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Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions

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Abstract

This paper investigates the existence of solutions for higher order fractional differential inclusions with fractional integral boundary conditions involving non-intersecting finite many strips of arbitrary length. Our study includes the cases when the right-hand side of the inclusion has convex as well non-convex values. Some standard fixed point theorems for multivalued maps are applied to establish the main results. An illustrative example is also presented.

Keywords: Fractional differential inclusions; nonlocal; integral boundary conditions; fixed point theorems
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1 Introduction

In this paper, we study a boundary value problem of a fractional differential inclusion with multi-strip fractional integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0, T], \\ x(0) = 0,\ x'(0) = 0, \dots,\ x^{(n-2)}(0) = 0,\ x(T) = \displaystyle\sum_{i=1}^{m} \gamma_i [I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i)], \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $i = 1, 2, \dots, m$, $0 < \zeta_1 < \eta_1 < \zeta_2 < \eta_2 < \dots < \zeta_m < \eta_m < T$, and $\gamma_i \in \mathbb{R}$ are appropriately chosen constants.

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The subject of initial and boundary value problems of fractional order differential equations has recently emerged as an important area of investigation due to its extensive applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, aerodynamics, viscoelasticity and damping, electro-dynamics of complex medium, wave propagation, blood flow phenomena, etc.([1, 9, 21, 23, 25, 26, 29]). Many researchers have contributed to the development of the existence theory for nonlinear fractional boundary value problems, for instance, see ([2]-[6], [12, 13, 18, 19, 28, 30]) and the references cited therein.

The present work is motivated by a recent paper [7] where a nonlocal strip condition of the form

$$x(1)=\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}x(s)ds,\quad 0<\zeta_i<\eta_i,<1,\ i=1,2,\ldots,(n-2).$$

is considered. In the present study, we have introduced Riemann-Liouville type fractional integral boundary conditions involving nonintersecting finite many strips of arbitrary length. Such boundary conditions can be interpreted in the sense that a controller at the right-end of the interval under consideration is influenced by a discrete distribution of finite many nonintersecting sensors (strips) of arbitrary length expressed in terms of Riemann-Liouville type integral boundary conditions. The results concerning the single valued case of (1.1) are reported in the paper [8].

We establish the new existence results for the problem (1.1), when the right hand side of the inclusion is convex as well as non-convex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods employed in the present work are well known, however their exposition in the framework of problem (1.1) is new.

The paper is organized as follows: Section 2 contains some preliminary concepts and results about multivalued maps while the main results are presented in Section 3.

2 Preliminaries

2.1 Fractional Calculus

First of all, we recall some basic definitions of fractional calculus [21, 25, 26] and then obtain an auxiliary result.

Definition 2.1 *For an at least n-times differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 *[21] For $q > 0$, the general solution of the fractional differential equation $D^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.3, it follows that

$$I^q D^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ ($n = [q] + 1$).

In the following, $AC^{n-1}([0, T], \mathbb{R})$ will denote the space of functions $x : [0, T] \to \mathbb{R}$ that are $(n-1)-$times absolutely continuously differentiable functions.

Definition 2.4 *A function $x \in AC^{n-1}([0, T], \mathbb{R})$ is called a solution of problem (1.1) if there exists a function $v \in L^1([0, T], \mathbb{R})$ with $v(t) \in F(t, x(t))$, a.e. $[0, T]$ such that $^cD^q x(t) = v(t)$, a.e. $[0, T]$ and $x(0) = 0,\ x'(0) = 0, \ldots,\ x^{(n-2)}(0) = 0,\ x(T) = \sum_{i=1}^{m} \gamma_i [I^{\beta i} x(\eta_i) - I^{\beta i} x(\zeta_i)]$.*

Lemma 2.5 *For $g \in C[0, T]$, the fractional boundary value problem*

$$\begin{cases} ^cD^q x(t) = g(t), \quad t \in [0, T],\ q \in (n-1,\ n] \\ x(0) = 0,\ x'(0) = 0, \ldots,\ x^{(n-2)}(0) = 0,\ x(T) = \sum_{i=1}^{m} \gamma_i [I^{\beta i} x(\eta_i) - I^{\beta i} x(\zeta_i)], \end{cases} \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds - \frac{t^{n-1}}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} g(s) ds \\ &+ \frac{t^{n-1}}{\lambda \Gamma(q)} \sum_{i=1}^{m} \frac{\gamma_i}{\Gamma(\beta_i)} \Big[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i - 1} (s-u)^{q-1} g(u) du ds \\ &- \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i - 1} (s-u)^{q-1} g(u) du ds \Big], \end{aligned} \tag{2.3}$$

where

$$\lambda = \left(T^{n-1} - \sum_{i=1}^{m} \gamma_i \frac{(\eta_i^{\beta_i+n-1} - \zeta_i^{\beta_i+n-1})\Gamma(n)}{\Gamma(\beta_i+n)} \right) \neq 0. \tag{2.4}$$

Proof. The general solution of fractional differential equations in (2.2) can be written as

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)ds - c_0 - c_1 t - \ldots - c_{n-1} t^{n-1}. \tag{2.5}$$

Using the given boundary conditions, it is found that $c_0 = 0,\ c_1 = 0,\ \ldots,\ c_{n-2} = 0$. Applying the Riemann-Liouville integral operator I^{β_i} on (2.5), we get

$$\begin{aligned} I^{\beta_i} x(t) &= \frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} \Big(\frac{1}{\Gamma(q)} \int_0^s (s-u)^{q-1} g(u)du - c_{n-1} s^{n-1} \Big) ds \\ &= \frac{1}{\Gamma(\beta_i)\Gamma(q)} \int_0^t \int_0^s (t-s)^{\beta_i-1} (s-u)^{q-1} g(u)duds \\ &\quad - c_{n-1} \frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} s^{n-1} ds. \end{aligned}$$

Using the condition $x(T) = \sum_{i=1}^{m} \gamma_i [I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i)]$, together with the fact that

$$\frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} s^{n-1} ds = \frac{t^{\beta_i+n-1}\Gamma(n)}{\Gamma(\beta_i+n)},$$

we obtain

$$\begin{aligned} &\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} g(s)ds - c_{n-1} T^{n-1} \\ &= \sum_{i=1}^{m} \frac{\gamma_i}{\Gamma(q)\Gamma(\beta_i)} \Big[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} g(u)duds \\ &- \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} g(u)duds \Big] - c_{n-1} \sum_{i=1}^{m} \gamma_i \frac{(\eta_i^{\beta_i+n-1} - \zeta_i^{\beta_i+n-1})\Gamma(n)}{\Gamma(\beta_i+n)}, \end{aligned}$$

which yields

$$\begin{aligned} c_{n-1} &= \frac{1}{\lambda\Gamma(q)} \int_0^T (T-s)^{q-1} g(s)ds \\ &\quad - \frac{1}{\lambda\Gamma(q)} \sum_{i=1}^{m} \frac{\gamma_i}{\Gamma(\beta_i)} \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} g(u)duds \right. \\ &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} g(u)duds \right], \end{aligned}$$

where λ is given by (2.4). Substituting the values of $c_0,\ c_1, \ldots, c_{n-2}, c_{n-1}$ in (2.5), we obtain (2.3). This completes the proof. □

2.2 Basic Material for Multivalued Maps

Here we outline some basic concepts of multivalued analysis. [15, 20, 27].

Let $C([0,T])$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned}
\mathcal{P}_{cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},\\
\mathcal{P}_{b}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},\\
\mathcal{P}_{cp}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \text{and}\\
\mathcal{P}_{cp,c}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.
\end{aligned}$$

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;

(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable;

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Definition 2.6 *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(*i*) *$t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;*

(*ii*) *$x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$.*

Further a Carathéodory function F is called L^1-Carathéodory if

(*iii*) *for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $\|x\|_\infty \leq \alpha$ and for a. e. $t \in [0,T]$.

For each $x \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 2.7 *([15, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 2.8 *([24]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator*

$$\Theta \circ S_F : C([0,T],X) \to \mathcal{P}_{cp,c}(C([0,T],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 2.9 *(Nonlinear alternative for Kakutani maps)[17]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map. Then either*

(*i*) *F has a fixed point in $\overline{U}$, or*

(*ii*) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Definition 2.10 *Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.*

Definition 2.11 *A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.*

Lemma 2.12 *([10]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and decomposable values. Then N has a continuous selection, that is, there exists a continuous function (single-valued) $h : Y \to L^1([0,T],\mathbb{R})$ such that $h(x) \in N(x)$ for every $x \in Y$.*

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [22]).

Definition 2.13 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

(**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x),N(y)) \le \gamma d(x,y) \text{ for each } x,y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 2.14 *([14]) Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Main Results

3.1 The Carathéodory Case

In this section, we are concerned with the existence of solutions for the problem (1.1) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map. For the forthcoming analysis, we set

$$\Omega = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|}\sum_{i=1}^{m}\gamma_i\frac{\eta_i^{q+\beta_i}-\zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}. \tag{3.1}$$

Theorem 3.1 *Suppose that*

(H_1) *the map $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x)\} \le p(t)\psi(\|x\|)$$

for each $(t,u) \in [0,T]\times\mathbb{R}$;

(H_3) *there exists a number $M>0$ such that*

$$\frac{M}{\psi(M)\|p\|_{L^1}\Omega}>1,$$

where Ω is given by (3.1).

Then the BVP (1.1) has at least one solution.

Proof. Let us introduce the operator $N: C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ as

$$N(x)=\left\{\begin{array}{l} h\in C([0,T],\mathbb{R}): \\ h(t)=\left\{\begin{array}{l} \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}v(s)ds-\dfrac{t^{n-1}}{\lambda\Gamma(q)}\displaystyle\int_0^T (T-s)^{q-1}v(s)ds \\ +\dfrac{t^{n-1}}{\lambda\Gamma(q)}\displaystyle\sum_{i=1}^m\dfrac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\displaystyle\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds \\ -\displaystyle\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds\Bigg], \end{array}\right. \end{array}\right\}$$

for $v\in S_{F,x}$. We will show that the operator N satisfies the assumptions of the nonlinear alternative of Leray- Schauder type. The proof consists of several steps. As a first step, we show that $N(x)$ *is convex for each* $x\in C([0,T],\mathbb{R})$. For that, let $h_1,h_2\in N(x)$. Then there exist $v_1,v_2\in S_{F,x}$ such that for each $t\in[0,T]$, we have

$$\begin{aligned} h_i(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_i(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_i(s)ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_i(u)duds \\ &-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_i(u)duds\Bigg],\quad i=1,2. \end{aligned}$$

Let $0\le\omega\le 1$. Then, for each $t\in[0,T]$, we have

$$\begin{aligned} &[\omega h_1+(1-\omega)h_2](t) \\ =\; &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}[\omega v_1(r)+(1-\omega)v_2(r)]ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}[\omega v_1(r)+(1-\omega)v_2(r)]duds \end{aligned}$$

$$- \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i - 1}(s-u)^{q-1}[\omega v_1(r) + (1-\omega)v_2(r)]duds\Bigg].$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\omega h_1 + (1-\omega)h_2 \in N(x)$.

Next, we show that $N(x)$ *maps bounded sets into bounded sets in* $C([0,T],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in N(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v(s)ds \\ &+ \frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i - s)^{\beta_i-1}(s-u)^{q-1}v(u)duds \\ &- \int_0^{\zeta_i}\int_0^s (\zeta_i - s)^{\beta_i-1}(s-u)^{q-1}v(u)duds\Bigg], \end{aligned}$$

and

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|v(s)|ds + \frac{t^{n-1}}{|\lambda|\Gamma(q)}\int_0^T (T-s)^{q-1}|v(s)|ds \\ &+ \frac{t^{n-1}}{|\lambda|\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Big[\int_0^{\eta_i}\int_0^s (\eta_i - s)^{\beta_i-1}(s-u)^{q-1}|v(u)|duds \\ &+ \int_0^{\zeta_i}\int_0^s (\zeta_i - s)^{\beta_i-1}(s-u)^{q-1}|v(u)|duds\Big] \\ &\le \psi(\|x\|)\|p\|_{L^1}\left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|}\sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1}\right\}. \end{aligned}$$

Then

$$\|h\| \le \psi(\rho)\|p\|_{L^1}\left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|}\sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1}\right\}.$$

Now we show that N *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ, as above, is a bounded set of $C([0,T],\mathbb{R})$. For each $h \in N(x)$, we obtain

$$\begin{aligned} &|h(t'') - h(t')| \\ = &\Bigg| \frac{1}{\Gamma(q)}\int_0^{t''} (t''-s)^{q-1}v(s)ds - \frac{1}{\Gamma(q)}\int_0^{t'} (t'-s)^{q-1}v(s)ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{[(t'')^{n-1}-(t')^{n-1}]}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v(s)ds \\
&+\frac{[(t'')^{n-1}-(t'')^{n-1}]}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Big[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds \\
&-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds\Big]\Bigg| \\
\leq\ & \frac{1}{\Gamma(q)}\int_0^{t'} |(t''-s)^{q-1}-(t'-s)^{q-1}|\psi(r)\sigma(s)ds+\frac{1}{\Gamma(q)}\int_{t'}^{t''}|t''-s|^{q-1}\psi(r)p(s)ds \\
&+\frac{|(t'')^{n-1}-(t')^{n-1}|}{|\lambda|\Gamma(q)}\int_0^T |T-s|^{q-1}\psi(r)p(s)ds \\
&+\frac{|(t'')^{n-1}-(t')^{n-1}|}{|\lambda|\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Big[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}du\psi(r)p(s)ds \\
&-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}\psi(r)p(s)duds\Big].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t''-t' \to 0$. As N satisfies the above three assumptions, therefore it follows by Ascoli-Arzelá theorem that $N : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that N *has a closed graph.* Let $x_n \to x_*, h_n \in N(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in N(x_*)$. Associated with $h_n \in N(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_n(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_n(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_n(s)ds \\
&+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_n(u)duds \\
&-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_n(u)duds\Bigg].
\end{aligned}
$$

Thus we have to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$
\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_*(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_*(s)ds \\
&+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_*(u)duds
\end{aligned}
$$

$$-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_*(u)duds\Bigg].$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ so that

$$\begin{aligned} v\mapsto\Theta(v) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T(T-s)^{q-1}v(s)ds \\ &\quad+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds \\ &\quad-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds\Bigg]. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\| \\ =\ &\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(v_n(s)-v_*(s))ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T(T-s)^{q-1}(v_n(s)-v_*(s))ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}(v_n(u)-v_*(u))duds \\ &-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}(v_n(u)-v_*(u))duds\Bigg], \end{aligned}$$

which tends to zero as $n\to\infty$.

Thus, it follows by Lemma 2.8 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, it follows that

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v_*(s)ds-\frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T(T-s)^{q-1}v_*(s)ds \\ &\quad+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_*(u)duds \\ &\quad-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_*(u)duds\Bigg], \end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Finally, *we discuss a priori bounds on solutions.* Let x be a solution of (1.1). Then, using the computations proving that $N(x)$ maps bounded sets into bounded sets, we have

$$\|x\| \leq \psi(\|x\|)\|p\|_{L^1}\left\{\frac{T^q}{\Gamma(q+1)}+\frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)}+\frac{T^{n-1}}{|\lambda|}\sum_{i=1}^{m}\gamma_i\frac{\eta_i^{q+\beta_i}-\zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1}\right\}.$$

Consequently, in view of (3.1), we get

$$\frac{\|x\|}{\psi(\|x\|)\|p\|_{L^1}\Omega}\leq 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U=\{x\in C([0,T],\mathbb{R}):\|x\|<M+1\}.$$

Note that the operator $N:\overline{U}\to\mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x\in\mu N(x)$ for some $\mu\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [17], we deduce that N has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

Example 3.2 *Let us consider the following* 4*−strip nonlocal boundary value problem:*

$$\begin{cases} {}^cD^{9/2}x(t)\in F(t,x(t)), \quad t\in[0,2], \\ x(0)=0,\ x'(0)=0,\ x''(0)=0,\ x'''(0)=0, \\ x(2)=\sum_{i=1}^{4}\gamma_i[I^{\beta_i}x(\eta_i)-I^{\beta_i}x(\zeta_i)], \end{cases} \tag{3.2}$$

where $q=9/2,\ n=5,\ T=2,\ \zeta_1=1/4,\ \eta_1=1/2,\ \zeta_2=2/3,\ \eta_2=1,\ \zeta_3=5/4,\ \eta_3=4/3,\ \zeta_4=3/2,\ \eta_4=7/4,\ \gamma_1=5,\ \gamma_2=10,\ \gamma_3=15,\ \gamma_4=25,\ \beta_1=5/4,\ \beta_2=7/4,\ \beta_3=9/4,\ \beta_4=11/4,$ *and* $F:[0,2]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$x\to F(t,x)=\left[\frac{\sqrt[3]{t}|x|^5}{(|x|^5+3)},\frac{\sqrt[3]{t}|x|}{2(|x|+1)}\right].$$

For $f\in F$*, we have*

$$|f|\leq\max\left(\frac{\sqrt[3]{t}|x|^5}{(|x|^5+3)},\frac{\sqrt[3]{t}|x|}{2(|x|+1)}\right)\leq\sqrt[3]{t},\quad x\in\mathbb{R}$$

with $p(t)=\sqrt[3]{t},\ \psi(\|x\|)=1.$

With the given values of the parameters involved, we find that

$$\lambda=\left(T^{n-1}-\sum_{i=1}^{m}\gamma_i\frac{(\eta_i^{\beta_i+n-1}-\zeta_i^{\beta_i+n-1})\Gamma(n)}{\Gamma(\beta_i+n)}\right)\simeq 9.334784,$$

and

$$\Omega = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|}\sum_{i=1}^{m}\gamma_i\frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \simeq 1.406972.$$

Using the above values in the condition (H_3) :

$$\frac{M}{\psi(M)\|p\|_{L^1}\Omega} > 1,$$

we find that $M > M_1 \simeq 1.055229$. Clearly, all the conditions of Theorem 3.1 are satisfied. Hence the conclusion of Theorem 3.1 applies to the problem (3.2).

3.2 The Lower Semi-Continuous Case

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the nonlinear alternative of Leray-Schauder type combined with the selection theorem of Bressan and Colombo for lower semi-continuous maps with decomposable values.

Theorem 3.3 *Assume that* $(H_2)-(H_3)$ *and the following conditions hold:*

(H_4) $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

- **(a)** $(t, x) \longmapsto F(t, x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
- **(b)** $x \longmapsto F(t, x)$ *is lower semicontinuous for each* $t \in [0, T]$*;*

Then the boundary value problem (1.1) has at least one solution on $[0, T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type ([16]). Then from Lemma 2.12, there exists a continuous function $f : C([0, T], \mathbb{R}) \to L^1([0, T], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, T], \mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad 0 < t < T, \\ x(0) = 0,\ x'(0) = 0, \ldots,\ x^{(n-2)}(0) = 0,\ x(T) = \displaystyle\sum_{i=1}^{m}\gamma_i[I^{\beta_i}x(\eta_i) - I^{\beta_i}x(\zeta_i)]. \end{cases} \tag{3.3}$$

Observe that if $x \in AC^{n-1}([0, T])$ is a solution of (3.3), then x is a solution to the problem (1.1). In order to transform the problem (3.3) into a fixed point problem, we define the operator $\overline{N}$ as

$$(\overline{N}x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(x(s))ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}f(x(s))ds$$

$$+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^{m}\frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s(\eta_i-s)^{\beta_i-1}(s-u)^{q-1}f(x(u))duds$$

$$-\int_0^{\zeta_i}\int_0^s(\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}f(x(u))duds\Bigg].$$

It can easily be shown that $\overline{N}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

3.3 The Lipschitz Case

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [14].

Theorem 3.4 *Assume that the following conditions hold:*

(H_5) *$F : [0,T]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [0,T]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$;*

(H_6) *$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ for almost all $t\in[0,T]$ and $x,\bar{x}\in\mathbb{R}$ with $m\in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,T]$.*

Then the boundary value problem (1.1) has at least one solution on $[0,T]$ if

$$\left\{\frac{T^q}{\Gamma(q+1)}+\frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)}+\frac{T^{n-1}}{|\lambda|}\sum_{i=1}^{m}\gamma_i\frac{\eta_i^{q+\beta_i}-\zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1}\right\}\|m\|_{L^1}<1.$$

Proof. We transform the boundary value problem (1.1) into a fixed point problem. Consider the operator $N : C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ defined at the begin of the proof of Theorem 3.1. We show that the operator N, satisfies the assumptions of Lemma 2.14. The proof will be given in two steps.

Step 1. $N(x)$ is nonempty and closed for every $v\in S_{F,x}$. Note that since the set-valued map $F(\cdot,x(\cdot))$ is measurable with the measurable selection theorem (e.g., [11, Theorem III.6]) it admits a measurable selection $v : I\to\mathbb{R}$. Moreover, by the assumption (H_6), we have

$$|v(t)|\le m(t)+m(t)|x(t)|,$$

i.e. $v\in L^1([0,T],\mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,y}\neq\emptyset$.

To show that $N(x) \in \mathcal{P}_{cl}((C[0,T],\mathbb{R}))$ for each $x \in C([0,T],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_n(s)ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_n(s)ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_n(u)duds \\ &-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_n(u)duds\Bigg]. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) \to u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v(s)ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds \\ &-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v(u)duds\Bigg]. \end{aligned}$$

Hence, $u \in N(x)$.

Step 2. Next we show that there exists $\gamma < 1$ such that

$$H_d(N(x), N(\bar{x})) \leq \gamma\|x-\bar{x}\| \quad \text{for each} \quad x, \bar{x} \in C([0,T],\mathbb{R}).$$

Let $x, \bar{x} \in C([0,T],\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_1(s)ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_1(s)ds \\ &+\frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_1(u)duds \\ &-\int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_1(u)duds\Bigg]. \end{aligned}$$

By (H_6), we have

$$H_d(F(t,x),F(t,\bar{x})) \leq m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w(t) \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w(t)| \leq m(t)|x(t)-\bar{x}(t)|, \quad t \in [0,T].$$

Define $\mathcal{W}:[0,T] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{W}(t) = \{w \in \mathbb{R} : |v_1(t)-w| \leq m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $\mathcal{W}(t) \cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [11]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{W}$. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t)-v_2(t)| \leq m(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0,T]$, let us define

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}v_2(s)ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}v_2(s)ds \\ &\quad + \frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}v_2(u)duds \\ &\quad - \int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}v_2(u)duds\Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} &|h_1(t)-h_2(t)| \\ \leq\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|v_1(s)-v_2(s)|ds - \frac{t^{n-1}}{\lambda\Gamma(q)}\int_0^T (T-s)^{q-1}|v_1(s)-v_2(s)|ds \\ & + \frac{t^{n-1}}{\lambda\Gamma(q)}\sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)}\Bigg[\int_0^{\eta_i}\int_0^s (\eta_i-s)^{\beta_i-1}(s-u)^{q-1}|v_1(u)-v_2(u)|duds \\ & - \int_0^{\zeta_i}\int_0^s (\zeta_i-s)^{\beta_i-1}(s-u)^{q-1}|v_1(u)-v_2(u)|duds\Bigg]. \end{aligned}$$

Hence,

$$\|h_1-h_2\| \leq \left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|}\sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i}-\zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right\}\|m\|_{L^1}\|x-\overline{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} & H_d(N(x), N(\bar{x})) \leq \gamma \|x - \bar{x}\| \\ \leq & \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda|\Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^{m} \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} |m\|_{L^1} \|x - \overline{x}\|. \end{aligned}$$

Since N is a contraction, it follows by Lemma 2.14 that N has a fixed point x which is a solution of (1.1). This completes the proof. $\square$

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