from formula (27) of Lemma 8, where we take $\bar\alpha = \alpha + 2k - 2n$.

Concerning the second identity, we need to show that for $k = 1,\dots,n$,

$$\frac{1}{4^k}\prod_{l=n-k+1}^{n}(2l-1-\alpha)^2\sum_{l=n-k+1}^{n}\frac{1}{(2l-1-\alpha)^2} = A_k^2 + \frac{B_k}{2^{k-1}}\prod_{l=n-k+1}^{n}(2l-1-\alpha),$$

which holds according to (28) if we let $\bar\alpha = \alpha + 2k - 2n$. □

Lemma 10. ([1]) *Let $y_1,\dots,y_n,\tilde y_1,\dots,\tilde y_n \in C^{n-1}$ be a system of linearly independent functions and let $X, \tilde X$ be the Wronski matrices of $y_1,\dots,y_n$, and $\tilde y_1,\dots,\tilde y_n$, respectively. Then*

$$[\tilde X^{-1}X]_{i,j} = \frac{W(\tilde y_1,\dots,\tilde y_{i-1},y_j,\tilde y_{i+1},\dots,\tilde y_n)}{W(\tilde y_1,\dots,\tilde y_n)}.$$

Lemma 11. ([1]) *Let $y_1, \dots, y_m \in C^{m-1}$ be an ordered system of functions (at ∞) and let $i_1, \dots, i_k \in \{1, \dots, m\}$ be such that $i_1 < i_2 < \cdots < i_k$, $j_1, \dots, j_k \in \{1, \dots, m\}$, and $j_1 < j_2 < \cdots < j_k$. Then*

$$\lim_{t\to\infty} \frac{W(y_{i_1}, \dots, y_{i_k})}{W(y_{j_1}, \dots, y_{j_k})} = 0, \quad k = 1, \dots, m,$$

whenever $i_1 \le j_1, \dots, i_k \le j_k$ and at least one of the inequalities is strict.

Lemma 12. ([1]) *Let $y_1, \dots, y_m \in C^{m-1}$, $r \in C^{m-1}$ and $r \neq 0$. Then*

$$W(ry_1, \dots, ry_m) = r^m W(y_1, \dots, y_m).$$

Lemma 13. *Let $y_1, \dots, y_n, \tilde{y}_1, \dots, \tilde{y}_n$ be an ordered system of solutions of (2) given by (25) and let $L := X^T(t)\tilde{U}(t) - U^T(t)\tilde{X}(t)$, where (X, U) is the principal solution of LHS associated with (2) generated by $y_1, \dots, y_n$ and $(\tilde{X}, \tilde{U})$ is the solution of this LHS generated by $\tilde{y}_1, \dots, \tilde{y}_n$. Let $L_{i,j}$, $i, j = 1, \dots, n$ be the entries of L. Then*

$$L_{n,1} = 4\tilde{\nu}_{n,\alpha},$$

where $4\tilde{\nu}_{n,\alpha}$ is given by (7).

Proof. We have $L_{n,1} = x_{[n]}^T \tilde{u}_{[1]} - u_{[n]}^T \tilde{x}_{[1]}$, where $x_{[n]}$, $u_{[n]}$ denote the n-th column of X, U respectively and $\tilde{x}_{[1]}$, $\tilde{u}_{[1]}$ denote the first column of $\tilde{X}$, $\tilde{U}$ respectively, i.e.,

$$x_{[n]} = \begin{pmatrix} y_n \\ y_n' \\ \vdots \\ y_n^{(n-1)} \end{pmatrix}, \quad u_{[n]} = \begin{pmatrix} (-1)^{n-1}(t^\alpha y_n^{(n)})^{(n-1)} + \cdots + \nu_1 t^{\alpha-2n+2} y_n' \\ \vdots \\ -(t^\alpha y_n^{(n)})' + \nu_{n-1} t^{\alpha-2} y_n^{(n-1)} \\ t^\alpha y_n^{(n)} \end{pmatrix}$$

and

$$\tilde{x}_{[1]} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_1' \\ \vdots \\ \tilde{y}_1^{(n-1)} \end{pmatrix}, \quad \tilde{u}_{[1]} = \begin{pmatrix} (-1)^{n-1}(t^\alpha \tilde{y}_1^{(n)})^{(n-1)} + \cdots + \nu_1 t^{\alpha-2n+2} \tilde{y}_1' \\ \vdots \\ -(t^\alpha \tilde{y}_1^{(n)})' + \nu_{n-1} t^{\alpha-2} \tilde{y}_1^{(n-1)} \\ t^\alpha \tilde{y}_1^{(n)} \end{pmatrix},$$

where $y_n = t^{\frac{2n-1-\alpha}{2}}$ and $\tilde{y}_1 = t^{\frac{2n-1-\alpha}{2}} \ln t$. By a direct computation and using the fact that L is a constant matrix (so that we don't need to take into account the terms with $\ln t$), we obtain $L_{n,1} = C_1 - C_2$, where

$$\begin{aligned} C_1 &= \frac{\prod_{k=1}^{n-1}(2k-1-\alpha)^2}{2^{2n-2}} + \frac{\nu_{n-1}\prod_{k=2}^{n-1}(2k-1-\alpha)^2}{2^{2n-4}} + \cdots + \frac{\nu_2}{2^2}(2n-3-\alpha)^2 + \nu_1 \\ &+ \\ &\vdots \\ &+ \sum_{k=2}^{n} \frac{1}{2k-1-\alpha}\left[\frac{\prod_{k=1}^{n}(2k-1-\alpha)^2}{2^{2n-2}(3-\alpha)} + \frac{\nu_{n-1}\prod_{k=2}^{n}(2k-1-\alpha)^2}{2^{2n-4}(3-\alpha)}\right] \\ &+ \frac{\prod_{k=1}^{n}(2k-1-\alpha)^2}{2^{2n-2}(1-\alpha)} \sum_{k=1}^{n} \frac{1}{2k-1-\alpha} \end{aligned}$$

and

$$\begin{aligned} C_2 &= \frac{1}{2n-1-\alpha}\left[\frac{\prod_{k=1}^{n}(2k-1-\alpha)^2}{2^{2n-2}(2n-3-\alpha)} + \cdots + \frac{\nu_2}{2^2}(2n-1-\alpha)^2(2n-3-\alpha)\right] \\ &+ \\ &\vdots \\ &+ \sum_{k=3}^{n} \frac{1}{2k-1-\alpha}\left[\frac{\prod_{k=1}^{n}(2k-1-\alpha)^2}{2^{2n-2}(3-\alpha)} + \frac{\nu_{n-1}\prod_{k=2}^{n}(2k-1-\alpha)^2}{2^{2n-4}(3-\alpha)}\right] \\ &+ \sum_{k=2}^{n} \frac{1}{2k-1-\alpha} \frac{\prod_{k=1}^{n}(2k-1-\alpha)^2}{2^{2n-2}(1-\alpha)}. \end{aligned}$$

Hence,

$$\begin{aligned} C_1 - C_2 &= \frac{1}{4^{n-1}} \prod_{k=1}^{n}(2k-1-\alpha)^2 \sum_{k=1}^{n} \frac{1}{(2k-1-\alpha)^2} \\ &+ \frac{\nu_{n-1}}{4^{n-2}} \prod_{k=2}^{n}(2k-1-\alpha)^2 \sum_{k=2}^{n} \frac{1}{(2k-1-\alpha)^2} \\ &+ \\ &\vdots \\ &+ \frac{\nu_2}{4}(2n-1-\alpha)^2(2n-3-\alpha)^2\left(\frac{1}{(2n-1-\alpha)^2} + \frac{1}{(2n-3-\alpha)^2}\right) \\ &+ \nu_1 \\ &= 4\tilde{\nu}_{n,\alpha}. \end{aligned}$$

□

References

[1] W. A. Coppel, Disconjugacy, Lectures Notes in Math., No. 220, Springer Verlag, Berlin-Heidelberg 1971.

[2] O. Došlý, *Oscillation criteria and the discreteness of the spectrum of self-adjoint, even order, differential operators*, Proc. Roy. Soc. Edinburgh **119A** (1991), 219–232.

[3] O. Došlý, *Oscillation criteria for self-adjoint linear differential equations*, Math. Nachr. **166** (1994), 141-153.

[4] O. Došlý, *Oscillation and spectral properties of a class of singular self-adjoint differential operators*, Math. Nachr. **188** (1997), 49-68.

[5] O. Došlý, *Constants in oscillation theory of higher order Sturm-Liouville differential equations*, Electron. J. Differ. Equ. **2002** (2002), No. 34, pp.1-12.

[6] O. Došlý, *Oscillatory properties of fourth order Sturm-Liouville differential equations,* Acta Univ. Palacki. Olomouc., Fac. Rer. Nat. Mathematica **41** (2002), 49-59.

[7] O. Došlý, *Oscillation and spectral properties of self-adjoint even order differential operators with middle terms*, E. J. Qualitative Theory Diff. Equ., Proc. 7'th Coll. Qualitative Theory of Diff. Equ., No. 7. (2004), pp.1-21.

[8] O. Došlý, S. Fišnarová, *Oscillation and nonoscillation of solutions to even order self-adjoint differential equations,* Electron. J. Differ. Equ. **2003** (2003), No. 115, pp.1-21.

[9] O. Došlý, J. Osička, *Kneser-type oscillation criteria for self-adjoint, two terms, differential equations*, Georgian J. Math. **2** (1995), 241-258.

[10] O. Došlý, J. Osička, *Oscillation and nonoscillation of higher order self-adjoint differential equations*, Czech. Math. J. **52 (127)** (2002), 833-849.

[11] O. Došlý, J. Osička, *Oscillatory properties of higher order Sturm-Liouville differential equations*, Studies Univ. Žilina, Math,. Ser. **15** (2002), 25-40.

[12] F. Fiedler, *Oscillation criteria for a class of* $2n$*-order ordinary differential operators*, J. Differential Equations, **42** (1982), 155-188.

[13] F. Fiedler, *Oscillation criteria for a special class of* $2n$*-order ordinary differential equations*, Math. Nachr. **131** (1987), 205-218.

[14] S. Fišnarová, *Oscillatory properties of fourth order self-adjoint differential equations,* Arch. Math. **40** (2004), 457-469.

[15] I. M. Glazman, Direct Methods of Qualitative Analysis of Singular Differential Operators, Davey, Jerusalem 1965.

[16] E. Müller-Pfeiffer, *An oscillation theorem for self-adjoint differential equations*, Math. Nachr. **108** (1982), 79-92.

[17] W. T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer Verlag, New York-Heidelberg-Berlin 1980.

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Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, CZ-662 95 Brno.

E-mail address: simona@math.muni.cz