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Oscillation results of higher order nonlinear neutral delay differential equations

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Abstract. In this paper we investigate the oscillation and asymptotic behavior of solutions to the n-th order neutral nonlinear differential equations of the form

$$[x(t) + g(t, x(\tau(t)))]^{(n)} + f(t, x(\sigma(t))) = 0.$$

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1 Introduction

In the last two decades, there has been a growing interest in the study of oscillation properties of solutions of higher order neutral differential equations, see [1–8] and the references cited therein. This interest is due to the appearance of these equations in many applications in natural science and technology. In particular, such equations appear in networks containing lossless transmission lines and in problems dealing with vibrating masses attached to an elastic bar, see Hale [3].

This paper is concerned with the oscillation of n-th order neutral type nonlinear differential equations of the form

$$[x(t) + g(t, x(\tau(t)))]^{(n)} + f(t, x(\sigma(t))) = 0, \tag{1.1}$$

where $n \geq 2$, and the following conditions are always assumed to hold:

(H1) $\tau(t)$, $\sigma(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\tau(t) < t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, where $\mathbb{R}_+ = [0, \infty)$;

(H2) $g\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, and $xg(t, x) > 0$;

(H3) $|g(t, x)| \leq M|x|$ for some $M > 0$;

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(H4) $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, and $xf(t,x) > 0$;

(H5) The function $f(t,x)$ can be written as $f(t,x) = h(t,x)k(x)$, where $h: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and $xh(t,x) > 0$, $k: \mathbb{R} \to \mathbb{R}$ is continuous, positive for $x \neq 0$, nondecreasing if $x > 0$, and nonincreasing if $x < 0$.

In this paper we establish new oscillation criteria for equation (1.1). This equation involves one delay in the differential part and one delay in the non-differential part. Many authors have used a frequent assumption for the differential part that is $g(t, x(\tau(t))) = p(t)x(\tau(t))$, where p is a continuous function, see [2,5–8]. Here the function $g(t,x)$ can be a nonlinear function of x. As well as the function $f(t,x)$ can be nonlinear function. Moreover, both bounded and unbounded solutions are considered.

We extend the arguments developed in Zafer [6], Dahiya and Zafer [1] and employ these ideas to establish sufficient conditions for the oscillation of (1.1). One can see that the results in [6] are included in our results.

As is customary, solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros and non-oscillatory otherwise.

This paper is organized as follows: Section 2 states some useful lemmas that are used in the proof of the results. Section 3 is devoted to our main results. In Section 4 some examples are given to illustrate the applicability of the new theorems.

2 Auxiliary lemmas

Lemma 2.1 ([4, p. 193]). *Let $y(t)$ be an n times differentiable function on $\mathbb{R}_+$ of constant sign, $y^{(n)}(t)$ be of constant sign and not identically equal to zero in any interval $[t_0, \infty)$, $t_0 \geq 0$, and $y(t)y^{(n)}(t) \leq 0$. Then:*

i. There exists a $t_1 \geq t_0$ such that $y^{(k)}(t)$, $k = 1, \dots, n-1$, is of constant sign on $[t_1, \infty)$,

ii. There exists an integer l, $0 \leq l \leq n-1$, with $n-l$ odd, such that

$$y(t)y^{(k)}(t) > 0, k = 0, 1, \dots, l, \quad t \geq t_1, \tag{2.1}$$

$$(-1)^{n+k-1} y(t)y^{(k)}(t) > 0, k = l+1, \dots, n-1, \quad t \geq t_1, \tag{2.2}$$

and

iii.

$$|y(t)| \geq \frac{(t-t_1)^{n-1}}{(n-1)\dots(n-l)} \left| y^{(n-1)}\left(2^{n-l-1}t\right) \right|, \quad t \geq t_1. \tag{2.3}$$

Lemma 2.2 ([5]). *Let $n \geq 3$ be an odd integer, $\beta(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $0 < \beta(t) \leq \beta_0$, and $y \in C^n(\mathbb{R}_+, \mathbb{R})$ such that $(-1)^i y^{(i)}(t) > 0$, $0 \leq i \leq n-1$, and $y^{(n)}(t) \leq 0$. Then*

$$y(t - \beta(t)) \geq \frac{(\beta(t))^{n-1}}{(n-1)!} y^{(n-1)}(t), \quad \text{for } t \geq \beta_0. \tag{2.4}$$

3 Main results

Theorem 3.1. *Assume that $\phi(t)$ is a nonnegative continuous function on $\mathbb{R}_+$, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on $\mathbb{R}_+$ with:*

$$|h(t,x)| \geq \phi(t) w\left(\frac{|x|}{[\sigma(t)]^{n-1}}\right), \tag{3.1}$$

$$\int_0^{\pm\alpha} \frac{dx}{w(x)} < \infty \text{, for every } \alpha > 0, \tag{3.2}$$

and

$$\int^{\infty} \phi(t)\, dt = \infty. \tag{3.3}$$

Then:

i. If n is even, every solution $x(t)$ of equation (1.1) is oscillatory.

ii. If n is odd, every unbounded solution $x(t)$ of equation (1.1) is oscillatory.

Proof. Assume that equation (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$.

Set

$$z(t) = x(t) + g(t, x(\tau(t))). \tag{3.4}$$

By (H2), $z(t) > x(t) > 0$ for $t \geq t_0 \geq 0$.

From (1.1) and (3.4) we have

$$z^{(n)}(t) = -f(t, x(\sigma(t))) < 0. \tag{3.5}$$

Thus $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ $(i = 0, 1, \dots, n-1)$ is strictly monotonic and of constant sign eventually.

By applying Lemma 2.1, $z(t)$ satisfies (2.1) and (2.2). If n is even, the integer l associated with $z(t)$ is odd, i.e. $l \geq 1$. But if n is odd then $l \in \{0, 2, \dots, n-1\}$, and since the solution $x(t)$ is unbounded for odd orders, then $z(t)$ is unbounded, and hence $l \geq 2$. Therefore, either n is odd or even, then $l \geq 1$. Hence $z(t)$ is increasing for $t \geq t_1 \geq t_0$.

From (H3) and (3.4) we have

$$z(t) \leq x(t) + Mx(\tau(t)). \tag{3.6}$$

Let $x(s) = \max\{x(t), x(\tau(t))\}$, where $\tau(t) \leq s \leq t$ and assume that $\tau(t) \geq t_1$ for $t \geq t_2 \geq t_1$. Using (3.6) with the fact that $z(t)$ is increasing we obtain

$$z(s) \leq z(t) \leq (1+M)x(s).$$

Hence

$$x(s) \geq \frac{z(s)}{1+M},$$

or

$$x(t) \geq \frac{z(t)}{1+M}, \quad t \geq t_2. \tag{3.7}$$

From (2.3), and the fact that $z(t)$ is increasing, we have

$$z(t) \geq z(2^{l-n+1}t) \geq \frac{2^{(l-n+1)(n-1)}}{(n-1)\dots(n-l)} (t-t_3)^{n-1} z^{(n-1)}(t), \quad t \geq t_3 = 2^{n-l-1}t_1.$$

Therefore, by choosing $t_4 > t_3$, sufficiently large, we have

$$z(t) \geq ct^{n-1} z^{(n-1)}(t), \quad t \geq t_4, \tag{3.8}$$

where $c > 0$ is an appropriate constant dependent upon n and l.

Let $t_5 \geq \max\{t_2, t_4\}$ be such that $\sigma(t) \geq t_5$ for all $t \geq t_6$. From (3.7), (3.8), and the decreasing character of $z^{(n-1)}(t)$ we then have

$$\frac{x(\sigma(t))}{[\sigma(t)]^{n-1}} \geq \frac{c}{1+M} z^{(n-1)}(t), \quad t \geq t_6. \tag{3.9}$$

Using (H5), (3.1) and (3.9), it follows from (3.5) that

$$z^{(n)}(t) + \phi(t) w\left(\frac{c}{1+M} z^{(n-1)}(t)\right) k(x(\sigma(t))) \leq 0. \tag{3.10}$$

Since $z(t) > 0$ and $z'(t) > 0$ $\lim_{t\to\infty} z(t) > 0$, this implies that $\liminf_{t\to\infty} x(t) \neq 0$. Let $\epsilon > 0$ such that $x(\sigma(t)) > \epsilon$ for $t \geq t_7 \geq t_6$.

Using the fact that $k(x)$ is nondecreasing it follows from (3.10) that for $t \geq t_7$

$$z^{(n)}(t) + \phi(t) w\left(\frac{c}{1+M} z^{(n-1)}(t)\right) k(\epsilon) \leq 0. \tag{3.11}$$

Setting $u(t) = \frac{c}{1+M} z^{(n-1)}(t)$, and integrating (3.11) divided by $w(u(t))$ from t_7 to t, we obtain

$$\frac{c}{1+M} k(\epsilon) \int_{t_7}^{t} \phi(v)\, dv \leq \int_{u(t)}^{u(t_7)} \frac{ds}{w(s)}. \tag{3.12}$$

Since $u'(t) < 0$, $u(t)$ is decreasing. And since $u(t) > 0$, it follows that $\lim_{t\to\infty} u(t) = L \geq 0$. If $L \neq 0$, then by (3.10) we must have

$$\int_{t_8}^{\infty} \phi(t)\, dt < \infty, \tag{3.13}$$

which contradicts (3.3). In the case when $L = 0$, letting $t \to \infty$ in (3.12) and using (3.2), we again obtain (3.13). Thus the proof is complete. □

In the next theorem besides conditions (H1)–(H5) we further assume that:

(H6) $0 < t - \sigma(t) \leq \sigma_0$, where σ_0 is positive constant;

(H7) The constant M in (H3) is assumed to be $0 < M < 1$.

Theorem 3.2. *Assume that $\phi(t)$ is a nonnegative continuous function on $\mathbb{R}_+$, and that $w(t) > 0$ for $t > 0$ is continuous and nondecreasing on $\mathbb{R}_+$ with:*

$$|h(t,x)| \geq \phi(t) w\left(\frac{|x|}{[t-\sigma(t)]^{n-1}}\right), \tag{3.14}$$

and

$$\int_0^{\pm\alpha} \frac{dx}{w\,(x)} < \infty \text{ , for every } \alpha > 0. \tag{3.15}$$

If n is odd and

$$\int^{\infty} \phi(t)dt = \infty, \tag{3.16}$$

then every bounded solution $x(t)$ of equation (1.1) is either oscillatory or tends to zero as $t \longrightarrow \infty$.

Proof. Assume that equation (1.1) has a bounded non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_0 \geq 0$.

Set $z(t)$ as in (3.4). By (H2), $z(t) > x(t) > 0$ for $t \geq t_0 \geq 0$. Then from (1.1) and (3.4) we have (3.5). Thus $z^{(n)}(t) < 0$. It follows that $z^{(i)}(t)$ $(i = 0, 1, \ldots, n-1)$ is strictly monotonic and of constant sign eventually.

From (H3) and (3.4) we have (3.6), which implies that $z(t)$ is bounded for $t \geq t_1 \geq t_0$.

By applying Lemma 2.1, there exists a $t_2 \geq t_1$ and an integer l with $n - l$ odd such that (2.1) and (2.2) are satisfied by $z(t)$ for $t \geq t_2$. Since n is odd and $z(t)$ is bounded then $l = 0$ (otherwise $z(t)$ is not bounded). Hence from relations (2.1) and (2.2) we have

$$(-1)^i z^{(i)}\,(t) > 0, \quad i = 0, 1, \ldots, n-1. \tag{3.17}$$

Thus $z(t)$ is decreasing for $t \geq t_2$.

From (3.6) and the fact that $z(t) > x(t)$, we obtain

$$x(t) \geq z(t) - Mx(\tau(t)) \geq z(t) - Mz(\tau(t)),$$

or

$$x(t) \geq z(\tau(t)) \left[\frac{z(t)}{z(\tau(t))} - M \right]. \tag{3.18}$$

From (3.17) and from $z(t) > 0$, $z'(t) < 0$ and $z''(t) > 0$, we have $\lim_{t\to\infty} z(t) = \lambda \geq 0$. Now, we consider two cases:

<u>Case I</u>: $\lambda > 0$. Since $z(t)$ is decreasing, for every $\varepsilon > 0$ there exists $t_3 \geq t_2$ such that

$$\lambda \leq z(t) \leq z(\tau(t)) \leq \lambda + \varepsilon$$

for all $t \geq t_3$. From this we can conclude that

$$\frac{z(t)}{z(\tau(t))} \geq \frac{\lambda}{\lambda + \varepsilon}, \quad \text{for } t \geq t_3.$$

Let us choose an $\varepsilon > 0$ and $\varepsilon_1 > 0$ such that $M + \varepsilon_1 \leq \dfrac{\lambda}{\lambda + \varepsilon}$. Thus

$$\frac{z(t)}{z(\tau(t))} \geq M + \varepsilon_1, \quad t \geq t_3.$$

Using this inequality in (3.18), and the fact that $z(t)$ is decreasing, we obtain

$$x(t) \geq \varepsilon_1 z(t), \quad t \geq t_3. \tag{3.19}$$

Since n is odd, from (H6) and (3.17) we can apply Lemma 2.2. Then we have

$$z(\sigma(t)) = z(t-(t-\sigma(t))) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t), \quad t \geq t_3 + \sigma_0.$$

Hence,

$$z(\sigma(t)) \geq \frac{[t-\sigma(t)]^{n-1}}{(n-1)!} z^{(n-1)}(t), \quad t \geq t_3 + \sigma_0. \tag{3.20}$$

Let $t_4 \geq t_3 + \sigma_0$ be such that $\sigma(t) \geq t_4$ for all $t \geq t_5$. From (3.19), and (3.20) we have

$$\frac{x(\sigma(t))}{[t-\sigma(t)]^{n-1}} \geq \frac{\varepsilon_1 z^{(n-1)}(t)}{(n-1)!}, \quad t \geq t_5. \tag{3.21}$$

Using (3.14), and (3.21), then it follows from (3.5) that

$$z^{(n)}(t) + \phi(t)\, w\left(c z^{(n-1)}(t)\right) k(x(\sigma(t))) \leq 0,$$

where $c = \dfrac{\varepsilon_1}{(n-1)!}$. Proceeding as in the proof of Theorem 3.1, we obtain a contradiction.

Case II: $\lambda = 0$, since $x(t) \leq z(t)$, $x(t)$ tends to zero as $t \longrightarrow \infty$, and this completes the proof.

□

4 Examples

Example 4.1. Consider the equation

$$\left[x(t) + (5+\sin^2 t)^2 \frac{x(t-2\pi)}{5+x^2(t-2\pi)}\right]'' + 9x^3(t-4\pi) = 0. \tag{4.1}$$

Here $|g(t,x)| = (5+\sin^2 t)^2 \left|\frac{x}{5+x^2}\right| < 7.2|x|$. By setting $h(t,x) = 9x^{\frac{1}{3}}$, $k(x) = x^{\frac{8}{3}}$, $w(x) = x^{\frac{1}{3}}$, $\phi(t) = 9(t-4\pi)^{\frac{1}{3}}$, we can see that all conditions of Theorem 3.1 are satisfied. Thus every solution of equation (4.1) is oscillatory. It is easy to check that $x(t) = \sin t$ is such a solution.

Example 4.2. Consider the equation

$$\left[x(t) + e^{2\pi}(4+e^{-t})x(t-2\pi)\right]^{(4)} + e^{2\pi}(20-e^{-t})x(t-2\pi) = 0. \tag{4.2}$$

Here $|g(t,x)| = e^{2\pi}(4+e^{-t})\,|x| \leq 5e^{2\pi}|x|$. If we take $h(t,x) = e^{2\pi}(20-e^{-t})x^{\frac{1}{3}}$, $k(x) = x^{\frac{2}{3}}$, $w(x) = x^{\frac{1}{3}}$, $\phi(t) = e^{2\pi}(20-e^{-t})(t-2\pi)$, we conclude from Theorem 3.1 that every solution of equation (4.2) is oscillatory. In fact, $x(t) = e^t \sin t$ is such an unbounded solution.

Example 4.3. Consider the equation

$$\left[x(t) + e^{\frac{3\pi}{2}}(1+e^{-t})x\left(t-\frac{3\pi}{2}\right)\right]''' + e^{2\pi}(4-e^{-t})x(t-2\pi) = 0. \tag{4.3}$$

Here $|g(t,x)| = e^{\frac{3\pi}{2}}(1+e^{-t})\,|x| \leq 2e^{\frac{3\pi}{2}}|x|$. By setting $h(t,x) = e^{2\pi}(4-e^{-t})x^{\frac{1}{3}}$, $k(x) = x^{\frac{2}{3}}$, $w(x) = x^{\frac{1}{3}}$, $\phi(t) = e^{2\pi}(4-e^{-t})(t-2\pi)^{\frac{2}{3}}$, we see that all conditions of Theorem 3.1 are satisfied. Hence every unbounded solution of equation (4.3) is oscillatory. Indeed, $x(t) = e^t \sin t$ is such an unbounded solution of the equation.

Example 4.4. Consider the equation

$$\left[x(t) + \frac{1+\sin^2 t}{4} \frac{x(t-2\pi)}{1+x^2(t-2\pi)}\right]''' + \frac{5}{4} x\left(t - \frac{3\pi}{2}\right) = 0. \tag{4.4}$$

Here $|g(t,x)| = \frac{1+\sin^2 t}{4} \left|\frac{x}{1+x^2}\right| < \frac{1}{2}|x|$. By taking $h(t,x) = \frac{5}{4}x^{\frac{1}{3}}$, $k(x) = x^{\frac{2}{3}}$, $w(x) = x^{\frac{1}{3}}$, $\phi(t) = \frac{5}{4}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}$, all conditions of Theorem 3.2 are satisfied. Thus every bounded solution of equation (4.4) is either oscillatory or tends to zero as $t \longrightarrow \infty$. In fact, $x(t) = \sin t$ is an oscillatory solution of the equation.

Example 4.5. Consider the equation

$$\left[x(t) + \frac{e^{-t}}{2} x\left(\frac{t}{2} - \ln 2\right)\right]''' + 24e^{4t-6}x^3(t-1) = 0. \tag{4.5}$$

Here $|g(t,x)| = \frac{e^{-t}}{2}|x| \leq \frac{1}{2}|x|$. All conditions of Theorem 3.2 are satisfied with $h(t,x) = 24e^{4t-6}x^{\frac{1}{3}}$, $k(x) = x^{\frac{8}{3}}$, $w(x) = x^{\frac{1}{3}}$, $\phi(t) = 24e^{4t-3}$. So every bounded solution of equation (4.5) is either oscillatory or tends to zero as $t \longrightarrow \infty$. Indeed, $x(t) = e^{-2t}$ is a solution that tends to zero as $t \longrightarrow \infty$.

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