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EXISTENCE THEORY AND QUALITATIVE PROPERTIES OF SOLUTIONS TO DOUBLE DELAY INTEGRAL EQUATIONS

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Abstract. In this work, we are concerned with nonlinear integral equations with two constant delays. According to the behavior of the data functions, existence and uniqueness results of a measurable solution, an exponentially stable solution, a bounded solution and an integrable solution are provided.

Keywords: measurable solution; exponentially stable solution; bounded solution; integrable solution.

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1. Introduction

The aim of this paper is to prove existence and uniqueness theorems for the nonlinear double delay integral equation

$$x(t) = \begin{cases} g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s) f\left(s, x(s)\right) ds, \ t \in [\tau_2, +\infty), \\ \\ \Phi(t), \quad t \in [0, \tau_2), \end{cases} \tag{1.1}$$

where the constant delays $\tau_2 > \tau_1 > 0$.

Equations of the type (1.1) are typical in the mathematical modeling of age structured populations in which, for example, the growth of two sizes of the same population is considered (see [1, 2, 5, 6]). In this case τ_1 and τ_2 represent the maturation and the maximal age, respectively.

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Problem (1.1) represents an integral formulation of the following nonlinear Gurtin–MacCamy model (see, for instance, [5, 8]).

$$\begin{aligned} B(t) &= \int_0^t K\left(t, t-\sigma, S\right) B(\sigma) d\sigma + F(t, S), \\ S(t) &= \int_0^t H(t, t-\sigma, S) B(\sigma) d\sigma + G(t, S), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} K(t,\sigma,S) &= \beta(\sigma,S)\Pi(\sigma,t,\sigma,S), H(t,\sigma,S) = \gamma(\sigma)\Pi(\sigma,t,\sigma,S), \\ \Pi(\sigma,t,x,S) &= \exp\left(-\int_0^x \mu(a-\sigma, S(t-\sigma))da\right), \\ F(t,S) &= \int_t^{+\infty} \beta(a,S)\Pi(a,t,t,S)p_0(a-t)da, \\ G(t,S) &= \int_t^{+\infty} \gamma(a)\Pi(a,t,t,S)p_0(a-t)da. \end{aligned}$$

We refer to [5, 8] for the meaning of all the data functions.

The unknown S in (1.2) can be transformed, under some initial conditions (see [8]) to a solution of the following double delay integral equation

$$S(t) = R_0 C \int_{t-a_+}^{t-a^m} \gamma(t-\sigma) \exp\left(-\int_0^{t-\sigma} \mu(a-t+\sigma, S(\sigma))da\right) \phi\left(S(\sigma)\right) S(\sigma) d\sigma, \tag{1.3}$$

where ϕ is a nonnegative decreasing function which is responsible for the reduction of fertility by crowding effect, a_+ and a^m are the maximum and the maturation age of the considered population.

On the other hand, J. Diblík and M. Růžičková [4] studied the exponential solutions of the following differential equation containing two delays $\tau > \delta \geq 0$

$$y'(t) = \beta(t)\left(y(t-\delta) - y(t-\tau)\right). \tag{1.4}$$

We note that if y is a solution of (1.4), then $x(t) = y(t) - y(t - (\tau - \delta))$ is a solution of the following double delay integral equation

$$x(t) = \int_{t-\tau}^{t-\delta} \beta(s+\delta)x(s)ds. \tag{1.5}$$

Also, if β is periodic with period $\tau - \delta$, then y is a solution of (1.5).

To our knowledge, there are a few papers concerning the existence and the uniqueness of the solution of (1.1). E. Messina et al. (see [7, 8, 9]) studied the existence and the uniqueness of the continuous solution of the following integral equation

$$x(t) = \begin{cases} g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t-s)h\left(x(s)\right)ds, & t \in [\tau_2, T], \\ \\ \Phi(t), & t \in [0, \tau_2), \end{cases}$$

where the functions g and k are continuous and the function h satisfies the Lipschitz condition.

However, many physical and biological models include data functions, which are discontinuous. For this reason, we devote our investigations, here, to extend the theory developed in [7, 8, 9] to study the existence and the uniqueness of a solution of (1.1), under simple and convenient conditions on the data functions, in more general spaces.

The paper is organized as follows. In Section 3, we prove a general existence principle. Section 4 is devoted to proving existence and uniqueness of a locally bounded solution, an exponentially stable solution and a bounded solution. In Section 5, we show existence and uniqueness of a locally integrable solution and an integrable solution. Finally, existence and uniqueness results of the solution of double delay convolution integral equations are discussed in Section 6.

2. Notations and some auxiliary facts

In this section, we provide some notations, definitions and auxiliary facts which will be needed for stating our results.

Denote by $L^1(\mathbb{R}^+)$ the set of all Lebesgue integrable functions on $\mathbb{R}^+$, endowed with the standard norm $\|x\|_{L^1(\mathbb{R}^+)} = \int_0^{+\infty} |x(t)|dt$ and by $L^\infty(\mathbb{R}^+)$ the set of all bounded functions on $\mathbb{R}^+$, endowed with the norm $\|x\|_{L^\infty(\mathbb{R}^+)} = \text{ess sup}\{|x(t)|, t \in \mathbb{R}^+\}$. Also, denote by $L^1_{Loc}(\mathbb{R}^+)$ the set of all Lebesgue integrable functions on any compact set of $\mathbb{R}^+$ and by $L^\infty_{Loc}(\mathbb{R}^+)$ the set of all bounded functions on any compact set of $\mathbb{R}^+$.

Let $\mathfrak{F}(\mathbb{R}^+, \mathbb{R})$ be the set of all measurable functions from a subset of $\mathbb{R}^+$ to $\mathbb{R}$.

Let $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function. We define the operator N_f on $\mathfrak{F}(\mathbb{R}^+, \mathbb{R})$ by $N_f x(t) = f(t, x(t)), t \in \mathbb{R}^+$. The operator N_f is said to be the Nemytskii operator associated to the function f.

Let $k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a given measurable function. We define the linear operator $\overline{K}$ on $\mathfrak{F}(\mathbb{R}^+, \mathbb{R})$ by the formula

$$\left(\overline{K}x\right)(t) = \begin{cases} \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s)x(s)ds, \ t \in [\tau_2, +\infty), \\ \\ 0, \quad t \in [0, \tau_2). \end{cases}$$

Let $E \subset \mathfrak{F}(\mathbb{R}^+, \mathbb{R})$ be a vectorial space satisfying the following property:

If $f \in E$ and $\emptyset \neq A \subset D(f)$, $D(f)$ is the domain of f, then the function: $f/_A$ (the restriction of f on A) belongs to E and if $f_1, f_2 \in E$ such that $D(f_1) \cap D(f_2) = \emptyset$, then the function $f : D(f_1) \cup D(f_2) \longrightarrow \mathbb{R}$ defined by

$$f(t) = \begin{cases} f_1(t), \ t \in D(f_1), \\ \\ f_2(t), \ t \in D(f_2), \end{cases} \qquad (*)$$

belongs also to E.

Remark 2.1. *If $f \in E$, then the function*

$$\widehat{f}(t) = \begin{cases} f(t), \ t \in D(f), \\ \\ 0, \ t \in \mathbb{R}^+ - D(f) \end{cases}$$

belongs to E.

We note that the spaces $L^\infty(\mathbb{R}^+), L^\infty_{Loc}(\mathbb{R}^+), L^1(\mathbb{R}^+), L^1_{Loc}(\mathbb{R}^+)$ satisfy the property $(*)$.

3. Existence of a measurable Solution

Let $E \subset \mathfrak{F}(\mathbb{R}^+, \mathbb{R})$ be a vectorial space satisfying the property $(*)$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(i) *$g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ and $\Phi : [0, \tau_2) \longrightarrow \mathbb{R}$ are measurable functions such that $\Phi, g \in E$.*

(ii) *$f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that the Nemytskii operator N_f transforms the space E into itself.*

(iii) *$k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a measurable function and the linear integral operator $\overline{K}$ generated by the function k transforms the space E into itself.*

Then Problem (1.1) *has a unique measurable solution defined on $\mathbb{R}^+$.*

Proof. It is clear that there exists a unique integer $r \geq 1$ such that $r\tau_1 \leq \tau_2 < (r+1)\tau_1$. We define the function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as follows: $x = x_n$ on the interval

$[0,(r+n)\tau_1)$ for $n \geq 1$ such that

$$x_1(t) = \begin{cases} \Phi(t), & if \quad t \in [0,\tau_2) \\ g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,\Phi(s))ds, & if \quad t \in [\tau_2,(r+1)\tau_1) \end{cases}$$
$$= \begin{cases} \Phi(t), & if \quad t \in [0,\tau_2) \\ g(t) + \left(\overline{K} N_f \Phi\right)(t), & if \quad t \in [\tau_2,(r+1)\tau_1), \end{cases} \tag{3.1}$$

and for $n \geq 2$

$$x_n(t) = \begin{cases} x_{n-1}(t), & if \quad t \in [0,(r+n-1)\tau_1) \\ g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,x_{n-1}(s))ds, & if \quad t \in [(r+n-1)\tau_1,(r+n)\tau_1) \end{cases}$$
$$= \begin{cases} x_{n-1}(t), & if \quad t \in [0,(r+n-1)\tau_1) \\ g(t) + \left(\overline{K} N_f x_{n-1}\right)(t)ds, & if \quad t \in [(r+n-1)\tau_1,(r+n)\tau_1). \end{cases} \tag{3.2}$$

We will prove that the sequence (x_n) is well defined and $x_n \in E$ for all $n \geq 1$.

1) We have $x_1 = \Phi \in E$ on $[0,\tau_2)$, and on $[\tau_2,(r+1)\tau_1)$ we have $x_1 = g + \overline{K} N_f \Phi \in E$. Then, by the property $(*)$, we deduce that $x_1 \in E$.

2) Assume that $x_{n-1} \in E$ for $n \geq 2$, hence by the definition of x_n, we get $x_n \in E$ on $[0,(r+n-1)\tau_1)$. Moreover, by the assumptions of Theorem 3.1, we deduce that $x_n = g + \overline{K} N_f x_{n-1} \in E$ on $[(r+n-1)\tau_1,(r+n)\tau_1)$.

Then, by the property $(*)$, we get $x_n \in E$.

Thus the sequence (x_n) is well defined and $x_n \in E$ for all $n \geq 1$, therefore the function x is measurable and defined on $\mathbb{R}^+$.

Now, we will prove that x is a solution of (1.1).

Step 1 : x is a solution on $[0,(r+1)\tau_1)$. By definition, x is a solution of (1.1) on $[0,\tau_2)$. Moreover, for $t \in [\tau_2,(r+1)\tau_1)$ we have $0 \leq t-\tau_2 < t-\tau_1 < r\tau_1 \leq \tau_2$

which implies that

$$x(t) = x_1(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,\Phi(s)) ds$$
$$= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,x(s)) ds.$$

Then x is a solution on $[0,(r+1)\tau_1)$.

Step 2 : x is a solution on $[(r+1)\tau_1,+\infty)$. For $t \in [(r+1)\tau_1,+\infty)$, there exists a unique integer $n \geq 1$ such that $(r+n)\tau_1 \leq t < (r+n+1)\tau_1$, hence

$$x(t) = x_{n+1}(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,x_n(s)) ds$$
$$= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,x(s)) ds.$$

Thus x is a solution on $[(r+1)\tau_1,+\infty)$.

For the uniqueness, let y be a solution of (1.1) on $\mathbb{R}^+$, we will prove that $x = y$ by the following induction.

1) $x = y$ on $[0,(r+1)\tau_1)$.

We have $x = y = \Phi$ on $[0,\tau_2)$ and for $t \in [\tau_2,(r+1)\tau_1)$ we have
$0 \leq t-\tau_2 < t-\tau_1 < r\tau_1 \leq \tau_2$, then $y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,\Phi(s)) ds = x(t)$,
we deduce that $x = y$ on $[0,(r+1)\tau_1)$.

2) Assume that $x = y$ on $[0,(r+n)\tau_1)$ for $n \geq 1$, and show that $x = y$ on $[0,(r+n+1)\tau_1)$.

Let $t \in [(r+n)\tau_1,(r+n+1)\tau_1)$, hence $0 \leq t-\tau_2 < t-\tau_1 < (r+n)\tau_1$.

Then,

$$y(t) = g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,y(s)) ds$$
$$= g(t) + \int_{t-\tau_2}^{t-\tau_1} k(t,s) f(s,x_{n-1}(s)) ds$$
$$= x_n(t) = x(t),$$

which implies that $x = y$ on $[0, (r + n + 1)\tau_1)$.

Then Problem (1.1) has a unique measurable solution defined on $\mathbb{R}^+$.

$\square$

Remark 3.2. *Under the conditions of Theorem 3.1, the solution x need not be in the space E as in the following counterexample.*

Example 3.3. *Consider the following double delay integral equation*

$$x(t) = \begin{cases} 1 + \displaystyle\int_{t-\tau_2}^{t-\tau_1} x(s)\, ds, \ t \in [\tau_2, +\infty), \\ \\ \qquad 0, \quad t \in [0, \tau_2), \end{cases} \tag{3.3}$$

such that $\tau_2 - \tau_1 \geq 1$, we have $\Phi(t) = 0, g(t) = 1, k(t, s) = 1$ and $f(t, x) = x$. Let $E = L^\infty(\mathbb{R}^+)$, it is clear that E satisfies the property $(*)$ and contains the functions Φ and g. Moreover, the operators $\overline{K}$ and N_f transform the space E into itself. Then, by Theorem 3.1, Problem (3.3) *has a unique measurable solution x defined on $\mathbb{R}^+$ by* (3.1) *and* (3.2)*. Hence, for all $t \in [\tau_2, (r+1)\tau_1)$, $x(t) = x_1(t) = 1$ and for all $t \in [(r + 1)\tau_1, (r + 2)\tau_1)$, $x(t) = x_2(t) = 1 + (\tau_2 - \tau_1)$. So, by using the iteration, we deduce that for $n \geq 2$ and $t \in [(r + n - 1)\tau_1, (r + n)\tau_1)$,*

$$x(t) = x_n(t) = \sum_{i=0}^{n-1} (\tau_2 - \tau_1)^i.$$

This implies that $\|x\|_{L^\infty(\mathbb{R}^+)} \geq n$ for all $n \geq 1$.

Consequently, we obtain $\|x\|_{L^\infty(\mathbb{R}^+)} = +\infty$ and $x \notin E$.

4. Existence of an Exponentially Stable Solution

We will need the following lemma.

Lemma 4.1. *Suppose that the following conditions are satisfied:*

(i) *$g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ and $\Phi : [0, \tau_2) \longrightarrow \mathbb{R}$ are measurable functions such that $\Phi \in L^\infty([0, \tau_2))$ and $g \in L^\infty_{Loc}([\tau_2, +\infty))$.*

(ii) *$f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function and there exist a constant b and a function $a \in L^\infty_{Loc}(\mathbb{R}^+)$ such that $|f(t,x)| \leq a(t) + b|x|$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.*

(iii) *$k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a measurable function and the linear integral operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.*

Then Problem (1.1) *has a unique solution in $L^\infty_{Loc}(\mathbb{R}^+)$.*

Proof. We have the vectorial space $L^\infty_{Loc}(\mathbb{R}^+)$ verifies the property $(*)$ and the functions $\Phi, g \in L^\infty_{Loc}(\mathbb{R}^+)$. Moreover, the assumption (ii) guarantees that the Nemytskii operator N_f transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself. Additionally to the assumption (iii), we deduce, by Theorem 3.1, that Problem (1.1) has a unique measurable solution x on $\mathbb{R}^+$ defined by $x = x_n$ on $[0,(r+n)\tau_1)$ for $n \geq 1$, where the sequence (x_n) is defined by (3.1) and (3.2). Moreover, the sequence $(x_n) \in L^\infty_{Loc}(\mathbb{R}^+)$, hence for all $n \geq 2$, we have $x \in L^\infty([0,(r+n-1)\tau_1])$, which implies that $x \in L^\infty_{Loc}(\mathbb{R}^+)$.
Thus Problem (1.1) has a unique solution in $L^\infty_{Loc}(\mathbb{R}^+)$. □

The following result gives a sufficient condition on k so that the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.

Proposition 4.2. *Assume that the function $t \longmapsto \int_{\tau_1}^{\tau_2} |k(t,t-s)|ds$ belongs to $L^\infty_{Loc}([\tau_2,+\infty))$, then the operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself.*

Proof. The operator $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself if and only if, for all $\alpha \geq \tau_2$ and for all $x \in L^\infty_{Loc}(\mathbb{R}^+)$, we have $\overline{K}x \in L^\infty([\tau_2,\alpha])$.
We have for all $t \in [\tau_2,\alpha]$

$$\begin{aligned} |\overline{K}x(t)| &\leq \int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|ds \\ &= \int_{\tau_1}^{\tau_2} |k(t,t-s)||x(t-s)|ds \end{aligned}$$

$$\leq \|x\|_{L^\infty([0,\alpha-\tau_1])} \int_{\tau_1}^{\tau_2} |k(t,t-s)|ds$$

and since $\displaystyle\int_{\tau_1}^{\tau_2} |k(t,t-s)|ds \in L^\infty([\tau_2,\alpha])$, then $\overline{K}x \in L^\infty([\tau_2,\alpha])$.
Thus, $\overline{K}$ transforms the space $L^\infty_{Loc}(\mathbb{R}^+)$ into itself. $\square$

Example 4.3. *Consider Problem* (1.1) *with* g, Φ *and* f *fulfilling the assumptions (i) and (ii) of Lemma 4.1 and* $k(t,s) = \frac{t+s}{t-s}e^s$. *Since*

$$\int_{\tau_1}^{\tau_2} |k(t,t-s)|ds = \left[2t\ln\left(\frac{\tau_2}{\tau_1}\right) - (\tau_2-\tau_1)\right] e^t \in L^\infty_{Loc}([\tau_2,+\infty)),$$

then, by Proposition 4.2 and Lemma 4.1, Problem (1.1) *has a unique solution* $x \in L^\infty_{Loc}(\mathbb{R}^+)$.

In the sequel, we will utilize the following definition.

Definition 4.4. *A measurable function* $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is called exponentially stable, if there are* $M \geq 0$ *and* $\gamma > 0$ *such that* $\forall t \in \mathbb{R}^+, |h(t)| \leq Me^{-\gamma t}$.

The following result gives the existence of an exponentially stable solution of Problem (1.1).

Theorem 4.5. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2,+\infty) \longrightarrow \mathbb{R}$ *is exponentially stable and* $\Phi \in L^\infty([0,\tau_2))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and an exponentially stable function* $a : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that* $|f(t,x)| \leq a(t)+b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $k : [\tau_2,+\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function such that the function*

$$t \longmapsto \int_{\tau_1}^{\tau_2} |k(t,t-s)|ds \in L^\infty_{Loc}([\tau_2,+\infty)).$$

(iv) *There exists* $c \geq \tau_2$ *such that* $b\alpha = b\left(\operatorname{ess\,sup}_{t\geq c} \int_{\tau_1}^{\tau_2} |k(t,t-s)|ds\right) < 1$.

Then Problem (1.1) *has a unique exponentially stable solution.*

Proof. By Proposition 4.2, the assumption (iii) guarantees that the operator $\overline{K}$ transforms $L^\infty_{Loc}(\mathbb{R}^+)$ into itself, then from the above assumptions, we deduce by Lemma 4.1, that Problem (1.1) has a unique solution $x \in L^\infty_{Loc}(\mathbb{R}^+)$. Moreover, there exist $\gamma_1, \gamma_2 > 0$ such that $|g(t)|e^{\gamma_1 t} \in L^\infty(\mathbb{R}^+)$ and $a(t)e^{\gamma_2 t} \in L^\infty(\mathbb{R}^+)$.
Now, let $0 < \gamma \leq \min(\gamma_1, \gamma_2)$, we have for all $t \geq c$

$$\begin{aligned}
|x(t)|e^{\gamma t} \leq & |g(t)|e^{\gamma t} + \int_{t-\tau_2}^{t-\tau_1} e^{\gamma(s+\tau_2)}|k(t,s)||f(s,x(s))|ds \\
\leq & |g(t)|e^{\gamma_1 t} + e^{\gamma_2\tau_2}\int_{t-\tau_2}^{t-\tau_1} |k(t,s)|a(s)e^{\gamma_2 s}ds \\
& + e^{\gamma\tau_2} b\int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|e^{\gamma s}ds \\
\leq & |g(t)|e^{\gamma_1 t} + e^{\gamma_2\tau_2}\int_{\tau_1}^{\tau_2} |k(t,t-s)|a(t-s)e^{\gamma_2(t-s)}ds \\
& + be^{\gamma\tau_2}\int_{\tau_1}^{\tau_2} |k(t,t-s)||x(t-s)|e^{\gamma(t-s)}ds \\
\leq & \|g(z)e^{\gamma_1 z}\|_{L^\infty(\mathbb{R}^+)} + \alpha e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \\
& + b\alpha e^{\gamma\tau_2}\|x(z)e^{\gamma z}\|_{L^\infty([c-\tau_2,t])} \\
\leq & \|g(z)e^{\gamma_1 z}\|_{L^\infty(\mathbb{R}^+)} + \alpha e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \\
& + b\alpha e^{\gamma_2\tau_2}\|x(z)e^{\gamma z}\|_{L^\infty([c-\tau_2,c])} + b\alpha e^{\gamma\tau_2}\|x(z)e^{\gamma z}\|_{L^\infty([c,t])},
\end{aligned}$$

hence, for all $t \geq c$

$$\begin{aligned}
\left(1 - b\alpha e^{\gamma\tau_2}\right)\|x(z)e^{\gamma z}\|_{L^\infty([c,t])} \leq & \|g(z)e^{\gamma_1 z}\|_{L^\infty(\mathbb{R}^+)} + \alpha e^{\gamma_2\tau_2}\|a(z)e^{-\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \\
& + b\alpha e^{\gamma_2\tau_2}\|x(z)e^{\gamma z}\|_{L^\infty([c-\tau_2,c])}.
\end{aligned}$$

Since $b\alpha < 1$, then there exists $0 < \gamma \leq \min(\gamma_1, \gamma_2)$ such that $(1 - b\alpha e^{\gamma\tau_2}) > 0$, which implies from the above estimate that $x(t)e^{\gamma t} \in L^\infty([c, +\infty])$, moreover $x(t)e^{\gamma t} \in L^\infty([0, c])$, it follows that $x(t)e^{\gamma t} \in L^\infty([0, +\infty))$.
Thus Problem (1.1) has a unique exponentially stable solution on $\mathbb{R}^+$. □

Example 4.6. *Consider Problem* (1.1) *with* g, Φ *and* f *fulfilling the assumptions (i) and (ii) of Theorem 4.5 and* $k(t,s) = \frac{1}{t+s}$, *hence*

$$\int_{\tau_1}^{\tau_2} |k(t,t-s)|ds = \ln\left(\frac{2t-\tau_1}{2t-\tau_2}\right) \in L^{\infty}_{Loc}([\tau_2,+\infty)).$$

Since, $\lim_{t\to+\infty} \ln\left(\frac{2t-\tau_1}{2t-\tau_2}\right) = 0$, *then there exists* $c \geq \tau_2$ *such that*

$$b\left(\operatorname{ess\,sup}_{t\geq c} \int_{\tau_1}^{\tau_2} |k(t,t-s)|ds\right) < 1.$$

Thus, by Theorem 4.5, Problem (1.1) *has a unique exponentially stable solution.*

Remark 4.7. *If we replace the expression "exponentially stable" by "bounded" in the assumptions (i) and (ii) of Theorem 4.5 and by setting* $\gamma = \gamma_1 = \gamma_2 = 0$ *in the proof, we obtain a unique bounded solution of* (1.1).

Before state the second result, we need the following lemma.

Lemma 4.8. [3](*Discrete Gronwall's inequality*) *Assume that* $(\alpha_n)_{n\geq 1}$ *and* $(q_n)_{n\geq 1}$ *are given non-negative sequences and the sequence* $(\varepsilon_n)_{n\geq 1}$ *satisfies*
$\varepsilon_1 \leq \beta$ *and*

$$\varepsilon_n \leq \beta + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \varepsilon_j, \ n \geq 2,$$

then

$$\varepsilon_n \leq \left(\beta + \sum_{j=1}^{n-1} q_j\right) \exp\left(\sum_{j=1}^{n-1} \alpha_j\right), \ n \geq 2.$$

Theorem 4.9. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ *is exponentially stable and* $\Phi \in L^{\infty}([0,\tau_2))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and an exponentially stable function* $a : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that* $|f(t,x)| \leq a(t) + b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function and* $|k(t,s)| \leq h(s)$ *such that* $h \in L^1_{Loc}(\mathbb{R}^+)$.

Then Problem (1.1) *has a unique solution* $x \in L^\infty_{Loc}(\mathbb{R}^+)$. *Moreover, there exist* $\gamma > 0, \lambda \geq 0$ *and* $\beta \geq 0$ *such that for all* $t \in \mathbb{R}^+$,

$$|x(t)|e^{\gamma t} \leq \left(\beta + \lambda \int_0^t h(s)ds\right) \exp\left(be^{\gamma\tau_2} \int_0^t h(s)ds\right). \tag{4.1}$$

Proof. We have, by the assumption (iii), for all $\alpha \geq \tau_2$ and for all $t \in [\tau_2, \alpha]$

$$\int_{\tau_1}^{\tau_2} |k(t, t-s)|ds \leq \int_{\tau_1}^{\tau_2} h(t-s)ds \leq \int_0^{\alpha-\tau_1} h(s)ds < +\infty.$$

Then, by Proposition 4.2, the operator $\overline{K}$ transforms $L^\infty_{Loc}(\mathbb{R}^+)$ into itself, hence from the above assumptions, we deduce by Lemma 4.1, that Problem (1.1) has a unique solution $x \in L^\infty_{Loc}(\mathbb{R}^+)$. Moreover, the solution is given by the following iteration: $x = x_n$ on the interval $[0, (r+n)\tau_1), n \geq 1$ such that

$$x_1(t) = \begin{cases} \Phi(t), & \text{if } \ t \in [0, \tau_2) \\ g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s)f(s, \Phi(s))ds, & \text{if } \ t \in [\tau_2, (r+1)\tau_1) \end{cases}$$

and for $n \geq 2$

$$x_n(t) = \begin{cases} x_{n-1}(t), & \text{if } \ t \in [0, (r+n-1)\tau_1) \\ g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} k(t,s)f(s, x_{n-1}(s))ds, & \text{if } \ t \in [(r+n-1)\tau_1, (r+n)\tau_1) \end{cases}$$

On the other hand, there exist $\gamma_1, \gamma_2 > 0$ such that $|g(t)|e^{\gamma_1 t} \in L^\infty(\mathbb{R}^+)$ and $a(t)e^{\gamma_2 t} \in L^\infty(\mathbb{R}^+)$.

Let $\gamma = \min(\gamma_1, \gamma_2)$ and define the sequence $(\epsilon_n)_{n\geq 1}$ as follows: for $n \geq 2$

$$\epsilon_n = \text{ess sup}\left\{|x(t)|e^{\gamma t}, t \in [(r+n-1)\tau_1, (r+n)\tau_1)\right\}$$

and $\epsilon_1 = \text{ess sup}\{|x(t)|e^{\gamma t}, t \in [0, (r+1)\tau_1)\}$.

Now, for $n \geq 2$ and $t \in [(r+n-1)\tau_1, (r+n)\tau_1)$, we have

$$|x(t)|e^{\gamma t} \leq |g(t)|e^{\gamma t} + \int_{t-\tau_2}^{t-\tau_1} e^{\gamma(s+\tau_2)}|k(t,s)|(a(s) + b|x(s)|)ds$$

$$
\begin{aligned}
&\le \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,+\infty))} + e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \int_0^{(r+n-1)\tau_1} h(s)ds \\
&\quad + be^{\gamma\tau_2}\int_0^{(r+n-1)\tau_1} h(s)|x(s)|e^{\gamma s}ds \\
&\le \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,+\infty))} + e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \int_0^{(r+1)\tau_1} h(s)ds \\
&\quad + e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \sum_{j=2}^{n-1}\int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)ds \\
&\quad + be^{\gamma\tau_2}\int_0^{(r+1)\tau_1} h(s)|x(s)|e^{\gamma s}ds + be^{\gamma\tau_2}\sum_{j=2}^{n-1}\int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)|x(s)|e^{\gamma s}ds \\
&\le \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,+\infty))} + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1}\alpha_j\epsilon_j,
\end{aligned}
$$

where

$q_1 = e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)}e^{\gamma_2\tau_2}\int_0^{(r+1)\tau_1} h(s)ds$ and for $j \ge 2$

$$
q_j = e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)}\int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)ds
$$

$\alpha_1 = be^{\gamma\tau_2}\int_0^{(r+1)\tau_1} h(s)ds$ and for $j \ge 2$

$$
\alpha_j = be^{\gamma\tau_2}\int_{(r+j-1)\tau_1}^{(r+j)\tau_1} h(s)ds.
$$

On the other hand, for $t \in [0,\tau_2)$, we have $|x(t)|e^{\gamma t} \le e^{\gamma\tau_2}\|\Phi\|_{L^\infty([0,\tau_2))}$, and for $t \in [\tau_2,(r+1)\tau_1)$, we have

$$
\begin{aligned}
|x(t)|e^{\gamma t} \le& \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,(r+1)\tau_1))} + e^{\gamma\tau_2}\|h\|_{L^1([0,r\tau_1))} \\
&\times \left(\|a(z)e^{\gamma_2 z}\|_{L^\infty([0,r\tau_1))} + be^{\gamma r\tau_1}\|\Phi\|_{L^\infty([0,r\tau_1))}\right),
\end{aligned}
$$

hence,

$$
\begin{aligned}
\epsilon_1 \le \max\{&e^{\gamma\tau_2}\|\Phi\|_{L^\infty([0,\tau_2))}, \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,(r+1)\tau_1))} \\
&+ e^{\gamma\tau_2}\|h\|_{L^1([0,r\tau_1))}\left(\|a(z)e^{\gamma_2 z}\|_{L^\infty([0,r\tau_1))} + be^{\gamma r\tau_1}\|\Phi\|_{L^\infty([0,r\tau_1))}\right)\} \equiv \rho.
\end{aligned}
$$

Let $\beta = \max\left\{\rho, \|g(z)e^{\gamma_1 z}\|_{L^\infty([\tau_2,+\infty))}\right\}$, then for all $n \geq 2$

$$\epsilon_n \leq \beta + \sum_{i=1}^{n-1} q_i + \sum_{i=1}^{n-1} \alpha_i \epsilon_i$$

with $\epsilon_1 \leq \beta$, we deduce, by Lemma 4.8, that for all $n \geq 2$

$$\begin{aligned}
\epsilon_n &\leq \left(\beta + \sum_{i=1}^{n-1} q_i\right) \exp\left(\sum_{i=1}^{n-1} \alpha_i\right) \\
&= \left(\beta + e^{\gamma_2 \tau_2} \|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)} \int_0^{(r+n-1)\tau_1} h(s)ds\right) \\
&\quad \times \exp\left(be^{\gamma\tau_2} \int_0^{(r+n-1)\tau_1} h(s)ds\right).
\end{aligned}$$

Then, for $\lambda = e^{\gamma_2\tau_2}\|a(z)e^{\gamma_2 z}\|_{L^\infty(\mathbb{R}^+)}$ and $t \in [(r+n-1)\tau_1, (r+n)\tau_1)$, we obtain

$$\begin{aligned}
|x(t)|e^{\gamma t} \leq \epsilon_n &\leq \left(\beta + \lambda \int_0^{(r+n-1)\tau_1} h(s)ds\right) \exp\left(be^{\gamma\tau_2} \int_0^{(r+n-1)\tau_1} h(s)ds\right) \\
&\leq \left(\beta + \lambda \int_0^t h(s)ds\right) \exp\left(be^{\gamma\tau_2} \int_0^t h(s)ds\right).
\end{aligned}$$

Moreover, for $t \in [0, (r+1)\tau_1)$, we obtain

$$|x(t)|e^{\gamma t} \leq \beta \leq \left(\beta + \lambda \int_0^t h(s)ds\right) \exp\left(be^{\gamma\tau_2} \int_0^t h(s)ds\right).$$

This completes the proof of the theorem. □

Remark 4.10. *1) If $h \in L^1(\mathbb{R}^+)$ we deduce, by the inequality* (4.1)*, that the solution is exponentially stable.*

2) If we replace the expression "exponentially stable" by "bounded" in the assumptions (i) and (ii) of Theorem 4.9, then, by setting $\gamma = \gamma_1 = \gamma_2 = 0$ in the proof, we obtain the inequality (4.1) *with $\gamma = 0$. Moreover, if $h \in L^1(\mathbb{R}^+)$, then the solution is bounded.*

Example 4.11. *Consider Problem* (1.1) *with* g, Φ *and* f *fulfilling the assumptions (i) and (ii) of Theorem 4.9 and* $k(t,s) = ts\ e^{-(t+s)}$. *Since*

$$|k(t,s)| \leq se^{-s} = h(s) \in L^1(\mathbb{R}^+),$$

then, by Theorem 4.9, Problem (1.1) *has a unique exponentially stable solution.*

5. Existence of an integrable Solution

Arguing as in Lemma 4.1, we deduce the following result.

Lemma 5.1. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ *and* $\Phi : [0, \tau_2) \longrightarrow \mathbb{R}$ *are measurable functions such that* $\Phi \in L^1([0, \tau_2))$ *and* $g \in L^1_{Loc}([\tau_2, +\infty))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and a function* $a \in L^1_{Loc}(\mathbb{R}^+)$ *such that* $|f(t,x)| \leq a(t) + b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function and the operator* $\overline{K}$ *transforms the space* $L^1_{Loc}(\mathbb{R}^+)$ *into itself.*

Then Problem (1.1) *has a unique solution in* $L^1_{Loc}(\mathbb{R}^+)$.

The following result gives a sufficient condition on k so that the operator $\overline{K}$ transforms the space $L^1_{Loc}(\mathbb{R}^+)$ into itself.

Proposition 5.2. *Let* $\widetilde{k}$ *be the function defined on* $\mathbb{R}^+$ *by* $\widetilde{k}(s) = \displaystyle\int_{\theta(s)}^{\tau_2} |k(t+s,s)|dt$ *such that*

$$\theta(s) = \begin{cases} \tau_1, \ s \geq \tau_2 - \tau_1, \\ \\ \tau_2 - s, \ 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

If the function $\widetilde{k} \in L^\infty_{Loc}(\mathbb{R}^+)$, *then the operator* $\overline{K}$ *transforms the space* $L^1_{Loc}(\mathbb{R}^+)$ *into itself.*

Proof. The operator $\overline{K}$ transforms the space $L^1_{Loc}(\mathbb{R}^+)$ into itself if and only if, for all $\alpha \geq 2\tau_2 - \tau_1$ and $x \in L^1_{Loc}(\mathbb{R}^+)$, we have $\overline{K}x \in L^1([\tau_2, \alpha])$.

Assume that $\widetilde{k} \in L^\infty_{Loc}(\mathbb{R}^+)$, then for $\alpha \geq 2\tau_2 - \tau_1$ and $x \in L^1_{Loc}(\mathbb{R}^+)$, we have

$$\begin{aligned}
\int_{\tau_2}^{\alpha} |\overline{K}x(t)|dt \leq& \int_{\tau_2}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|dsdt \\
\leq& \int_{\tau_2}^{2\tau_2-\tau_1} \int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|dsdt \\
&+ \int_{2\tau_2-\tau_1}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|dsdt \\
\leq& \int_{\tau_2}^{2\tau_2-\tau_1} \int_{t-\tau_2}^{\tau_2-\tau_1} |k(t,s)||x(s)|dsdt \\
&+ \int_{\tau_2}^{2\tau_2-\tau_1} \int_{\tau_2-\tau_1}^{t-\tau_1} |k(t,s)||x(s)|dsdt \\
&+ \int_{2\tau_2-\tau_1}^{\alpha} \int_{t-\tau_2}^{t-\tau_1} |k(t,s)||x(s)|dsdt \\
\leq& \int_{0}^{\tau_2-\tau_1} \int_{\tau_2}^{s+\tau_2} |k(t,s)||x(s)|dtds \\
&+ \int_{\tau_2-\tau_1}^{2\tau_2-2\tau_1} \int_{s+\tau_1}^{2\tau_2-\tau_1} |k(t,s)||x(s)|dtds \\
&+ \int_{\tau_2-\tau_1}^{\alpha-\tau_1} \int_{s+\tau_1}^{s+\tau_2} |k(t,s)||x(s)|dtds \\
\leq& \int_{0}^{\tau_2-\tau_1} |x(s)| \int_{\tau_2-s}^{\tau_2} |k(t+s,s)|dtds \\
&+ \int_{\tau_2-\tau_1}^{2\tau_2-2\tau_1} |x(s)| \int_{\tau_1}^{\tau_2} |k(t+s,s)|dtds \\
&+ \int_{\tau_2-\tau_1}^{\alpha-\tau_1} |x(s)| \int_{\tau_1}^{\tau_2} |k(t+s,s)|dtds \\
\leq& \|\widetilde{k}\|_{L^\infty([0,\tau_2-\tau_1])} \|x\|_{L^1([0,\tau_2-\tau_1])} \\
&+ \|\widetilde{k}\|_{L^\infty([\tau_2-\tau_1, 2(\tau_2-\tau_1)])} \|x\|_{L^1([\tau_2-\tau_1, 2(\tau_2-\tau_1)])} \\
&+ \|\widetilde{k}\|_{L^\infty([\tau_2-\tau_1, \alpha-\tau_1])} \|x\|_{L^1([\tau_2-\tau_1, \alpha-\tau_1])}.
\end{aligned}$$

This shows that $\overline{K}x \in L^1([\tau_2, \alpha])$.

Thus, $\overline{K}$ transforms the space $L^1_{Loc}(\mathbb{R}^+)$ into itself. $\square$

Example 5.3. *Consider Problem* (1.1) *with* g, Φ *and* f *fulfilling the assumptions (i) and (ii) of Lemma 5.1 and* $k(t,s) = (t-s)e^s$. *Since*

$$\widetilde{k}(s) = \begin{cases} (\tau_2 - \tau_1)e^s, \ s \geq \tau_2 - \tau_1, \\ \\ se^s, \ 0 \leq s \leq \tau_2 - \tau_1, \end{cases}$$

then $\widetilde{k} \in L^\infty_{Loc}(\mathbb{R}^+)$, *this implies, by Proposition 5.2 and Lemma 5.1, that Problem* (1.1) *has a unique solution* $x \in L^1_{Loc}(\mathbb{R}^+)$.

The following result gives the existence of an integrable solution of (1.1).

Theorem 5.4. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ *and* $\Phi : [0, \tau_2) \longrightarrow \mathbb{R}$ *are measurable functions such that* $g \in L^1([\tau_2, \infty))$ *and* $\Phi \in L^1([0, \tau_2))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and a function* $a \in L^1(\mathbb{R}^+)$ *such that* $|f(t,x)| \leq a(t) + b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $k : [\tau_2, +\infty) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function such that* $\widetilde{k} \in L^\infty(\mathbb{R}^+)$.

(iv) *There exists* $c \geq \tau_2 - \tau_1$ *such that* $b\|\widetilde{k}\|_{L^\infty([c,+\infty))} < 1$.

Then Problem (1.1) *has a unique solution* $x \in L^1(\mathbb{R}^+)$.

Proof. By Proposition 5.2, the assumption (iii) guarantees that the operator $\overline{K}$ transforms $L^1_{Loc}(\mathbb{R}^+)$ into itself, then from the above assumptions, we deduce by Lemma 5.1, that Problem (1.1) has a unique solution $x \in L^1_{Loc}(\mathbb{R}^+)$.

We will show that $x \in L^1(\mathbb{R}^+)$. We have for all $t \geq c + \tau_2$

$$\int_{c+\tau_2}^{t} |x(s)|ds \leq \int_{c+\tau_2}^{t} |g(s)|ds + \int_{c+\tau_2}^{t} \int_{s-\tau_2}^{s-\tau_1} |k(s,r)||a(r)|drds$$

$$
\begin{aligned}
&+ b\int_{c+\tau_2}^{t}\int_{s-\tau_2}^{s-\tau_1} |k(s,r)||x(r)|dr \\
&\leq \int_{c+\tau_2}^{t} |g(s)|ds + \int_{c}^{t-\tau_1}\int_{\tau_1}^{\tau_2} |k(r+s,s)|a(s)drds \\
&+ b\int_{c}^{t-\tau_1}\int_{\tau_1}^{\tau_2} |k(r+s,s)||x(s)|drds \\
&\leq \|g\|_{L^1(\mathbb{R}^+)} + \|\widetilde{k}\|_{L^\infty(\mathbb{R}^+)}\|a\|_{L^1(\mathbb{R}^+)} + b\|\widetilde{k}\|_{L^\infty([c,t])}\|x\|_{L^1([c,t])} \\
&\leq \|g\|_{L^1(\mathbb{R}^+)} + \|\widetilde{k}\|_{L^\infty(\mathbb{R}^+)}\|a\|_{L^1(\mathbb{R}^+)} + b\|\widetilde{k}\|_{L^\infty([c,+\infty)])}\|x\|_{L^1([c,c+\tau_2])} \\
&+ b\|\widetilde{k}\|_{L^\infty([c,+\infty))}\|x\|_{L^1([c+\tau_2,t])},
\end{aligned}
$$

hence, for all $t \geq c+\tau_2$

$$
\begin{aligned}
\left(1 - b\|\widetilde{k}\|_{L^\infty([c,+\infty))}\right)\int_{c+\tau_2}^{t} |x(s)|ds \leq \|g\|_{L^1(\mathbb{R}^+)} + \|\widetilde{k}\|_{L^\infty(\mathbb{R}^+)}\|a\|_{L^1(\mathbb{R}^+)} \\
+ b\|\widetilde{k}\|_{L^\infty([c,+\infty)])}\|x\|_{L^1([c,c+\tau_2])}.
\end{aligned}
$$

This shows that $x \in L^1([\tau_2,+\infty))$, moreover $\Phi \in L^1([0,\tau_2))$ and $x \in L^1([\tau_2, c+\tau_2])$. Then Problem (1.1) has a unique integrable solution on $\mathbb{R}^+$. □

Example 5.5. *Consider Problem* (1.1) *with g, Φ and f fulfilling the assumptions (i) and (ii) of Theorem 5.4 and $k(t,s) = (t+s)e^{-t}$, hence*

$$
\widetilde{k}(s) = \begin{cases} \left[(\tau_1+1)e^{-\tau_1} - (\tau_2+1)e^{-\tau_2}\right]e^{-s} + 2\left(e^{-\tau_1} - e^{-\tau_2}\right)se^{-s}, \ s \geq \tau_2 - \tau_1, \\ \\ e^{-\tau_2}\left[\tau_2 + s + 1 - \tau_2 e^{-s} - e^{-s} - 2se^{-s}\right], \ 0 \leq s \leq \tau_2 - \tau_1. \end{cases}
$$

We have, $\widetilde{k}$ is continuous and $\lim_{s\to+\infty} \widetilde{k}(s) = 0$, then $\widetilde{k}$ is bounded and there exists $c \geq \tau_2$ such that $b\|\widetilde{k}\|_{L^\infty([c,+\infty))} < 1$.

Thus, by Theorem 5.4, Problem (1.1) *has a unique solution $x \in L^1(\mathbb{R}^+)$.*

6. Double delay convolution integral equations

Consider the following nonlinear double delay integral equation:

$$x(t) = \begin{cases} g(t) + \displaystyle\int_{t-\tau_2}^{t-\tau_1} h(t-s) f\left(s, x(s)\right) ds, \ t \in [\tau_2, +\infty), \\ \\ \Phi(t), \quad t \in [0, \tau_2), \end{cases} \tag{6.1}$$

where $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a measurable function.

We have (6.1) is of the form (1.1) such that $k(t, s) = h(t - s)$.
Then $\displaystyle\int_{\tau_1}^{\tau_2} |k(t, t-s)| ds = \int_{\tau_1}^{\tau_2} |h(s)| ds$ and

$$\widetilde{k}(s) = \begin{cases} \displaystyle\int_{\tau_1}^{\tau_2} |h(t)| dt, \ s \geq \tau_2 - \tau_1, \\ \\ \displaystyle\int_{\tau_2 - s}^{\tau_2} |h(t)| dt, \ 0 \leq s \leq \tau_2 - \tau_1. \end{cases}$$

The following result is directly yielded by applying Theorem 4.5 and by using Remark 4.7.

Theorem 6.1. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ *is exponentially stable (resp. bounded) and* $\Phi \in L^\infty([0, \tau_2))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and an exponentially stable function (resp. bounded)* a *such that* $|f(t, x)| \leq a(t) + b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function such that* $\displaystyle\int_{\tau_1}^{\tau_2} |h(t)| dt < +\infty$ *and* $b \displaystyle\int_{\tau_1}^{\tau_2} |h(t)| dt < 1$.

Then Problem (6.1) *has a unique exponentially stable (resp. bounded) solution.*

Also, by applying Theorem 5.4, the following result takes place.

Theorem 6.2. *Suppose that the following conditions are satisfied:*

(i) $g : [\tau_2, +\infty) \longrightarrow \mathbb{R}$ *and* $\Phi : [0, \tau_2) \longrightarrow \mathbb{R}$ *are measurable functions such that* $g \in L^1([\tau_2, \infty))$ *and* $\Phi \in L^1([0, \tau_2))$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a measurable function and there exist a constant* b *and a function* $a \in L^1(\mathbb{R}^+)$ *such that* $|f(t, x)| \leq a(t) + b|x|$ *for all* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$.

(iii) $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is a measurable function such that* $\displaystyle\int_{\tau_1}^{\tau_2} |h(t)|dt < +\infty$ *and* $\displaystyle b\int_{\tau_1}^{\tau_2} |h(t)|dt < 1$.

Then Problem (6.1) *has a unique solution* $x \in L^1(\mathbb{R}^+)$.

Finally, we consider the following double delay integral equations of the form (1.3)

$$x(t) = \begin{cases} R_0 C \displaystyle\int_{t-\tau_2}^{t-\tau_1} \gamma(t-\sigma) \exp\left(-\int_0^{t-\sigma} \mu(a-t+\sigma, x(\sigma))da\right) \phi\left(x(\sigma)\right) x(\sigma) d\sigma, \\ \qquad\qquad t \in [\tau_2, +\infty), \\ \\ \Phi(t), \quad t \in [0, \tau_2). \end{cases} \tag{6.2}$$

Problem (6.2) will be studied under the following assumptions:

(1) $R_0, C \in \mathbb{R}^+$.

(2) γ is a nonnegative function on $\mathbb{R}^+$.

(3) $\mu(a, b) = \alpha(a)$ such that α is a nonnegative function.

(4) ϕ is a nonnegative decreasing function on $\mathbb{R}^+$.

(5) Φ is a nonnegative function on $[0, \tau_2)$.

Then Problem (6.2) is a double delay convolution integral equation of the form (6.1) such that

$$h(s) = R_0 C \gamma(s) \exp\left(-\int_0^s \alpha(a-s)da\right), f(s, x) = \phi(x)x.$$

Moreover; it is clear, by the above assumptions, that if (6.2) has a measurable solution x on $\mathbb{R}^+$, then x is nonnegative.

The following corollaries are directly yielded by applying Theorem 6.1 (resp. Theorem 6.2).

Corollary 6.3. *Suppose that the following conditions are satisfied:*

(i) $\Phi \in L^\infty([0,\tau_2))$.

(ii) $\gamma, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are measurable functions such that*

$$R_0 C\phi(0)\int_{\tau_1}^{\tau_2} \gamma(t)\exp\left(-\int_0^t \alpha(a-t)da\right)dt < 1.$$

Then Problem (6.1) *has a unique nonnegative bounded solution.*

Corollary 6.4. *Suppose that the following conditions are satisfied:*

(i) $\Phi \in L^1([0,\tau_2))$.

(ii) $\gamma, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are measurable functions such that*

$$R_0 C\phi(0)\int_{\tau_1}^{\tau_2} \gamma(t)\exp\left(-\int_0^t \alpha(a-t)da\right)dt < 1.$$

Then Problem (6.1) *has a unique nonnegative solution* $x \in L^1(\mathbb{R}^+)$.

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