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ON THE INTEGRAL CHARACTERIZATION OF PRINCIPAL SOLUTIONS FOR HALF-LINEAR ODE

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Abstract. We discuss a new integral characterization of principal solutions for half-linear differential equations, introduced in the recent paper of S. Fišnarová and R. Mařik, Nonlinear Anal. 74 (2011), 6427–6433. We study this characterization in the framework of the existing results and we show when this new integral characterization with a parameter α is equivalent with two extremal cases of the integral characterization used in the literature. We illustrate our results on the Euler and Riemann-Weber differential equations.

1. Introduction

Consider the half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) := |x|^{p-1}\operatorname{sgn} x, \tag{1}$$

where $p > 1$, r, c are continuous functions on $[t_0, \infty)$ and $r(t) > 0$. Denote by q the conjugate number to p, i.e. $q = p/(p-1)$ and set

$$J_r = \int^{\infty} \frac{dt}{r^{q-1}(t)}, \quad J_c = \int^{\infty} |c(t)|\, dt.$$

It is well-known, see, e.g., [9, Theorem 1.2.3], that if c is positive and both integrals J_r and J_c are divergent, then (1) is oscillatory. Throughout the paper we suppose that (1) is nonoscillatory, that is, all its solutions are either positive or negative for large t. Since the solution space of (1) is homogeneous, we consider its positive solutions only.

Some recent trends in the qualitative theory of ODE's consist in the extension of properties of linear second order Sturm-Liouville equations, see, e.g., [9]. One of them is related to the notion of principal solution of (1). More precisely, when (1) is nonoscillatory, following [13, 16], a nontrivial solution h of (1) is called *a principal solution* if for every nontrivial solution x of (1) such that $x \neq \lambda h$, $\lambda \in \mathbb{R}$, we have

$$\frac{h'(t)}{h(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t. \tag{2}$$

As in the linear case, a principal solution h exists and it is unique up to a nonzero constant multiplicative factor. For this reason, in the following we will denote it by *the principal*

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solution. Any nontrivial solution $x \neq \lambda h$ is called *a nonprincipal solution.* The problem of an integral characterization of principal solutions of (1), similar to that in the linear case, was initiated in 1988 when Mirzov's paper [16] was published, see also [7, 9] for more details. For instance, in [8], see also [7, Proposition 2], the following integral characterization of the principal solution has been suggested as an extension of that given in the linear case, see [14, Chap. XI].

Theorem A. *Let* (1) *be nonoscillatory and* h *be its positive solution satisfying* $h'(t) \neq 0$ *for large* t. *Then:*

(i) *Let* $p \in (1, 2]$. *If*

$$Q := \int^{\infty} \frac{dt}{r(t)h^2(t)|h'(t)|^{p-2}} = \infty. \tag{3}$$

holds, then h *is the principal solution of* (1).

(ii) *Let* $p \geq 2$. *If* h *is the principal solution of* (1), *then* $Q = \infty$.

(iii) *Suppose that* $p \geq 2$, $J_r = \infty$, *the function* $\gamma(t) := \int_t^{\infty} c(s)\, ds$ *exists, and* $\gamma(t) \geq 0$, *but* $\gamma(t) \not\equiv 0$ *for large* t. *Then* h *is the principal solution of* (1) *if and only if* $Q = \infty$.

Note that Theorem A-(iii) was stated in [8] without the assumption $p \geq 2$. When $c(t) > 0$, the implication

$$h \text{ is the principal solution} \quad \Longrightarrow \quad Q = \infty \tag{4}$$

may fail to hold for $p \in (1, 2)$ as Example 2 below shows, see also an example in [7].

When c is negative for large t, in [1, Theorem 3.2] it is shown that a solution h of (1) is the principal solution if and only if

$$\int^{\infty} \frac{dt}{r^{q-1}(t)h^2(t)} = \infty. \tag{5}$$

Later on, in [3, Theorem 7], it is proved that (5) is necessary and sufficient for h to be the principal solution of (1) when $J_r < \infty$ and $J_c < \infty$, independently of the sign of c. As it follows from the proof of [3, Theorem 7], when $J_r < \infty$ and $J_c < \infty$, condition (5) is equivalent with

$$N := \int^{\infty} \frac{dt}{r^{q-1}(t)h^q(t)} = \infty. \tag{6}$$

Finally, other contributions to the problem of integral characterizations of principal solutions can be found in [2, 4, 7, 16]. Observe that in these papers, under some additional conditions, either sufficient conditions or necessary conditions are presented.

As a reaction on the fact that the implication (4) does not generally hold when c is eventually positive and $p \in (1, 2)$, the following alternative integral characterization has been proposed recently in [11, Theorem 4.1].

Theorem B. *Let* (1) *be nonoscillatory and* h *be its positive solution satisfying* $h'(t) \neq 0$ *for large* t.

(i) *Let* $p \in (1,2]$. *If*

$$Q^{[\alpha]} := \int^{\infty} \frac{dt}{r^{\alpha-1}(t) h^{\alpha}(t) |h'(t)|^{(p-1)(\alpha-q)}} = \infty \tag{7}$$

holds for some $\alpha \in [2,q]$, *then* h *is the principal solution.*

(ii) *Let* $p \geq 2$. *If* h *is the principal solution, then* (7) *holds for every* $\alpha \in [q,2]$.

For the extremal cases of the parameter α, namely for $\alpha = 2$ and $\alpha = q$, we have

$$Q^{[2]} = Q \quad \text{and} \quad Q^{[q]} = N,$$

where Q is given by (3) and N by (6). Hence, the integral characterization $Q^{[\alpha]}$ creates, roughly speaking, a bridge between the characterizations Q and N. Moreover, the role of the parameter α in Theorem B suggests that the situation is different for $p \leq 2$ (in which $Q^{[\alpha]}$ may diverge for *some* α) and $p \geq 2$ (in which $Q^{[\alpha]}$ may diverge for *every* α).

Nevertheless, when the function c changes its sign, (1) can have positive solutions with changing-sign derivatives, as the following example illustrates, and this fact makes Theorem B inapplicable.

Example 1. Consider the equation

$$((x')^3)' + \frac{3\sin^2 t \cos t}{(\cos t - 2)^3}\, x^3 = 0, \tag{8}$$

Since $x(t) = 2 - \cos t$ is a solution of (8) and x' does not have a fixed sign, Theorem B cannot be used.

The main aim here is to show that, when $c(t) > 0$ for large t, the new characterization $Q^{[\alpha]}$ introduced in [11] is equivalent with two extremal cases, that is the integrals Q or N. As a consequence, we will obtain that the opposite implication in the claim (ii) of Theorem B is valid under an additional condition. The obtained results are illustrated by two critical cases, namely the half-linear Euler and Riemann-Weber differential equations. Finally, an application of the obtained results to the so-called reciprocal equation completes the paper.

2. Main results

Here we study the integral characterizations Q, N and $Q^{[\alpha]}$ of principal solutions defined by (3), (6) and (7), respectively.

When $p > 2$, $J_r = \infty$, $J_c < \infty$ and $c(t) > 0$ for large t, in view of Theorem A-(iii), the integral Q gives a necessary and sufficient condition for h to be a principal solution of (1). For this reason, throughout the paper we assume

$$1 < p < 2, \quad c(t) > 0 \text{ for large } t, \quad J_r = \infty, \quad J_c < \infty. \tag{H1}$$

In view of (H1) and Theorem A, the problem of integral characterization of principal solutions of (1) reduces to the problem whether at least one of the integrals Q, N and $Q^{[\alpha]}$ diverges when h is the principal solution of (1).

Let h be a solution of (1) and denote by $h^{[1]}$ its quasi-derivative, i.e. $h^{[1]}(t) = r(t)\Phi(h'(t))$. Consider the function

$$G_h(t) = h(t)h^{[1]}(t).$$

Since $J_r = \infty$, any eventually positive solution h of (1) satisfies $h'(t) > 0$ for large t. Indeed $h^{[1]}$ is decreasing for large t, say $t \geq T$; if there exists $t_1 > T$ such that $h^{[1]}(t_1) < 0$, we obtain $h^{[1]}(t) < h^{[1]}(t_1) < 0$ for $t \geq t_1$, or

$$h(t) < h^{[1]}(t_1) \int_{t_1}^{t} \frac{1}{r^{q-1}(s)} ds,$$

which contradicts the positiveness of h. Hence, also the function G_h is positive for large t. The role of the function G_h is given by the following result.

Lemma 1. [4, Lemma 1] *Assume that $J_r = \infty$, $c(t) > 0$ for large t and* (1) *is nonoscillatory. If x is a nonprincipal solution of* (1)*, then*

$$\limsup_{t\to\infty} G_x(t) = \infty.$$

Our next lemma shows how the integral characterizations Q, N and $Q^{[\alpha]}$ can be formulated in terms of the function G.

Lemma 2. *Let* (1) *be nonoscillatory and h be its positive solution satisfying $h'(t) \neq 0$ for large t. Then the following identities hold:*

$$Q^{[\alpha]} = \int^{\infty} \frac{h'(t)}{h(t)\big(G_h(t)\big)^{\alpha-1}}\, dt,$$

$$Q = \int^{\infty} \frac{h'(t)}{h(t)G_h(t)}\, dt, \quad N = \int^{\infty} \frac{h'(t)}{h(t)\big(G_h(t)\big)^{q-1}}\, dt.$$

Proof. The assertion follows by a direct calculation. □

Hence, if h is a nonoscillatory solution of (1) such that $\lim_{t\to\infty} G_h(t) = c$, where $c > 0$, then integrals Q, N and $Q^{[\alpha]}$ have the same behavior, i.e. either all are divergent or all are convergent. In the remaining cases when $\lim_{t\to\infty} G_h(t)$ is zero or infinity, the following inequalities hold.

Theorem 1. *Assume* (H1)*. Let* (1) *be nonoscillatory and h be its solution.*

(i) *If $\lim_{t\to\infty} G_h(t) = 0$, then for every $\alpha \in [2, q]$ we have*

$$N \geq Q^{[\alpha]} \geq Q.$$

(ii) *If $\lim_{t\to\infty} G_h(t) = \infty$, then for every $\alpha \in [2, q]$ we have*

$$N \leq Q^{[\alpha]} \leq Q. \tag{9}$$

Proof. Claim (i). Since for every $\alpha \in [2, q]$ it holds $1 \le \alpha - 1 \le q - 1$, we have for large t

$$\frac{1}{G_h(t)} \le \Big(\frac{1}{G_h(t)}\Big)^{\alpha-1} \le \Big(\frac{1}{G_h(t)}\Big)^{q-1}$$

and from Lemma 2 the assertion follows.

Claim (ii) can be proved using a similar argument. □

A partial answer to the question when the implication (4) holds is given by the following theorem.

Theorem 2. *Assume* (H1). *Let* (1) *be nonoscillatory and h be its solution.*

(i) *If h is unbounded and*

$$\limsup_{t\to\infty} G_h(t) < \infty, \tag{10}$$

then h is the principal solution and $Q^{[\alpha]} = \infty$ for every $\alpha \in [2, q]$; in particular $Q = \infty$ and $N = \infty$.

(ii) *If h is bounded, then h is the principal solution and $N = \infty$. Moreover, $\lim_{t\to\infty} G_h(t) = 0$ and*

$$\int^{\infty} \frac{1}{r^{q-1}(t)}\Big(\int_t^{\infty} c(s)\,ds\Big)^{q-1} dt < \infty. \tag{11}$$

(iii) *If h is bounded and, in addition,*

$$\int^{\infty} c(t)\Big(\int^{t} \frac{ds}{r^{q-1}(s)}\Big)^{p-1} dt = \infty, \tag{12}$$

then $Q^{[\alpha]} = \infty$ for every $\alpha \in [2, q]$; in particular $Q = \infty$.

To prove this theorem, the following lemma will be needed.

Lemma 3. [5, Lemma 1] *Let a, b be continuous positive functions on $[T, \infty)$, $\int_T^{\infty} b(t)\,dt < \infty$, and λ, μ be real positive constants. If $\mu > \lambda$ and*

$$\int_T^{\infty} b(t)\Big(\int_T^{t} a(s)\,ds\Big)^{\lambda} dt = \infty,$$

then

$$\int_T^{\infty} a(t)\Big(\int_t^{\infty} b(s)\,ds\Big)^{1/\mu} dt = \infty.$$

Proof of Theorem 2. Claim (i). By [4, Lemma 1], h is the principal solution. The second conclusion follows immediately from Lemma 2.

Claim (ii). By [4, Corollary 1], h is the principal solution. Since h is eventually increasing and $J_r = \infty$, from (6) we get $N = \infty$. Since h is bounded and $h^{[1]}$ is eventually positive decreasing, we have $\lim_{t\to\infty} h^{[1]}(t) = 0$. Indeed, if $\lim_{t\to\infty} h^{[1]}(t) = d > 0$, then $h^{[1]}(t) > d$

for large t and integrating this inequality, we get a contradiction with the boundedness of h. Finally, the statement (11) follows from [15, Theorem 4.2].

Claim (iii). Assume (12). We have for large t

$$\Phi(h'(t)) = \frac{1}{r(t)} \int_t^\infty c(s)\Phi(h(s))ds$$

or

$$(h'(t))^{2-p} = \left(\frac{1}{r(t)} \int_t^\infty c(s)\Phi(h(s))ds \right)^{(2-p)/(p-1)}.$$

Since h is eventually increasing, there exists $k_1 > 0$ such that we have for large t

$$(h'(t))^{2-p} \geq k_1 \left(\frac{1}{r(t)} \int_t^\infty c(s)ds \right)^{(2-p)/(p-1)}$$

or, because $q - 1 = (p-1)^{-1}$,

$$\frac{(h'(t))^{2-p}}{r(t)} \geq k_1 \left(\frac{1}{r(t)} \right)^{q-1} \left(\int_t^\infty c(s)ds \right)^{(2-p)/(p-1)}. \tag{13}$$

Set $\mu = (p-1)/(2-p)$ and $\lambda = p - 1$. Since $1 < p < 2$, we get $\mu > \lambda$. Thus from (12) and Lemma 3 with

$$a(t) = r^{1-q}(t), \quad b(t) = c(t),$$

we obtain

$$\int_0^\infty \frac{1}{r^{q-1}(t)} \left(\int_t^\infty c(s)\, ds \right)^{(2-p)/(p-1)} dt = \infty.$$

Thus from here and (13) we get

$$\int_T^\infty \frac{dt}{r(t)\,(h'(t))^{p-2}} = \int_T^\infty \frac{(h'(t))^{2-p}}{r(t)} dt = \infty.$$

Since h is increasing and bounded, we have $Q = \infty$. Finally, since $\lim_{t\to\infty} G_h(t) = 0$, from Theorem 1 we obtain $Q^{[\alpha]} = \infty$ for every $\alpha \in [2, q]$. □

Remark 1. Theorem 2 assumes boundedness of solutions of (1). When c is eventually positive, $I_r = \infty$ and $I_c < \infty$, a necessary and sufficient condition for (1) to have bounded (principal) solutions is well-known, see, e.g. [6, Theorem 4-i$_1$)], and reads as follows: *Assume* (1) *nonoscillatory,* $c(t) > 0$ *for large* t*,* $J_r = \infty$ *and* $J_c < \infty$*. Then* (1) *has bounded solutions if and only if* (11) *holds.*

When (H1) holds, we get the following improvement of Theorem B.

Corollary 1. *Assume* (H1) *and* (11)*. Then a solution* h *of* (1) *is the principal solution if and only if* $Q^{[\alpha]} = \infty$ *for some* $\alpha \in [2, q]$.

Proof. Since (11) holds, equation (1) has a bounded nonoscillatory solution, see [15, Theorem 4.1]. Now the assertion follows from Theorem 2-(ii) and Theorem B. □

The following example is taken from [2, Corrigendum] and shows that the implication (4) may fail to hold not only for Q but also for $Q^{[\alpha]}$ with $\alpha \in [2, q)$.

Example 2. Consider the equation

$$(\Phi(x'))' + e^{-t}\Phi(x) = 0 \tag{14}$$

with $1 < p < 2$. This equation is nonoscillatory and has both bounded and unbounded solutions, as it follows, for instance, from [15, Theorems 4.1,4.2]. If h is the principal solution of (14), in view of [6, Theorem 2-(i$_1$)], h is bounded and h' is eventually positive and satisfies $\lim_{t\to\infty} h'(t) = 0$. Integrating (14) for large t we have

$$\Phi(h'(t)) = \int_t^\infty e^{-s}\Phi(h(s))ds.$$

Thus, in virtue of the boundedness of h, there exist two positive constants $k_1 < k_2$ such that for large t

$$k_1 e^{-t/(p-1)} < h'(t) < k_2 e^{-t/(p-1)}.$$

Hence, we have as $t \to \infty$

$$Q \sim \int^\infty (h'(t))^{2-p}\, dt < \infty.$$

Notice that the same happens for $Q^{[\alpha]}$ with $\alpha \in [2, q)$, since a standard calculation gives as $t \to \infty$

$$\frac{h'(t)}{h(t)\big(G_h(t)\big)^{\alpha-1}} \sim e^{-(q-\alpha)t}\ .$$

Thus, from Lemma 2 we get

$$Q^{[\alpha]} < \infty \text{ for every } \alpha \in [2, q).$$

On the other hand, $N = Q^{[q]} = \infty$. This fact also illustrates how, in general, in Corollary 1 the stronger statement "$Q^{[\alpha]} = \infty$ *for every* $\alpha \in [2, q]$" can fail.

Remark 2. Let h be a solution of (1). When $\lim_{t\to\infty} G_h(t) = 0$, in view of Theorem 1 we have

$$Q = \infty \implies N = \infty \text{ and } Q^{[\alpha]} = \infty \text{ for every } \alpha \in [2, q]. \tag{15}$$

Observe that $\lim_{t\to\infty} G_h(t) = 0$ may occur not only when the principal solution h is bounded, but also when every solution of (1) is unbounded and the Euler half-linear equation discussed in the next section is a typical example.

We will show also that the principal solution can satisfy $\lim_{t\to\infty} G_h(t) = \infty$ and the implication (15) can fail. A typical example of this fact is the Riemann-Weber equation which will also be studied in the following section.

3. Examples

The following example illustrates Theorem 2.

Example 3. Consider the half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0 \quad (\gamma > 0). \tag{16}$$

First, let $\gamma = \gamma_p$ where γ_p is the critical oscillation constant $\gamma_p = ((p-1)/p)^p$. In this case $h(t) = t^{(p-1)/p}$ is a solution of (16) and any linearly independent solution x behaves asymptotically (up to a multiplicative factor) as the function $t^{(p-1)/p}\log^{2/p} t$, see [9, Section 1.4.2]. Moreover,

$$\frac{th'(t)}{h(t)} = \frac{p-1}{p}, \qquad \frac{tx'(t)}{x(t)} \sim \frac{p-1}{p} + \frac{2}{p\log t} \quad \text{as } t \to \infty.$$

Hence, from (2) h is the principal solution of (16) and

$$G_h(t) = ((p-1)/p)^{p-1}.$$

By Theorem 2 we obtain $Q = \infty$ and $N = \infty$, i.e., $Q^{[\alpha]} = \infty$ for every $\alpha \in [2, q]$. Clearly, this result can be also verified by a direct computation.

Now let $\gamma < \gamma_p$. In this case, again from [9, Section 1.4.2], the function

$$F(v) = |v|^p - \Phi(v) + \gamma/(p-1)$$

has two real roots, namely λ_1, λ_2 with $\lambda_1 < (p-1)/p < \lambda_2$. Moreover, $h(t) = t^{\lambda_1}$ is the principal solution of (16) and any linearly independent solution x behaves asymptotically as t^{λ_2} (again up to a multiplicative factor). We have

$$\lim_{t\to\infty} G_h(t) = 0,$$

thus, by Theorem 2, we obtain $Q = \infty$ and $N = \infty$, i.e., $Q^{[\alpha]} = \infty$ for every $\alpha \in [2, q]$. The same conclusion follows by a direct computation observing that $\lambda_1 q < 1$.

The following example presents a typical equation for which $\lim_{t\to\infty} G_h(t) = \infty$ and $N < \infty$ for every solution.

Example 4. Consider the half-linear Riemann-Weber differential equation

$$\big(\Phi(x')\big)' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right]\Phi(x) = 0, \tag{17}$$

where

$$\gamma_p = \left(\frac{p-1}{p}\right)^p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

By [10, Corollary 1], equation (17) has a solution satisfying

$$h(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t\big(1 + o(\log^{-1} t)\big) \text{ as } t \to \infty,$$

and every linearly independent solution x satisfies

$$x(t) \sim Ct^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \log^{\frac{2}{p}} (\log t), \; C \in \mathbb{R}, \quad \text{as } t \to \infty.$$

Moreover, by [10, Theorem 4] with

$$\delta(t) = \frac{1}{4 \log^2 t}, \quad z_h(t) = \sqrt{t}, \quad z_x(t) = \sqrt{t} \log t,$$

i.e. (in notation of [10])

$$\xi_h(t) = \frac{z_h'(t)}{z_h(t)} = \frac{1}{2t}, \quad \xi_x = \frac{z_x'(t)}{z_x(t)} = \frac{1}{2t}\left(1 + \frac{2}{\log t}\right),$$

we have as $t \to \infty$

$$\frac{th'(t)}{h(t)} \sim \frac{p-1}{p} + \frac{1}{p \log t}, \quad \frac{tx'(t)}{x(t)} \sim \frac{p-1}{p} + \frac{1}{p \log t} + \frac{2}{p \log(\log t)}.$$

Hence, from (2) h is the principal solution of (17). Using the fact that

$$G_h(t) = h^p(t) \left(\frac{h'(t)}{h(t)}\right)^{p-1},$$

we have $G_h(t) \sim \log t \to \infty$ as $t \to \infty$. From here and Lemma 1 we get $Q = \infty$. Moreover,

$$N = \int^\infty \frac{1}{h^q(t)}\, dt \sim \int^\infty \frac{1}{(t^{p-1} \log t)^{1/(p-1)}}\, dt = \int^\infty \frac{dt}{t \log^{q-1} t},$$

and so, because $p < 2$, i.e. $q > 2$, we have $N < \infty$.

Summarizing, the integral characterization Q of the principal solution of (1) remains an open problem when all solutions h of (1) satisfy $\lim G_h(t) = \infty$. By Theorem 2-(ii), this means that all solutions are unbounded and (11) does not hold. In view of Example 2, we conjecture:

Conjecture. Assume (H1) and

$$\int^\infty \frac{1}{r^{q-1}(t)} \left(\int_t^\infty c(s)\, ds\right)^{q-1} dt = \infty. \tag{18}$$

Then the implication (4) holds.

4. An application

Here, as an application of the above result, we discuss the opposite case to (H1), namely the case

$$c(t) > 0 \;\text{ for large } t, \quad J_r < \infty, \quad J_c = \infty. \tag{H2}$$

Then Theorem A-(iii) is not applicable, but this case can be treated in a similar way by means of the so-called reciprocity principle, see [2, Section 2] or [6, Section 3]. Consider the so called reciprocal equation

$$(c^{1-q}(t)\Phi^*(y'))' + r^{1-q}(t)\Phi^*(y) = 0, \qquad \Phi^*(y) := |y|^{q-1}\operatorname{sgn} y, \tag{19}$$

which is obtained from (1), by interchanging the functions r, c with c^{1-q}, r^{1-q}, respectively, and replacing the index p with its conjugate q. Observe that in (19) the role of the integral J_r is played by J_c and vice-versa. Moreover, for any solution y of (19), the quasi-derivative $y^{[1]} = c^{1-q}(t)\Phi^*(y')$ is solution of (1) and vice-versa, for any solution x of (1), the quasi-derivative $x^{[1]}$ is a solution of (19). Using the property that h is the principal solution of (1) if and only if $\tilde{h} = h^{[1]}$ is the principal solution of (19), see, e.g., [2, Proposition 1], the above results can be easily formulated also when the case (H2) occurs.

Let y be a solution of (19). Then it is easy to verify that the integral characterizations $Q, N, Q^{[\alpha]}$ read for (19) as follows.

$$\begin{aligned} \widetilde{Q} &= \int^{\infty} \frac{dt}{c^{1-q}(t)y^2(t)|y'(t)|^{q-2}}, \\ \widetilde{N} &= \int^{\infty} \frac{c(t)}{y^p(t)}dt, \\ \widetilde{Q}^{[\alpha]} &= \int^{\infty} \frac{dt}{c^{(1-q)(\alpha-1)}(t)y^{\alpha}(t)|y'(t)|^{(q-1)(\alpha-p)}}, \end{aligned}$$

respectively. Since, as already claimed, for any solution y of (19), the function $y^{[1]}$ is solution of (1), a standard calculation yields that the integrals $\widetilde{Q}, \widetilde{N}, \widetilde{Q}^{[\alpha]}$ can be written as

$$\begin{aligned} R &= \int^{\infty} \frac{c(t)\Phi(h(t))}{h(t)(h^{[1]}(t))^2}\, dt, \\ P &= \int^{\infty} \frac{c(t)}{(h^{[1]}(t))^p}dt, \\ R^{[\alpha]} &= \int^{\infty} \frac{dt}{c^{(1-q)(\alpha-1)}(t)y^{\alpha}(t)|y'(t)|^{(q-1)(\alpha-p)}}, \end{aligned}$$

respectively, where h is a solution of (1).

Thus, the reciprocity principle leads to other possible integral characterizations of principal solutions of (1), namely the integrals $R, P, R^{[\alpha]}$. Similarly to the previous situation, roughly speaking the integral $R^{[\alpha]}$ is a bridge between R and P, because

$$R^{[2]} = R \quad \text{and} \quad R^{[p]} = P.$$

Applying Theorem A-(iii) to the reciprocal equation (19) we get the following result.

Corollary 2. *Let* (1) *be nonoscillatory,* $1 < p < 2$, *and* (H2) *hold. Then* h *is the principal solution of* (1) *if and only if* $R = \infty$.

Moreover, when the case (H2) holds, any solution x of (1) satisfies $x(t)x^{[1]} < 0$ for large t, see, e.g., [4, Proposition 2]. Hence the function G_x is negative for large t. Since $y = x^{[1]}$ is solution of (19) and

$$x(t)x^{[1]} = -y(t)y^{[1]}(t), \tag{20}$$

in studying the integral characterization of principal solutions, the role of the function G remains almost the same as the one illustrated in Section 2 for the case $J_r = \infty$, $J_c < \infty$. More precisely, the following holds.

Lemma 4. *Let* (1) *be nonoscillatory and* (H2) *hold. If x is a nonprincipal solution of* (1), *then*

$$\liminf_{t\to\infty} G_x(t) = -\infty.$$

Proof. The assertion follows from (20) and Lemma 1. □

When $p > 2$ and the case (H2) occurs, the following extension of Theorem 2 holds.

Theorem 3. *Assume $p > 2$ and* (H2)*. Let* (1) *be nonoscillatory and h be its solution.*

(i) *If $h^{[1]}$ is unbounded and*

$$\liminf_{t\to\infty} G_h(t) > -\infty,$$

then h is the principal solution and $R^{[\alpha]} = \infty$ for every $\alpha \in [2, p]$; in particular $R = \infty$ and $P = \infty$.

(ii) *If $h^{[1]}$ is bounded, then h is the principal solution and $P = \infty$. Moreover,* $\lim_{t\to\infty} G_h(t) = 0$ *and*

$$\int^{\infty} c(t)\Big(\int_t^{\infty} r^{1-q}(s)\, ds\Big)^{p-1} dt < \infty. \tag{21}$$

(iii) *If $h^{[1]}$ is bounded and, in addition,*

$$\int^{\infty} r^{1-q}(t)\,\Big(\int^{t} c(s) ds\Big)^{q-1} dt = \infty,$$

then $R^{[\alpha]} = \infty$ for every $\alpha \in [2, p]$; in particular $R = \infty$.

Proof. Consider the reciprocal equation (19). Since $p > 2$, we have $1 < q < 2$. Hence, the assertion follows by applying Theorem 2 to (19) and using [2, Proposition 1], with minor changes. The details are left to the reader. □

Remark 3. The integral characterization R has been already considered in [4]. Hence Theorem 3 complements [4, Theorem 3-i$_2$),Theorem 4]. As follows from the proof of Theorem 3 in [4], if $J_r = \infty$, then $Q \le R$ for any solution of (1). This inequality completes (9).

Remark 4. Theorem 3 assumes boundedness of quasi-derivatives of solutions of (1). When (H2) holds, a necessary and sufficient condition on this topic is well-known, see, e.g. [6, Theorem 4-(i_2)], and reads as follows: *Assume* (H2) *and* (1) *nonoscillatory. Then* (1) *has solutions with bounded quasi-derivative if and only if* (21) *holds.*

Similarly to Corollary 1, the following result gives another improvement of Theorem B.

Corollary 3. *Assume* (H2), $p > 2$ *and* (21). *Then a solution* h *of* (1) *is the principal solution if and only if* $R^{[\alpha]} = \infty$ *for some* $\alpha \in [2, p]$.

Proof. Since (21) holds, equation (1) has nonoscillatory solutions with bounded quasiderivative, see e.g., [4, Theorem A] or [15, Theorem 4.1]. Now the assertion follows applying Theorem 2-(ii) and Theorem B to the reciprocal equation (19) and using again [2, Proposition 1], with minor changes. □

Analogously, if (H2) holds, it remains an open problem the statement *"If* $Q = \infty$*, then* h *is the principal solution"* when all solutions h of (1) tend to zero and $|G_h(t)|$ is unbounded. Thus Conjecture 1 reads as

Conjecture 1'. Assume (H2) and

$$\int^{\infty} c(t)\Big(\int_t^{\infty} r^{1-q}(s)\, ds\Big)^{p-1} dt = \infty. \tag{22}$$

If $Q = \infty$, then h is the principal solution of (1).

We conclude this paper by summarizing integral characterizations P, R in terms of the function G_h.

Corollary 4. *Let* (1) *be nonoscillatory and either* $J_r = \infty$ *or* $J_c = \infty$. *In addition, if* (18) *holds, suppose* $p \geq 2$ *and if* (22) *holds, suppose* $1 < p \leq 2$. *Then a solution* h *of* (1) *is the principal solution if and only if*

$$\int^{\infty} \frac{r(t)(h'(t))^p + c(t)(h(t))^p}{G_h^2(t)}\, dt = \infty.$$

Proof. By [4, Corollary 2], a solution h of (1) is principal if and only if $P + R = \infty$. The integral R can be written using the function G_h as

$$R = \int^{\infty} \frac{c(t)}{r(t)} \frac{\Phi(h(t)}{\Phi(h'(t))} \frac{1}{G_h(t)}\, dt, \tag{23}$$

so from here and Lemma 2, we get the conclusion. □

Note added in proof. After this paper was written, we pointed out that the integral characterization $Q^{[\alpha]}$ introduced in [11] is discussed also in [12]. When c is positive for large t, then Theorem 2 extends [12, Corollary 1] where additional assumptions on h are posed.

The interesting case is when $\lim_{t\to\infty} G(t) = \infty$ for all solutions of (1), as Riemann-Weber equation illustrates, but this case is not treated in [12].

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