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Asymptotic Behavior of Positive Solutions of a Class of Systems of Second Order Nonlinear Differential Equations

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Abstract. The two-dimensional system of nonlinear differential equations

$$x'' = p(t)y^{\alpha}, \quad y'' = q(t)x^{\beta}, \tag{A}$$

with positive exponents α and β satisfying $\alpha\beta < 1$ is analyzed in the framework of regular variation. Under the assumption that $p(t)$ and $q(t)$ are nearly regularly varying it is shown that system (A) may possess three types of positive solutions $(x(t), y(t))$ which are strongly monotone in the sense that (i) both components are strongly decreasing, (ii) both components are strongly increasing, and (iii) one of the components is strongly decreasing, while the other is strongly increasing. The solutions in question are sought in the three classes of nearly regularly varying functions of positive or negative indices. It is also shown that if we make a stronger assumption that $p(t)$ and $q(t)$ are regularly varying, then the solutions from the above three classes are fully regularly varying functions, too.

Key words. systems of nonlinear differential equations, positive solutions, strongly increasing (decreasing) solutions, asymptotic behavior, regularly varying functions.

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1 Introduction

This paper is concerned with positive solutions of the two-dimensional system of second order nonlinear differential equations

$$x'' = p(t)y^{\alpha}, \quad y'' = q(t)x^{\beta}, \tag{A}$$

where

(a) α and β are positive constants;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[t_0, \infty)$, $t_0 > 0$.

By a positive solution of (A) we mean a vector function $(x(t), y(t))$ on an interval of the form $[T, \infty)$, $T \geq t_0$, with positive components $x(t)$ and $y(t)$ satisfying system (A) for $t \geq T$. Our aim is to acquire as precise information as possible about the asymptotic behavior of positive solutions of (A). Let $(x(t), y(t))$ be a positive solution of (A) existing on $[T, \infty)$. Then, we see from (A) that $x''(t) > 0$ and $y''(t) > 0$, so that $x'(t)$ and $y'(t)$ are increasing for $t \geq T$ and tend to finite or infinite limits as $t \to \infty$. If $x'(t)$ is eventually positive, then either $\lim_{t\to\infty} x'(t) = \text{const} > 0$ or $\lim_{t\to\infty} x'(t) = \infty$, in which case $x(t)$ satisfies

$$\text{(I)} \quad \lim_{t\to\infty} \frac{x(t)}{t} = \infty \quad \text{or} \quad \text{(II)} \quad \lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0$$

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respectively, while if $x'(t)$ is eventually negative, then $\lim_{t\to\infty} x'(t) = 0$, in which case $x(t)$ satisfies

$$\text{(III)} \quad \lim_{t\to\infty} x(t) = \text{const} > 0 \quad \text{or} \quad \text{(IV)} \quad \lim_{t\to\infty} x(t) = 0,$$

respectively. Naturally the same is true of the asymptotic behavior of the component $y(t)$. A function $x(t)$ (or $y(t)$) is referred to as *primitive* if it is of type (II) or (III), and as *non-primitive* it it is of type (I) or (IV).

Positive solutions of (A) may exhibit a variety of asymptotic behavior at infinity depending on which of the four cases (I), (II), (III) and (IV) holds for each of their components. In view of the symmetry of $x(t)$ and $y(t)$ there are ten different types of asymptotic behavior beginning with type (I,I) and ending with type (IV,IV), of which the three types (II,II), (II,III) and (III,III) are special in the sense that the existence of solutions of these types for (A) can be completely characterized without difficulty.

Proposition 1.1. (i) *System* (A) *has solutions* $(x(t), y(t))$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0, \quad \lim_{t\to\infty} \frac{y(t)}{t} = \text{const} > 0, \tag{1.1}$$

if and only if

$$\int_{t_0}^{\infty} t^{\alpha} p(t)dt < \infty, \quad \int_{t_0}^{\infty} t^{\beta} q(t)dt < \infty. \tag{1.2}$$

(ii) *System* (A) *has solutions* $(x(t), y(t))$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0, \quad \lim_{t\to\infty} y(t) = \text{const} > 0, \tag{1.3}$$

if and only if

$$\int_{t_0}^{\infty} p(t)dt < \infty, \quad \int_{t_0}^{\infty} t^{\beta+1} q(t)dt < \infty. \tag{1.4}$$

(iii) *System* (A) *has solutions* $(x(t), y(t))$ *such that*

$$\lim_{t\to\infty} x(t) = \text{const} > 0, \quad \lim_{t\to\infty} y(t) = \text{const} > 0, \tag{1.5}$$

if and only if

$$\int_{t_0}^{\infty} tp(t)dt < \infty, \quad \int_{t_0}^{\infty} tq(t)dt < \infty. \tag{1.6}$$

To prove each of these statements it suffices to construct a suitable system of integral equations from (A) and solve it routinely by means of the Schauder-Tychonoff fixed point theorem. The proof may be omitted.

Of the remaining types of solutions of (A) which seem to be difficult to deal with in the case of general positive continuous $p(t)$ and $q(t)$ we take up the extreme three types, i.e., (I,I), (IV,IV) and (I,IV) types of solutions and show that, if analyzed in the framework of regular variation, it is possible to indicate the situation in which system (A) possesses solutions of these types having accurate order of growth or decay as $t \to \infty$. More specifically, the exponents α and β in (A) are restricted to the case $\alpha\beta < 1$, the coefficients $p(t)$ and $q(t)$ in (A) are assumed to be nearly regularly varying, and the above-mentioned types of solutions are sought in the classes of nearly regularly varying solutions of suitable but definite indices, positive or negative.

The present work was motivated by the recent progress of the asymptotic analysis of positive solutions of nonlinear differential equations by means of regular variation which was triggered by the publication of Marić's book [8]; see, for example, the papers [3 - 7]. A prototype of existence results we are going to prove here is Theorem 8 from [6] concerning the fourth order sublinear differential equation of the Thomas-Fermi type

$$x^{(4)} = q(t)|x|^{\beta}\text{sgn } x, \quad 0 < \beta < 1, \tag{1.7}$$

(equivalent to a special case of (A) where $p(t) \equiv 1$ and $\alpha = 1$) which states that if $q(t)$ is regularly varying of index $\sigma \in (-\infty, -4) \cup (-\beta - 3, -2\beta - 2) \cup (-3\beta - 1, \infty)$ and ρ is given by

$$\rho = \frac{\sigma + 4}{1 - \beta},$$

then (1.7) possesses solutions $x(t)$ satisfying

$$a\left[\frac{t^4 q(t)}{\rho(\rho - 1)(2 - \rho)(3 - \rho)}\right]^{\frac{1}{1-\beta}} \le x(t) \le A\left[\frac{t^4 q(t)}{\rho(\rho - 1)(2 - \rho)(3 - \rho)}\right]^{\frac{1}{1-\beta}}, \quad t \ge T, \tag{1.8}$$

for some constants $T \ge t_0$ and $a, A > 0$. Our purpose here is to generalize the above result to nonlinear systems of the form (A) with "nearly regularly varying" coefficients $p(t)$ and $q(t)$ and to show that the solutions of (A) satisfying (1.8) are actually fully regularly varying if it is assumed that both $p(t)$ and $q(t)$ are regularly varying functions.

2 Regularly varying functions

For the reader's convenience we recall here the definition of regularly varying functions, basic terminologies and notations, and Karamata's integration theorem which will play a central role in establishing the main results of this paper.

Definition 2.1. A measurable function $f : (0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for } \forall\lambda > 0,$$

or equivalently it is expressed in the form

$$f(t) = c(t) \exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\right\}, \quad t \ge t_0,$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV instead of $\mathrm{RV}(0)$ and call members of SV *slowly varying functions*. By definition any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$. So, the class SV of slowly varying functions is of fundamental importance in theory of regular variation. Typical examples of slowly varying functions are: all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbf{R}, \quad \text{and} \quad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\left\{(\log t)^{\frac{1}{3}} \cos\, (\log t)^{\frac{1}{3}}\right\}$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0.$$

A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) = \mathrm{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) is used to denote the set of all trivial $\mathrm{RV}(\rho)$-functions (or the set of all nontrivial $\mathrm{RV}(\rho)$-functions).

The following proposition, known as Karamata's integration theorem, is particularly useful in handling slowly and regularly varying functions analytically and is extensively used throughout the paper.

Proposition 2.1. *Let $L(t) \in \mathrm{SV}$. Then,*
(i) *if $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,,$$

and

$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0,$$

provided $L(s)/s$ is integrable near the infinity in the latter case.

A measurable function $f : (0,\infty) \to (0,\infty)$ is called *regularly bounded* if for any $\lambda_0 > 1$ there exist positive constants m and M such that

$$1 < \lambda < \lambda_0 \quad \Longrightarrow \quad m \le \frac{f(\lambda t)}{f(t)} \le M \quad \text{for all large } t.$$

The totality of regularly bounded functions is denoted by RO. It is clear that $\mathrm{RV}(\rho) \subset$ RO for any $\rho \in \mathbf{R}$. Any function which is bounded both from above and from below by positive constants is regularly bounded. For example, $2+\sin t$ and $2+\sin(\log t)$ are regularly bounded. Note that $2+\sin t$ and $2+\sin(\log t)$ are not slowly varying, whereas $2+\sin(\log_n t)$, $n \geq 2$, are slowly varying.

We now define the class of nearly regularly varying functions which is a useful subclass of RO including all regularly varying functions. To this end it is convenient to introduce the following

Notation 2.1. Let $f(t)$ and $g(t)$ be two positive continuous functions defined in a neighborhood of infinity, say for $t \geq T$. We use the notation $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants m and M such that

$$mg(t) \leq f(t) \leq Mg(t) \quad \text{for} \quad t \geq T.$$

Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. It is easy to see that if $f(t) \asymp g(t)$, $t \to \infty$, and if $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

Definition 2.2. If positive continuous function $f(t)$ satisfies $f(t) \asymp g(t)$, $t \to \infty$, for some $g(t)$ which is regularly varying of index ρ, then $f(t)$ is called a *nearly regularly varying function of index ρ.*

Since $2+\sin t \asymp 2+\sin(\log_n t)$, $t \to \infty$, for all $n \geq 2$, the function $2+\sin t$ is nearly slowly varying, and the same is true of $2+\sin(\log t)$. If $g(t) \in \mathrm{RV}(\rho)$, then the functions $(2+\sin t)g(t)$ and $(2+\sin(\log t))g(t)$ are nearly regularly varying of index ρ, but not regularly varying of index ρ.

The reader is referred to Bingham et al [1] for the most complete exposition of theory of regular variation and its applications and to Marić [8] for the comprehensive survey of results up to 2000 on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

3 Strongly decreasing solutions of (A)

We begin with the problem of existence of *strongly decreasing solutions* of system (A), by which we mean positive solutions of type (IV,IV), that is, those solutions $(x(t), y(t))$ such that

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0. \tag{3.1}$$

Let $(x(t), y(t))$ be such a solution of (A) on $[T, \infty)$, $T > t_0$. Since $x'(t)$ and $y'(t)$ tend to 0 as $t \to \infty$, integrating two equations in (A) twice from t to ∞, we have

$$x(t) = \int_t^\infty \int_s^\infty p(r)y(r)^\alpha drds, \quad y(t) = \int_t^\infty \int_s^\infty q(r)x(r)^\beta drds, \quad t \geq T. \tag{3.2}$$

Our aim is to obtain strongly decreasing solutions of (A) as solutions of the system of integral equations (3.2) by means of fixed point techniques. For this purpose essential use is made of the fact that regularly varying solutions of the system of asymptotic relations

$$x(t) \sim \int_t^\infty \int_s^\infty p(r)y(r)^\alpha drds, \quad y(t) \sim \int_t^\infty \int_s^\infty q(r)x(r)^\beta drds, \quad t \to \infty. \tag{3.3}$$

which is regarded as an approximation at infinity of (A), can be completely characterized provided $\alpha\beta < 1$ and $p(t)$ and $q(t)$ are regularly varying.

Let $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$ (resp. $y(t) = t^\sigma \eta(t)$, $\eta(t) \in \mathrm{SV}$) satisfy (3.1). Then, it is clear that either $\rho < 0$, or $\rho = 0$ and $\xi(t) \to 0$ as $t \to \infty$ (resp. either $\sigma < 0$, or $\sigma = 0$ and $\eta(t) \to 0$ as $t \to \infty$). In this section we focus only on the case where $\rho < 0$ and $\sigma < 0$, leaving the remaining two possibilities to our later investigations.

3.1 Integral asymptotic relations

Lemma 3.1. *Let $\alpha\beta < 1$ and suppose that $p(t) \in \mathrm{RV}(\lambda)$ and $q(t) \in \mathrm{RV}(\mu)$ are expressed as*

$$p(t) = t^\lambda l(t), \quad q(t) = t^\mu m(t), \quad l(t),\ m(t) \in \mathrm{SV}. \tag{3.4}$$

System (3.3) has regularly varying solutions of index (ρ, σ) with $\rho < 0$ and $\sigma < 0$ if and only if (λ, μ) satisfies the system of inequalities

$$\lambda + 2 + \alpha(\mu + 2) < 0, \quad \beta(\lambda + 2) + \mu + 2 < 0, \tag{3.5}$$

in which case ρ and σ are given by

$$\rho = \frac{\lambda + 2 + \alpha(\mu + 2)}{1 - \alpha\beta}, \quad \sigma = \frac{\beta(\lambda + 2) + \mu + 2}{1 - \alpha\beta}, \tag{3.6}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the formula

$$x(t) \sim \Big[\frac{t^{2(\alpha+1)} p(t) q(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{2(\beta+1)} p(t)^\beta q(t)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{3.7}$$

where $\Delta(\tau) = -\tau(1 - \tau)$ for $\tau < 0$.

PROOF. (The "only if" part) Suppose that (3.3) has a regularly varying solution $(x(t), y(t))$ of negative index (ρ, σ) which is expressed in the form

$$x(t) = t^\rho \xi(t), \quad y(t) = t^\sigma \eta(t), \quad \xi(t),\ \eta(t) \in \mathrm{SV}, \quad t \geq T. \tag{3.8}$$

Since the functions $p(t)y(t)^\alpha = t^{\lambda+\alpha\sigma} l(t)\eta(t)^\alpha$ and $q(t)x(t)^\beta = t^{\mu+\beta\rho} m(t)\xi(t)^\beta$ are integrable twice on $[T, \infty)$ we see that $\lambda + \alpha\sigma \leq -2$ and $\mu + \beta\rho \leq -2$, in which case we have by Karamata's integration theorem ((ii) of Proposition 2.1)

$$\int_t^\infty p(s)y(s)^\alpha ds = \int_t^\infty s^{\lambda+\alpha\sigma} l(s)\eta(s)^\alpha ds \sim \frac{t^{\lambda+\alpha\sigma+1} l(t)\eta(t)^\alpha}{-(\lambda + \alpha\sigma + 1)}, \quad t \to \infty,$$

and hence

$$\int_t^\infty \int_s^\infty p(r)y(r)^\alpha dr ds \sim \int_t^\infty \frac{s^{\lambda+\alpha\sigma+1} l(s)\eta(s)^\alpha}{-(\lambda + \alpha\sigma + 1)} ds, \quad t \to \infty. \tag{3.9}$$

This excludes the possibility $\lambda + \alpha\sigma = -2$. In fact, if this equality would hold, then (3.9) and the first relation of (3.3) would imply that

$$x(t) \sim \int_t^\infty \frac{s^{-1} l(s)\eta(s)^\alpha}{-(\lambda + \alpha\sigma + 1)} ds \in \mathrm{SV}, \quad t \to \infty,$$

which shows that the regularity index of $x(t)$ is zero: $\rho = 0$, an impossibility. Thus, we must have $\lambda+\alpha\sigma < -2$. In this case, applying (ii) of Proposition 2.1 to (3.9) and combining the result with the first relation in (3.3), we obtain the asymptotic equivalence

$$x(t) \sim \frac{t^{\lambda+\alpha\sigma+2} l(t)\eta(t)^\alpha}{[-(\lambda+\alpha\sigma+1)][-(\lambda+\alpha\sigma+2)]}, \quad t \to \infty. \tag{3.10}$$

The same argument applies to the second relation in (3.3) and leads to the conclusion that $\mu+\beta\rho < -2$ holds and $y(t)$ satisfies the asymptotic relation

$$y(t) \sim \frac{t^{\mu+\beta\rho+2} m(t)\xi(t)^\beta}{[-(\mu+\beta\rho+1)][-(\mu+\beta\rho+2)]}, \quad t \to \infty. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\rho = \lambda + \alpha\sigma + 2, \quad \sigma = \mu + \beta\rho + 2,$$

from which ρ and σ are determined uniquely by the formula (3.6). This clearly implies (3.5).

As is easily checked, (3.10) and (3.11) can be rewritten as

$$x(t) \sim \frac{t^{\lambda+2} l(t)y(t)^\alpha}{(-\rho)(1-\rho)} = \frac{t^2 p(t)y(t)^\alpha}{\Delta(\rho)}, \quad y(t) \sim \frac{t^{\mu+2} m(t)x(t)^\beta}{(-\sigma)(1-\sigma)} = \frac{t^2 q(t)x(t)^\beta}{\Delta(\sigma)},$$

as $t \to \infty$, which in turn is transformed into

$$x(t) \sim \Big[\frac{t^{2(\alpha+1)} p(t)q(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{2(\beta+1)} p(t)^\beta q(t)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty,$$

establishing the asymptotic formula (3.7) for $(x(t), y(t))$.

(The "if" part) Suppose that (λ, μ) satisfies (3.5) and define (ρ, σ) by (3.6). We define $(X(t), Y(t))$ by

$$X(t) = \Big[\frac{t^{2(\alpha+1)} p(t)q(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = \Big[\frac{t^{2(\beta+1)} p(t)^\beta q(t)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \geq t_0, \tag{3.12}$$

which can be rewritten as

$$X(t) = t^\rho \Big[\frac{l(t)m(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = t^\sigma \Big[\frac{l(t)^\beta m(t)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}.$$

It suffices to prove that

$$\int_t^\infty \int_s^\infty p(r)Y(r)^\alpha dr ds \sim X(t), \quad \int_t^\infty \int_s^\infty q(r)X(r)^\beta dr ds \sim Y(t), \quad t \to \infty. \tag{3.13}$$

Using Karamata's integration theorem, we compute as follows:

$$\int_t^\infty p(s)Y(s)^\alpha ds = \int_t^\infty s^{\lambda+\alpha\sigma} l(s) \Big[\frac{l(s)^\beta m(s)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}} ds$$

$$= \int_t^\infty s^{\rho-2} l(s) \Big[\frac{l(s)^\beta m(s)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}} ds \sim \frac{t^{\rho-1} l(t)}{1-\rho} \Big[\frac{l(t)^\beta m(t)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}}, \quad t \to \infty,$$

and hence

$$\int_t^\infty \int_s^\infty p(r) Y(r)^\alpha dr ds \sim \frac{t^\rho l(t)}{(-\rho)(1-\rho)} \Big[\frac{l(t)^\beta m(t)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}} = X(t), \quad t \to \infty.$$

Similarly we obtain

$$\int_t^\infty \int_s^\infty q(r) X(r)^\beta dr ds \sim \frac{t^\sigma m(t)}{(-\sigma)(1-\sigma)} \Big[\frac{l(t) m(t)^\alpha}{\Delta(\rho) \Delta(\sigma)^\alpha}\Big]^{\frac{\beta}{1-\alpha\beta}} = Y(t), \quad t \to \infty.$$

This ensures the truth of (3.13). This completes the proof of Lemma 3.1.

3.2 Strongly decreasing solutions of (A)

We now consider system (A) with nearly regularly varying $p(t)$ and $q(t)$ and show that strongly decreasing solutions of (A) can be found in the class of nearly regularly varying functions of negative indices.

Theorem 3.1. *Let $\alpha\beta < 1$ and let $p(t)$ and $q(t)$ be nearly regularly varying of index λ and μ, respectively, such that*

$$p(t) \asymp t^\lambda l(t), \quad q(t) \asymp t^\mu m(t), \quad l(t),\ m(t) \in \mathrm{SV}. \tag{3.14}$$

Suppose that λ and μ satisfy (3.5) *and define ρ and σ by* (3.6). *Then, system* (A) *possesses nearly regularly varying solutions $(x(t), y(t))$ of index (ρ, σ) with the property that*

$$x(t) \asymp t^\rho \Big[\frac{l(t) m(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \asymp t^\sigma \Big[\frac{l(t)^\beta m(t)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{3.15}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau < 0$.

PROOF. Let $p_\lambda(t)$ and $q_\mu(t)$ denote the functions

$$p_\lambda(t) = t^\lambda l(t) \in \mathrm{RV}(\lambda), \quad q_\mu(t) = t^\mu m(t) \in \mathrm{RV}(\mu). \tag{3.16}$$

By hypothesis there exist positive constants k, l, K and L such that

$$k p_\lambda(t) \le p(t) \le K p_\lambda(t), \quad l q_\mu(t) \le q(t) \le L q_\mu(t), \quad t \ge t_0. \tag{3.17}$$

Define the function $(X_\lambda(t), Y_\mu(t))$ by the formula (3.12) with $p(t)$ and $q(t)$ replaced by $p_\lambda(t)$ and $q_\mu(t)$, respectively. Since by Lemma 3.1 $(X_\lambda(t), Y_\mu(t))$ satisfies the asymptotic relation (3.3), i.e. (3.13), there exists $T > t_0$ such that for $t \ge T$

$$\begin{aligned} &\frac{1}{2} X_\lambda(t) \le \int_t^\infty \int_s^\infty p_\lambda(r) Y_\mu(r)^\alpha dr ds \le 2 X_\lambda(t), \\ &\frac{1}{2} Y_\mu(t) \le \int_t^\infty \int_s^\infty q_\mu(r) X_\lambda(r)^\beta dr ds \le 2 Y_\mu(t). \end{aligned} \tag{3.18}$$

We may assume that $X_\lambda(t)$ and $Y_\mu(t)$ are non-increasing for $t \geq T$ because it is known ([1, Theorem 1.5.3]) that any regularly varying function of negative index is asymptotic to a monotone non-increasing function. Let us now choose (a,b), $(A,B) \in \mathbf{R^2}$ so that $a < A$, $b < B$ and

$$a \leq \frac{1}{2}kb^\alpha, \quad b \leq \frac{1}{2}la^\beta, \quad 2KB^\alpha \leq A, \quad 2LA^\beta \leq B. \tag{3.19}$$

It is elementary to see that such a choice of (a,b), (A,B) is really possible. For example, one may choose as follows:

$$a = (2^{-(1+\alpha)}kl^\alpha)^{\frac{1}{1-\alpha\beta}}, \quad A = (2^{(1+\alpha)}KL^\alpha)^{\frac{1}{1-\alpha\beta}},$$

$$b = (2^{-(1+\beta)}k^\beta l)^{\frac{1}{1-\alpha\beta}}, \quad B = (2^{(1+\beta)}K^\beta L)^{\frac{1}{1-\alpha\beta}}.$$

We define $\mathcal{X}$ to be the subset of $C[T,\infty)\times C[T,\infty)$ consisting of vector functions $(x(t), y(t))$ satisfying

$$aX_\lambda(t) \leq x(t) \leq AX_\lambda(t), \quad bY_\mu(t) \leq y(t) \leq BY_\mu(t), \quad t \geq T. \tag{3.20}$$

It is clear that $\mathcal{X}$ is a closed convex subset of $C[T,\infty)\times C[T,\infty)$. Furthermore, define the mapping $\Phi : \mathcal{X} \to C[T,\infty)\times C[T,\infty)$ by

$$\Phi(x(t), y(t)) = (\mathcal{F}y(t), \mathcal{G}x(t)), \quad t \geq T, \tag{3.21}$$

where $\mathcal{F}$ and $\mathcal{G}$ denote the integral operators

$$\mathcal{F}y(t) = \int_t^\infty \int_s^\infty p(r)y(r)^\alpha dr ds, \quad \mathcal{G}x(t) = \int_t^\infty \int_s^\infty q(r)x(r)^\beta dr ds, \quad t \geq T. \tag{3.22}$$

It can be checked that Φ fulfils the hypotheses of the Schauder-Tychonoff fixed point theorem.

(i) $\Phi(\mathcal{X}) \subset \mathcal{X}$. If $(x(t), y(t)) \in \mathcal{X}$, then using (3.18), (3.19) and (3.20), we see that

$$\mathcal{F}y(t) \geq \int_t^\infty \int_s^\infty kp_\lambda(r)(bY_\mu(r))^\alpha dr ds \geq \frac{1}{2}kb^\alpha X_\lambda(t) \geq aX_\lambda(t),$$

$$\mathcal{F}y(t) \leq \int_t^\infty \int_s^\infty Kp_\lambda(r)(BY_\mu(r))^\alpha dr ds \leq 2KB^\alpha X_\lambda(t) \leq AX_\lambda(t),$$

$$\mathcal{G}x(t) \geq \int_t^\infty \int_s^\infty lq_\mu(r)(aX_\lambda(r))^\beta dr ds \geq \frac{1}{2}la^\beta Y_\mu(t) \geq bY_\mu(t),$$

$$\mathcal{G}x(t) \leq \int_t^\infty \int_s^\infty Lq_\mu(r)(AX_\lambda(r))^\beta dr ds \leq 2LA^\beta Y_\mu(t) \leq BY_\mu(t),$$

for $t \geq T$. This shows that $(\mathcal{F}y(t), \mathcal{G}x(t)) \in \mathcal{X}$.

(ii) $\Phi(\mathcal{X})$ is relatively compact. From the inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ it follows that $\Phi(\mathcal{X})$ is uniformly bounded on $[T,\infty)$. The inequalities

$$0 \geq (\mathcal{F}y)'(t) \geq -B^\alpha \int_t^\infty p(s)Y_\mu(s)^\alpha ds, \quad 0 \geq (\mathcal{G}x)'(t) \geq -A^\beta \int_t^\infty q(s)X_\lambda(s)^\beta ds,$$

holding for all $(x(t), y(t)) \in \mathcal{X}$ guarantee that $\Phi(\mathcal{X})$ is equicontinuous on $[T, \infty)$. The relative compactness of $\Phi(\mathcal{X})$ then follows from the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_n(t), y_n(t))\}$ be a sequence in $\mathcal{X}$ converging to $(x(t), y(t)) \in \mathcal{X}$ uniformly on any compact subinterval of $[T, \infty)$. We need to prove that $\Phi(x_n(t), y_n(t)) \to \Phi(x(t), y(t))$, that is,

$$\mathcal{F}y_n(t) \to \mathcal{F}y(t), \quad \mathcal{G}x_n(t) \to \mathcal{G}x(t) \quad \text{as } n \to \infty$$

uniformly on compact subintervals of $[T, \infty)$. But this follows immediately from the Lebesgue dominated convergence theorem applied to the integrals in the following inequalities holding for $t \geq T$

$$|\mathcal{F}y_n(t) - \mathcal{F}y(t)| \leq \int_t^\infty sp(s)|y_n(s)^\alpha - y(s)^\alpha| ds,$$

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \leq \int_t^\infty sq(s)|x_n(s)^\beta - x(s)^\beta| ds.$$

Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of a function $(x(t), y(t)) \in \mathcal{X}$ such that $\Phi(x(t), y(t)) = (x(t), y(t)),\ t \geq T$, that is,

$$x(t) = \mathcal{F}y(t) = \int_t^\infty \int_s^\infty p(r)y(r)^\alpha dr ds, \quad y(t) = \mathcal{G}x(t) = \int_t^\infty \int_s^\infty q(r)x(r)^\beta dr ds,$$

for $t \geq T$. It follows that $(x(t), y(t))$ gives a strongly decreasing solution of system (A). The membership $(x(t), y(t)) \in \mathcal{X}$ implies that $(x(t), y(t))$ is a nearly regularly varying of negative index (ρ, σ). This completes the proof of Theorem 3.1.

As for the solutions constructed in Theorem 3.1, their regularity can be characterized completely under the stronger assumption that $p(t)$ and $q(t)$ are regularly varying functions.

The generalized L'Hospital's rule given in the following lemma (see [2]) plays a crucial role in the proof of this theorem.

Lemma 3.2. *Let $f(t), g(t) \in \mathrm{C}^1[T, \infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \text{and } g'(t) > 0 \text{ for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and } g'(t) < 0 \text{ for all large } t.$$

Then,

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leq \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \quad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

Theorem 3.2. *Suppose that $p(t)$ and $q(t)$ are regularly varying of indices λ and μ, respectively. System* (A) *possesses regularly varying solutions $(x(t), y(t))$ such that*

$$x(t) \in \mathrm{RV}(\rho), \quad y(t) \in \mathrm{RV}(\sigma), \quad \rho < 0, \quad \sigma < 0$$

if and only if (3.5) *holds, in which case ρ and σ are given by* (3.6) *and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the formulas*

$$(3.23)\qquad x(t) \sim \left[\frac{t^{2(\alpha+1)}p(t)q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \left[\frac{t^{2(\beta+1)}p(t)^{\beta}q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty.$$

PROOF OF THEOREM 3.2.

(The "only if" part) It follows from Lemma 3.1.

(The "if" part) Suppose that (3.5) holds and define the negative constants ρ and σ by (3.6). By Theorem 3.1 system (A) has a nearly regularly varying solution $(x(t), y(t))$ on $[T, \infty)$ such that

$$(3.24)\qquad aX(t) \le x(t) \le AX(t), \quad bY(t) \le y(t) \le BY(t), \quad t \ge T,$$

for some positive constants T, a, A, b and B, where

$$(3.25)\quad X(t) = \left[\frac{t^{2(\alpha+1)}p(t)q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}} \in \mathrm{RV}(\rho), \quad Y(t) = \left[\frac{t^{2(\beta+1)}p(t)^{\beta}q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha\beta}} \in \mathrm{RV}(\sigma).$$

It is clear that $(x(t), y(t))$ satisfies

$$(3.26)\qquad x(t) = \int_t^{\infty}\int_s^{\infty} p(r)y(r)^{\alpha}drds, \quad y(t) = \int_t^{\infty}\int_s^{\infty} q(r)x(r)^{\beta}drds, \quad t \ge T.$$

Let $U(t)$ and $V(t)$ denote the functions defined by

$$(3.27)\qquad U(t) = \int_t^{\infty}\int_s^{\infty} p(r)Y(r)^{\alpha}drds, \quad V(t) = \int_t^{\infty}\int_s^{\infty} q(r)X(r)^{\beta}drds, \quad t \ge t_0.$$

Note that $U(t)$ and $V(t)$ satisfy the asymptotic relations

$$(3.28)\qquad U(t) \sim X(t), \quad V(t) \sim Y(t), \quad t \to \infty.$$

Put

$$(3.29)\qquad k = \liminf_{t\to\infty}\frac{x(t)}{U(t)}, \quad K = \limsup_{t\to\infty}\frac{x(t)}{U(t)}, \quad l = \liminf_{t\to\infty}\frac{y(t)}{V(t)}, \quad L = \limsup_{t\to\infty}\frac{y(t)}{V(t)}.$$

From (3.24) and (3.26) we see that $0 < k \le K < \infty$ and $0 < l \le L < \infty$. Applying the generalized L'Hospital rule twice, we obtain

$$k = \liminf_{t\to\infty}\frac{x(t)}{U(t)} \ge \liminf_{t\to\infty}\frac{x''(t)}{U''(t)} = \liminf_{t\to\infty}\frac{p(t)y(t)^{\alpha}}{p(t)Y(t)^{\alpha}}$$

$$= \liminf_{t\to\infty}\left(\frac{y(t)}{Y(t)}\right)^{\alpha} = \liminf_{t\to\infty}\left(\frac{y(t)}{V(t)}\right)^{\alpha} = \left(\liminf_{t\to\infty}\frac{y(t)}{V(t)}\right)^{\alpha} = l^{\alpha},$$

and

$$l = \liminf_{t\to\infty}\frac{y(t)}{V(t)} \ge \liminf_{t\to\infty}\frac{y''(t)}{V''(t)} = \liminf_{t\to\infty}\frac{q(t)x(t)^{\beta}}{q(t)X(t)^{\beta}}$$

$$= \liminf_{t\to\infty}\left(\frac{x(t)}{X(t)}\right)^{\beta} = \liminf_{t\to\infty}\left(\frac{x(t)}{U(t)}\right)^{\beta} = \left(\liminf_{t\to\infty}\frac{x(t)}{U(t)}\right)^{\beta} = k^{\beta},$$

where (3.28) has been used in the final step of each of the above computations. Since $\alpha\beta < 1$, the inequalities $k \geq l^\alpha$ and $l \geq k^\beta$ thus obtained imply

$$1 \leq k < \infty, \quad 1 \leq l < \infty. \tag{3.30}$$

Similarly, we obtain $K \leq L^\alpha$ and $L \leq K^\beta$, from which it follows that

$$0 < K \leq 1, \quad 0 < L \leq 1. \tag{3.31}$$

From (3.30) and (3.31) we conclude that $k = K = 1$ and $l = L = 1$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{U(t)} = 1, \quad \lim_{t\to\infty} \frac{y(t)}{V(t)} = 1,$$

which combined with (3.28) shows that

$$x(t) \sim U(t) \sim X(t), \quad y(t) \sim V(t) \sim Y(t), \quad t \to \infty.$$

This completes the proof.

Example 3.1. Consider the system (A) with

$$p(t) \asymp 2t^{2\alpha-3}(\log t)^{-(\alpha+1)}, \quad q(t) \asymp 6t^{\beta-4}(\log t)^{\beta+1}, \quad t \to \infty. \tag{3.32}$$

Since

$$\lambda + 2 + \alpha(\mu + 2) = \alpha\beta - 1, \quad \beta(\lambda + 2) + \mu + 2 = 2(\alpha\beta - 1),$$

(3.6) gives $\rho = -1$ and $\sigma = -2$, which implies that $\Delta(-1) = 2$ and $\Delta(-2) = 6$. Using these constants and (3.27) in (3.15), we conclude from Theorem 3.1 that the system (A) under consideration possesses a strongly decreasing solution $(x(t), y(t))$ which is nearly regularly varying of index $(-1, -2)$ such that

$$x(t) \asymp (t \log t)^{-1}, \quad y(t) \asymp t^{-2} \log t, \quad t \to \infty.$$

In case condition (3.32) is strengthened to

$$p(t) \sim 2t^{2\alpha-3}(\log t)^{-(\alpha+1)}, \quad q(t) \sim 6t^{\beta-4}(\log t)^{\beta+1}, \quad t \to \infty,$$

then Theorem 3.2 guarantees the existence of a strongly decreasing regularly varying solution $(x(t), y(t))$ of index (-1,-2) such that

$$x(t) \sim (t \log t)^{-1}, \quad y(t) \sim t^{-2} \log t, \quad t \to \infty.$$

If in particular

$$p(t) = 2t^{2\alpha-3}(\log t)^{-(\alpha+1)}\Big(1 + \frac{3}{2\log t} + \frac{1}{(\log t)^2}\Big), \quad q(t) = 6t^{\beta-4}(\log t)^{\beta+1}\Big(1 - \frac{5}{6\log t}\Big),$$

then this system has an exact regularly varying solution $((t \log t)^{-1}, t^{-2} \log t)$.

4 Strongly increasing solutions of (A)

We turn our attention to *strongly increasing solutions* of system (A), by which we mean positive solutions of type (I,I), that is, those solutions $(x(t), y(t))$ such that

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty, \quad \lim_{t\to\infty} \frac{y(t)}{t} = \infty. \tag{4.1}$$

Let $(x(t), y(t))$ be one such solution of (A) on $[T, \infty)$. Note that $x'(t)$ and $y'(t)$ tend to infinity as $t \to \infty$. Integrating (A) twice on $[T, t]$ gives

$$x(t) = x_0 + x_1(t-T) + \int_T^t \int_T^s p(r)y(r)^\alpha dr ds, \quad y(t) = y_0 + y_1(t-T) + \int_T^t \int_T^s q(r)x(r)^\beta dr ds, \tag{4.2}$$

for $t \geq T$, where $x_0 = x(T)$, $x_1 = x'(T)$, $y_0 = y(T)$ and $y_1 = y'(T)$. The wanted strongly increasing solutions of (A) are obtained by solving the system of integral equations (4.2) with the help of the Schauder-Tychonoff fixed point theorem. For this purpose an essential role is played by some of the basic properties of regularly varying solutions satisfying (4.1) and the system of integral asymptotic relations

$$x(t) \sim \int_T^t \int_T^s p(r)y(r)^\alpha dr ds, \quad y(t) \sim \int_T^t \int_T^s q(r)x(r)^\beta dr ds, \quad t \to \infty, \tag{4.3}$$

in which $\alpha\beta < 1$ and $p(t)$ and $q(t)$ are regularly varying. Let $(x(t), y(t))$ be a regularly varying solution of (4.3) which is expressed as

$$x(t) = t^\rho \xi(t), \quad y(t) = t^\sigma \eta(t), \quad \xi(t),\ \eta(t) \in \mathrm{SV}. \tag{4.4}$$

Then, it is easy to see that (4.1) holds for $(x(t), y(t))$ if $\rho > 1$, or if $\rho = 1$ and $\xi(t) \to \infty$ as $t \to \infty$ on the one hand, and if $\sigma > 1$, or if $\sigma = 1$ and $\eta(t) \to \infty$ as $t \to \infty$ on the other. In this section we consider only the case where $\rho > 1$ and $\sigma > 1$, leaving the other possibilities to a later occasion because of computational difficulty.

4.1 Integral asymptotic relations

This subsection is concerned with the asymptotic system (4.3) which is regarded as an approximation of the system of integral equations (4.2). As a result of the analysis of (4.3) by means of regular variation full knowledge can be acquired of its regularly varying solutions satisfying (4.1) as the following lemma shows.

Lemma 4.1. *Let $\alpha\beta < 1$ and suppose that $p(t) \in \mathrm{RV}(\lambda)$ and $q(t) \in \mathrm{RV}(\mu)$ are expressed in the form* (4.4)*. System* (4.3) *has regularly varying solutions of index (ρ, σ) with $\rho > 1$ and $\sigma > 1$ if and only if (λ, μ) satisfies the system of inequalities*

$$\lambda + 1 + \alpha(\beta + \mu + 2) > 0, \quad \beta(\alpha + \lambda + 2) + \mu + 1 > 0, \tag{4.5}$$

in which case ρ and σ are given by (3.6)*, and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the formula*

$$x(t) \sim \Big[\frac{t^{2(1+\alpha)}p(t)q(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{2(1+\beta)}p(t)^\beta q(t)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{4.6}$$

where $\Delta(\tau) = \tau(\tau - 1)$ *for* $\tau > 1$.

PROOF. (The "only if" part) Suppose that (4.3) has a regularly varying solution $(x(t), y(t))$ on $[T, \infty)$ which is expressed in the form (4.4) with $\rho > 1$ and $\sigma > 1$. Then,

$$(4.7) \qquad x(t) \sim \int_T^t \int_T^s r^{\lambda+\alpha\sigma} l(r)\eta(r)^\alpha dr ds, \quad y(t) \sim \int_T^t \int_T^s r^{\mu+\beta\rho} m(r)\xi(r)^\beta dr ds,$$

from which we have for $t \to \infty$

$$(4.8) \qquad \frac{x(t)}{t} \sim \int_T^t s^{\lambda+\alpha\sigma} l(s)\eta(s)^\alpha ds, \quad \frac{y(t)}{t} \sim \int_T^t s^{\mu+\beta\rho} m(s)\xi(s)^\beta ds.$$

Since both integrals in (4.8) diverge as $t \to \infty$, it holds that $\lambda+\alpha\sigma \geq -1$ and $\mu+\beta\rho \geq -1$. But the cases $\lambda + \alpha\rho = -1$ and $\mu + \beta\sigma = -1$ are impossible, because if these equalities would hold, then Karamata's integration theorem would imply that

$$x(t) \sim t \int_T^t s^{-1} l(s)\eta(s)^\alpha ds \in \mathrm{RV}(1), \quad y(t) \sim t \int_T^t s^{-1} m(s)\xi(s)^\beta ds \in \mathrm{RV}(1), \quad t \to \infty,$$

(cf. (iii) of Proposition 2.1). Consequently, we must have $\lambda + \alpha\rho > -1$ and $\mu + \beta\sigma > -1$, in which case, applying Karamata's integration theorem to (4.7), we find that

$$\int_T^t s^{\lambda+\alpha\sigma} l(s)\eta(s)^\alpha ds \sim \frac{t^{\lambda+\alpha\sigma+1} l(t)\eta(t)^\alpha}{\lambda+\alpha\sigma+1}, \quad \int_T^t s^{\mu+\beta\rho} m(s)\xi(s)^\beta ds \sim \frac{t^{\mu+\beta\rho+1} m(t)\xi(t)^\beta}{\mu+\beta\rho+1},$$

and

$$(4.9) \quad x(t) \sim \frac{t^{\lambda+\alpha\sigma+2} l(t)\eta(t)^\alpha}{(\lambda+\alpha\sigma+1)(\lambda+\alpha\sigma+2)}, \quad y(t) \sim \frac{t^{\mu+\beta\rho+2} m(t)\xi(t)^\beta}{(\mu+\beta\rho+1)(\mu+\beta\rho+2)}, \quad t \to \infty.$$

This shows that $x(t)$ and $y(t)$ are regularly varying of indices $\lambda+\alpha\sigma+2 > 1$ and $\mu+\beta\rho+2 > 1$, respectively, and so we must have

$$\rho = \lambda + \alpha\sigma + 2, \quad \sigma = \mu + \beta\rho + 2,$$

from which it readily follows that ρ and σ are given by (3.6). Since $\rho > 1$ and $\sigma > 1$, (3.6) determines the range of (λ, μ) to be the subset of $\mathbf{R^2}$ defined by the inequalities

$$\lambda + 1 + \alpha(\beta + \mu + 2) > 0, \quad \beta(\alpha + \lambda + 2) + \mu + 1 > 0.$$

Noting that (4.9) can be rewritten as

$$x(t) \sim \frac{t^{\lambda+2} l(t) y(t)^\alpha}{\Delta(\rho)} = \frac{t^2 p(t) y(t)^\alpha}{\Delta(\rho)}, \quad y(t) \sim \frac{t^{\mu+2} m(t) x(t)^\beta}{\Delta(\sigma)} = \frac{t^2 q(t) x(t)^\beta}{\Delta(\sigma)},$$

we easily conclude that $x(t)$ and $y(t)$ satisfy

$$x(t) \sim \frac{t^{2(\alpha+1)} p(t) q(t)^\alpha x(t)^{\alpha\beta}}{\Delta(\rho)\Delta(\sigma)^\alpha}, \quad y(t) \sim \frac{t^{2(\beta+1)} p(t)^\beta q(t) y(t)^{\alpha\beta}}{\Delta(\rho)^\beta \Delta(\sigma)}, \quad t \to \infty,$$

from which the asymptotic formula (4.6) immediately follows.

(The "if" part) Suppose that (λ,μ) satisfies (4.5) and define (ρ,σ) by (3.6). We define $(X(t),Y(t))$ by

$$X(t)=\Big[\frac{t^{2(\alpha+1)}p(t)q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}},\quad Y(t)=\Big[\frac{t^{2(\beta+1)}p(t)^{\beta}q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}},\tag{4.10}$$

which can also be expressed as

$$X(t)=t^{\rho}\Big[\frac{l(t)m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}},\quad Y(t)=t^{\sigma}\Big[\frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}.\tag{4.11}$$

Using Karamata's integration theorem, we see that for any $b\geq t_0$

$$\int_b^t p(s)Y(s)^{\alpha}ds=\int_b^t s^{\lambda+\alpha\sigma}l(s)\Big[\frac{l(s)^{\beta}m(s)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}}ds$$
$$=\int_b^t s^{\rho-2}l(s)\Big[\frac{l(s)^{\beta}m(s)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}}ds\sim\frac{t^{\rho-1}l(t)}{\rho-1}\Big[\frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}},$$

and

$$\int_b^t\int_b^s p(r)Y(r)^{\alpha}drds\sim\int_b^t\frac{s^{\rho-1}l(s)}{\rho-1}\Big[\frac{l(s)^{\beta}m(s)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}}ds$$
$$\sim\frac{t^{\rho}l(t)}{\rho(\rho-1)}\Big[\frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{\alpha}{1-\alpha\beta}}=X(t),\quad t\to\infty.$$

Similarly, we obtain for any $b\geq t_0$

$$\int_b^t\int_b^s q(r)X(r)^{\beta}drds\sim\frac{t^{\sigma}m(t)}{\sigma(\sigma-1)}\Big[{\frac{l(t)m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)}}^{\alpha}\Big]^{\frac{\beta}{1-\alpha\beta}}=Y(t),\quad t\to\infty,$$

Thus, it follows that

$$\int_b^t\int_b^s p(r)Y(r)^{\alpha}drds\sim X(t),\quad\int_b^t\int_b^s q(r)X(r)^{\beta}drds\sim Y(t),\quad t\to\infty,\tag{4.12}$$

for any $b\geq t_0$, which shows that $(X(t),Y(t))$ is a solution of (4.3). This completes the proof of Lemma 4.1.

4.2 Strongly increasing solutions of (A)

It is shown that if $p(t)$ and $q(t)$ are nearly regularly varying, then strongly increasing solutions of system (A) can be found in the class of nearly regularly varying functions of indices greater than 1.

Theorem 4.1. *Let $\alpha\beta<1$ and let $p(t)$ and $q(t)$ be nearly regularly varying of index λ and μ, respectively, which are expressed in the form*

$$p(t)\asymp t^{\lambda}l(t),\quad q(t)\asymp t^{\mu}m(t),\quad l(t),\ m(t)\in\mathrm{SV},\quad t\to\infty.$$

Suppose that λ and μ satisfy (4.5) *and define $\rho > 1$ and $\sigma > 1$ by* (3.6)*. Then, system* (A) *possesses nearly regularly varying solutions $(x(t), y(t))$ of index (ρ, σ) with the property that*

$$x(t) \asymp t^{\rho}\Big[\frac{l(t)m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \asymp t^{\sigma}\Big[\frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{4.13}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau > 1$.

PROOF. Put $p_{\lambda}(t) = t^{\lambda}l(t)$ and $q_{\mu}(t) = t^{\mu}m(t)$. There exist positive constants k, K, l and L such that

$$kp_{\lambda}(t) \le p(t) \le Kp_{\lambda}(t), \quad lq_{\mu}(t) \le q(t) \le Lq_{\mu}(t), \quad t \ge t_0.$$

Define the vector function $(X_{\lambda}(t), Y_{\mu}(t))$ by (4.10) with $p(t)$ and $q(t)$ replaced by $p_{\lambda}(t)$ and $q_{\mu}(t)$, respectively. Since $(X_{\lambda}(t), Y_{\mu}(t))$ satisfies (4.12), there exists $T_0 > t_0$ such that

$$\int_{T_0}^{t}\int_{T_0}^{s} p_{\lambda}(r)Y_{\mu}(r)^{\alpha}drds \le 2X_{\lambda}(t), \quad \int_{T_0}^{t}\int_{T_0}^{s} q_{\mu}(r)X_{\lambda}(r)^{\beta}drds \le 2Y_{\mu}(t), \quad t \ge T_0. \tag{4.14}$$

We may assume that $X_{\lambda}(t)$ and $Y_{\mu}(t)$ are non-decreasing for $t \ge T_0$ because any regularly varying function of positive index is asymptotic to a monotone non-decreasing function (cf [1, Theorem 1.5.3]). Noting that

$$\int_{T_0}^{t}\int_{T_0}^{s} p_{\lambda}(r)Y_{\mu}(r)^{\alpha}drds \sim X_{\lambda}(t), \quad \int_{T_0}^{t}\int_{T_0}^{s} q_{\mu}(r)X_{\lambda}(r)^{\beta}drds \sim Y_{\mu}(t), \quad t \to \infty,$$

one can choose $T_1 > T_0$ such that

$$\int_{T_0}^{t}\int_{T_0}^{s} p_{\lambda}(r)Y_{\mu}(r)^{\alpha}drds \ge \frac{1}{2}X_{\lambda}(t), \quad \int_{T_0}^{t}\int_{T_0}^{s} q_{\mu}(r)X_{\lambda}(r)^{\beta}drds \ge \frac{1}{2}Y_{\mu}(t), \quad t \ge T_1. \tag{4.15}$$

Let us choose (a, b), $(A, B) \in \mathbf{R^2}$ so that $a < A$, $b < B$ and the following inequalities hold:

$$a \le \frac{1}{2}kb^{\alpha}, \quad b \le \frac{1}{2}la^{\beta}, \quad 4KB^{\alpha} \le A, \quad 4LA^{\beta} \le B, \tag{4.16}$$

$$2aX_{\lambda}(T_1) \le AX_{\lambda}(T_0), \quad 2bY_{\mu}(T_1) \le BY_{\mu}(T_0). \tag{4.17}$$

It is easy to check that such choice of (a, b) and (A, B) is possible by taking, if necessary, k and l sufficiently small and K and L sufficiently large. Let $\mathcal{X}$ denote the closed convex subset of $C[T_0, \infty)\times C[T_0, \infty)$ consisting of the vector functions $(x(t), y(t))$ such that

$$aX_{\lambda}(t) \le x(t) \le AX_{\lambda}(t), \quad bY_{\mu}(t) \le y(t) \le BY_{\mu}(t), \quad t \ge T_0, \tag{4.18}$$

and consider the mapping $\Phi : \mathcal{X} \to C[T_0, \infty)\times C[T_0, \infty)$ defined by

$$\Phi(x(t), y(t)) = (\mathcal{F}y(t), \mathcal{G}x(t)), \quad t \ge T_0, \tag{4.19}$$

where

$$\mathcal{F}y(t) = x_0 + \int_{T_0}^{t}\int_{T_0}^{s} p(r)y(r)^{\alpha}drds, \quad \mathcal{G}x(t) = y_0 + \int_{T_0}^{t}\int_{T_0}^{s} q(r)x(r)^{\beta}drds, \tag{4.20}$$

for $t \ge T_0$, and x_0 and y_0 are positive constants satisfying

$$aX_\lambda(T_1) \le x_0 \le \frac{1}{2}AX_\lambda(T_0), \quad bY_\mu(T_1) \le y_0 \le \frac{1}{2}BY_\mu(T_0). \tag{4.21}$$

We prove that Φ is a continuous self-map of $\mathcal{X}$ and sends $\mathcal{X}$ into a relatively compact subset of $C[T_0,\infty)\times C[T_0,\infty)$.

(i) $\Phi(\mathcal{X}) \subset \mathcal{X}$. Let $(x(t), y(t)) \in \mathcal{X}$. Using (4.14) - (4.21), we compute as follows:

$$\mathcal{F}y(t) \ge x_0 \ge aX_\lambda(T_1) \ge aX_\lambda(t), \quad T_0 \le t \le T_1,$$

$$\begin{aligned}\mathcal{F}y(t) &\ge \int_{T_0}^t\int_{T_0}^s p(r)y(r)^\alpha drds \ge \int_{T_0}^t\int_{T_0}^s kp_\lambda(r)(bY_\mu(r))^\alpha drds\\ &\ge \frac{1}{2}kb^\alpha X_\lambda(t) \ge aX_\lambda(t), \quad t \ge T_1,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}y(t) &\le \frac{1}{2}AX_\lambda(T_0) + \int_{T_0}^t\int_{T_0}^s Kp_\lambda(r)(BY_\mu(r))^\alpha drds\\ &\le \frac{1}{2}AX_\lambda(t) + 2KB^\alpha X_\lambda(t) \le \frac{1}{2}AX_\lambda(t) + \frac{1}{2}AX_\lambda(t) = AX_\lambda(t), \quad t \ge T_0.\end{aligned}$$

This shows that $aX_\lambda(t) \le \mathcal{F}y(t) \le AX_\lambda(t)$ for $t \ge T_0$. Likewise, it can be shown that $bY_\mu(t) \le \mathcal{G}x(t) \le BY_\mu(t)$ for $t \ge T_0$.

(ii) $\Phi(\mathcal{X})$ is relatively compact. The local uniform boundedness of $\Phi(\mathcal{X})$ on $[T_0,\infty)$ follows from the inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$. The local equicontinuity of $\Phi(\mathcal{X})$ on $[T_0,\infty)$ is a consequence of the inequalities

$$0 \le (\mathcal{F}y)'(t) \le B^\alpha \int_{T_0}^t p(s)Y_\mu(s)^\alpha ds, \quad 0 \le (\mathcal{G}x)'(t) \le A^\beta \int_{T_0}^t q(s)X_\lambda(s)^\beta ds,$$

holding for $t \ge T_0$ and for all $(x(t), y(t)) \in \mathcal{X}$.

(iii) Φ is continuous. Let $(x_n(t), y_n(t))$ be a sequence in $\mathcal{X}$ converging to $(x(t), y(t)) \in \mathcal{X}$ as $n \to \infty$ uniformly on any compact subinterval of $[T_0,\infty)$. Noting that

$$|\mathcal{F}y_n(t) - \mathcal{F}y(t)| \le t\int_{T_0}^t p(s)|y_n(s)^\alpha - y(s)^\alpha|ds,$$

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \le t\int_{T_0}^t q(s)|x_n(s)^\beta - x(s)^\beta|ds,$$

and applying the Lebesgue dominated convergence theorem to the right-hand sides of the above inequalities, it follows that

$$\mathcal{F}y_n(t) \to \mathcal{F}y(t), \quad \mathcal{G}x_n(t) \to \mathcal{G}x(t), \quad n \to \infty,$$

uniformly on compact subintervals of $[T_0,\infty)$. This implies the continuity of Φ.

Therefore, the Schauder-Tychonoff fixed point theorem guarantees the existence of an element $(x(t), y(t)) \in \mathcal{X}$ such that $(x(t), y(t)) = \Phi(x(t), y(t))$, $t \ge T_0$, that is,

$$x(t) = \mathcal{F}y(t) = x_0 + \int_{T_0}^t\int_{T_0}^s p(r)y(r)^\alpha drds, \quad y(t) = \mathcal{G}x(t) = y_0 + \int_{T_0}^t\int_{T_0}^s q(r)x(r)^\beta drds$$

for $t \ge T_0$, which is a special case of the system of integral equations (4.2). Thus, $(x(t), y(t))$ is a solution of the system of differential equations (A) on $[T_0, \infty)$. Since $(x(t), y(t))$ is a member of $\mathcal{X}$, it is a nearly regularly varying solution of index (ρ, σ) with $\rho > 1$ and $\sigma > 1$, which clearly provides a strongly increasing solution of (A). This completes the proof of Theorem 4.1.

As the next theorem demonstrates, the full regularity of the strongly increasing solutions obtained in Theorem 4.1 can be proved with the help of the generalized L'Hospital rule (Lemma 3.2), if we make a stronger assumption that the coefficients $p(t)$ and $q(t)$ are regularly varying functions. Thus, the existence of regularly varying solutions with indices greater than one is characterized completely in this particular case. The proof is similar to that of Theorem 3.2 and we omit it.

Theorem 4.2. *Suppose that $p(t)$ and $q(t)$ are regularly varying of indices λ and μ, respectively. System* (A) *possesses regularly varying solutions $(x(t), y(t))$ such that*

$$x(t) \in \mathrm{RV}(\rho), \quad y(t) \in \mathrm{RV}(\sigma), \quad \rho > 1, \ \sigma > 1,$$

if and only if (4.5) *holds, in which case ρ and σ are given by* (3.6) *and the asymptotic behavior of any such solution is governed by the formulas*

$$x(t) \sim \left[\frac{t^{2(\alpha+1)} p(t) q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \left[\frac{t^{2(\beta+1)} p(t)^{\beta} q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{4.22}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau > 1$.

Example 4.1. Consider the system (A) with

$$p(t) \sim 2t^{-3\alpha} \exp\bigl((\alpha+1)\sqrt{\log t}\bigr), \quad q(t) \sim 6t^{1-2\beta} \exp\bigl(-(\beta+1)\sqrt{\log t}\bigr), \quad t \to \infty. \tag{4.23}$$

Here $\lambda = -3\alpha$ and $\mu = 1 - 2\beta$, and hence

$$\lambda + 1 + \alpha(\beta + \mu + 2) = 1 - \alpha\beta > 0, \quad \beta(\alpha + \lambda + 2) + \mu + 1 = 2(1 - \alpha\beta) > 0,$$

which implies that condition (4.5) holds. It is easy to see that (3.6) defines $\rho = 2$ and $\sigma = 3$, so that $\Delta(\sigma) = 2$ and $\Delta(\sigma) = 6$. On the other hand, since $l(t) = 2\exp\bigl((\alpha+1)\sqrt{\log t}\bigr)$ and $m(t) = 6\exp\bigl(-(\beta+1)\sqrt{\log t}\bigr)$, we obtain

$$\frac{l(t)m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}} = \exp\bigl((1-\alpha\beta)\sqrt{\log t}\bigr), \quad \frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)} = \exp\bigl((\alpha\beta - 1)\sqrt{\log t}\bigr).$$

Combining the above calculations, we conclude from Theorem 4.2 that the system (A) under consideration possesses a strongly increasing solution $(x(t), y(t))$ such that

$$x(t) \sim t^2 \exp(\sqrt{\log t}), \quad y(t) \sim t^3 \exp(-\sqrt{\log t}), \quad t \to \infty.$$

As is easily checked, if

$$p(t) = 2t^{-3\alpha} \exp\bigl((\alpha+1)\sqrt{\log t}\bigr)\Bigl(1 + \frac{3}{4\sqrt{\log t}} + \frac{1}{8\log t} - \frac{1}{8\log t\sqrt{\log t}}\Bigr),$$

$$q(t) = 6t^{1-2\beta} \exp\bigl(-(\beta+1)\sqrt{\log t}\bigr)\Bigl(1 - \frac{5}{12\sqrt{\log t}} + \frac{1}{24\log t} + \frac{1}{24\log t\sqrt{\log t}}\Bigr),$$

then system (A) has an exact strongly increasing solution $(t^2\exp(\sqrt{\log t}), t^3\exp(-\sqrt{\log t}))$, components of which are regularly varying of indices 2 and 3, respectively.

5 Mixed strongly monotone solutions of (A)

The purpose of the final section is to indicate the situation in which the system (A) possesses a positive solution $(x(t), y(t))$ such that

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} \frac{y(t)}{t} = \infty. \tag{5.1}$$

Such a solution is referred to as a *mixed strongly monotone solution* of (A). As in the preceding sections it is assumed that $\alpha\beta < 1$ and that $p(t)$ and $q(t)$ are nearly regularly varying functions. Mixed strongly monotone solutions of (A) are sought as solutions of the system of integral equations of the form

$$x(t) = \int_t^\infty \int_s^\infty p(r)y(r)^\alpha drds, \quad y(t) = y_0 + \int_T^t \int_T^s q(r)x(r)^\beta drds, \quad t \geq T, \tag{5.2}$$

for some constants $T > t_0$ and $y_0 > 0$, belonging to the class of nearly regularly varying vector functions of indices $\rho < 0$ and $\sigma > 1$. The construction of such solutions of (A) is based on the accurate asymptotic behavior of *regularly varying solutions* of the system of asymptotic relations

$$x(t) \sim \int_t^\infty \int_s^\infty p(r)y(r)^\alpha drds, \quad y(t) \sim \int_T^t \int_T^s q(r)x(r)^\beta drds, \quad t \to \infty, \tag{5.3}$$

with *regularly varying coefficients* $p(t)$ and $q(t)$.

5.1 Integral asymptotic relations

Lemma 5.1. *Let $\alpha\beta < 1$ and suppose that $p(t)$ and $q(t)$ are regularly varying functions of indices λ and μ, respectively. System of relations* (5.3) *has regularly varying solutions of index (ρ, σ) with $\rho < 0$ and $\sigma > 1$ if and only if*

$$\lambda + 2 + \alpha(\mu + 2) < 0, \quad \beta(\alpha + \lambda + 2) + \mu + 1 > 0, \tag{5.4}$$

in which case (ρ, σ) is given by (3.6) *and the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the formula*

$$x(t) \sim \Big[\frac{t^{2(1+\alpha)}p(t)q(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{2(1+\beta)}p(t)^\beta q(t)}{\Delta(\rho)^\beta\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{5.5}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau \in (-\infty, 0) \cup (1, \infty)$.

SKETCH OF PROOF. Since the right asymptotic relation (resp. the left asymptotic relation) in (5.3) can be analyzed exactly as in Lemma 3.1 (resp. Lemma 4.1), we need only to give a brief sketch of the proof, in which use is made of the expressions (3.4) and (3.8) for $p(t)$, $q(t)$, $x(t)$ and $y(t)$.

(The "only if" part) Suppose that (5.3) has a regularly varying solution $(x(t), y(t))$ on $[T, \infty)$ of index (ρ, σ) with $\rho < 0$ and $\sigma > 1$. Analysis of the relation

$$x(t) \sim \int_t^\infty \int_s^\infty r^{\lambda+\alpha\sigma} l(r)\eta(r)^\alpha drds, \quad t \to \infty,$$

shows that $\lambda + \alpha\sigma < -2$ and

$$x(t) \sim \frac{t^{\lambda+\alpha\sigma+2} l(t)\eta(t)^{\alpha}}{[-(\lambda+\alpha\sigma+1)][-(\lambda+\alpha\sigma+2)]} \in \mathrm{RV}(\lambda+\alpha\sigma+2), \quad t \to \infty, \tag{5.6}$$

while analysis of the relation

$$y(t) \sim \int_T^t \int_T^s r^{\mu+\beta\rho} m(r)\xi(r)^{\beta} dr ds, \quad t \to \infty,$$

shows that $\mu + \beta\rho > -1$ and

$$y(t) \sim \frac{t^{\mu+\beta\rho+2} m(t)\xi(t)^{\beta}}{(\mu+\beta\rho+1)(\mu+\beta\rho+2)} \in \mathrm{RV}(\mu+\beta\rho+2), \quad t \to \infty. \tag{5.7}$$

Therefore, we have $\rho = \lambda + \alpha\sigma + 2$ and $\sigma = \mu + \beta\rho + 2$, from which we see that (ρ, σ) must be given by (3.6). The inequalities $\rho < 0$ and $\sigma > 1$ require that the range of (λ, μ) be given by (5.4). The asymptotic formula (5.5) is an immediate consequence of the fact that (5.6) and (5.7) are rewritten as

$$x(t) \sim \frac{t^2 p(t) y(t)^{\alpha}}{\Delta(\rho)}, \quad y(t) \sim \frac{t^2 q(t) x(t)^{\beta}}{\Delta(\sigma)}, \quad t \to \infty.$$

(The "if" part) Suppose that (λ, μ) satisfies (5.4) and define $\rho < 0$ and $\sigma > 1$ by (3.6). Then, it can be shown that the regularly varying function $(X(t), Y(t))$ of index (ρ, σ) given by

$$X(t) = \Big[\frac{t^{2(1+\alpha)} p(t) q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = \Big[\frac{t^{2(1+\beta)} p(t)^{\beta} q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}},$$

satisfies the system of asymptotic relations (5.3).

5.2 Mixed strongly monotone solutions of (A)

It is shown that system (A) with nearly regularly varying coefficients may have strongly monotone solutions of the mixed type which are nearly regularly varying vector functions of indices $\rho < 0$ and $\sigma > 1$.

Theorem 5.1. *Let $\alpha\beta < 1$ and let $p(t)$ and $q(t)$ be nearly regularly varying of index λ and μ, respectively. Suppose that λ and μ satisfy* (5.4) *and define $\rho < 0$ and $\sigma > 1$ by* (3.6). *Then, system* (A) *possesses nearly regularly varying solutions $(x(t), y(t))$ of index (ρ, σ) with the property that*

$$x(t) \asymp t^{\rho}\Big[\frac{l(t) m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \asymp t^{\sigma}\Big[\frac{l(t)^{\beta} m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{5.8}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau \in (-\infty, 0) \cup (1, \infty)$.

SKETCH OF PROOF. By hypothesis there exist regularly varying functions

$$p_{\lambda}(t) = t^{\lambda} l(t) \in \mathrm{RV}(\lambda), \quad q_{\mu}(t) = t^{\mu} m(t) \in \mathrm{RV}(\mu),$$

such that $p(t) \asymp p_\lambda(t)$ and $q(t) \asymp q_\mu(t)$ as $t \to \infty$. Define the regularly varying functions $X_\lambda(t)$ and $Y_\mu(t)$ by

$$X_\lambda(t) = t^\rho \Big[\frac{l(t)m(t)^\alpha}{\Delta(\rho)\Delta(\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y_\mu(t) = t^\sigma \Big[\frac{l(t)^\beta m(t)}{\Delta(\rho)^\beta \Delta(\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \geq t_0,$$

which satisfy the asymptotic relations

$$(5.9) \qquad \int_t^\infty \int_s^\infty p_\lambda(r) Y_\mu(r)^\alpha drds \sim X_\lambda(t), \quad \int_b^t \int_b^s q_\mu(r) X_\lambda(r)^\beta drds \sim Y_\mu(t), \quad t \to \infty,$$

for any fixed $b \geq t_0$.

Using (5.9), one can choose first $T > t_0$ so that

$$(5.10) \quad \frac{1}{2} X_\lambda(t) \leq \int_t^\infty \int_s^\infty p_\lambda(r) Y_\mu(r)^\alpha drds \leq 2X_\lambda(t), \quad \int_T^t \int_T^s q_\mu(r) X_\lambda(r)^\beta drds \leq 2Y_\mu(t),$$

for $t \geq T$, and then $T_1 > T$ so that

$$(5.11) \qquad \int_T^t \int_T^s q_\mu(r) X_\lambda(r)^\beta drds \geq \frac{1}{2} Y_\mu(t) \quad \text{for } t \geq T_1.$$

We may assume that $X_\lambda(t)$ is decreasing and $Y_\mu(t)$ is increasing for $t \geq T$. Using the positive constants k, l, K and L such that

$$k p_\lambda(t) \leq p(t) \leq K p_\lambda(t), \quad l q_\mu(t) \leq q(t) \leq L q_\mu(t), \quad t \geq t_0,$$

choose $(a, b),\ (A, B) \in \mathbf{R^2}$ so that $a < A$, $b < B$, and

$$a \leq \frac{1}{2} k b^\alpha, \quad b \leq \frac{1}{2} l a^\beta, \quad 2KB^\alpha \leq A, \quad 4LA^\beta \leq B, \quad 2bY_\mu(T_1) \leq BY_\mu(T).$$

Let us now define the mapping $\Phi(x(t), y(t)) = (\mathcal{F}y(t), \mathcal{G}x(t))$, where

$$(5.12) \quad \mathcal{F}y(t) = \int_t^\infty \int_s^\infty p(r) y(r)^\alpha drds, \quad \mathcal{G}x(t) = y_0 + \int_T^t \int_T^s q(r) x(r)^\beta drds, \quad t \geq T,$$

with a constant y_0 satisfying $bY_\mu(T_1) \leq y_0 \leq \frac{1}{2} BY_\mu(T)$, and let Φ act on the set $\mathcal{X}$ comprised of vector functions $(x(t), y(t)) \in C[T, \infty) \times C[T, \infty)$ satisfying

$$aX_\lambda(t) \leq x(t) \leq AX_\lambda(t), \quad bY_\mu(t) \leq y(t) \leq BY_\mu(t), \quad t \geq T.$$

It is clear that $\mathcal{X}$ which is a closed convex subset of $C[T, \infty) \times C[T, \infty)$ and it can be verified in a routine manner that Φ is a continuous self-map of $\mathcal{X}$ and sends $\mathcal{X}$ into a relatively compact subset of $C[T, \infty) \times C[T, \infty)$. Consequently, Φ has a fixed point $(x(t), y(t)) \in \mathcal{X}$ which satisfies the integral equation

$$x(t) = \int_t^\infty \int_s^\infty p(r) y(r)^\alpha drds, \quad y(t) = y_0 + \int_T^t \int_T^s q(r) x(r)^\beta drds, \quad t \geq T.$$

This implies that $(x(t), y(t))$ is a positive solution of system (A) belonging to the class of nearly regularly varying functions of index (ρ, σ). Since $\rho < 0$ and $\sigma > 1$, this solution provides a mixed strongly monotone solution of (A). This sketches the proof of Theorem 5.1.

As in the preceding sections, under a stronger assumption that both coefficients in (A) are regularly varying functions, the following theorem characterizing the existence of fully regularly varying mixed strongly monotone solutions can be proved. The proof is similar to that of Theorem 3.2 and we omit it.

Theorem 5.2. *Suppose that $p(t)$ and $q(t)$ are regularly varying of indices λ and μ, respectively. System* (A) *possesses regularly varying solutions $(x(t), y(t))$ such that*

$$x(t) \in \mathrm{RV}(\rho), \quad y(t) \in \mathrm{RV}(\sigma), \quad \rho < 0,\ \sigma > 1,$$

if and only if (5.4) *holds, in which case ρ and σ are given by* (3.6) *and the asymptotic behavior of any such solution is governed by the formulas*

$$x(t) \sim \left[\frac{t^{2(\alpha+1)}p(t)q(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \left[\frac{t^{2(\beta+1)}p(t)^{\beta}q(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty, \tag{5.13}$$

where $\Delta(\tau) = \tau(\tau - 1)$ for $\tau \in (-\infty, 0) \cup (1, \infty)$.

Example 5.1. Consider system (A) with

$$p(t) \sim 2t^{-(2\alpha+3)}(\log t)^{\alpha+1}, \quad q(t) \sim 2t^{\beta}(\log t)^{-(\beta+1)}, \quad t \to \infty. \tag{5.14}$$

Condition (5.4) is satisfied since

$$\lambda + 2 + \alpha(\mu + 2) = \alpha\beta - 1 < 0, \quad \beta(\alpha + \lambda + 2) + \mu + 1 = 1 - \alpha\beta > 0,$$

and (3.6) gives $\rho = -1$ and $\sigma = 2$, which implies $\Delta(\rho) = \Delta(\sigma) = 2$. Since

$$\frac{l(t)m(t)^{\alpha}}{\Delta(\rho)\Delta(\sigma)^{\alpha}} = (\log t)^{1-\alpha\beta}, \quad \frac{l(t)^{\beta}m(t)}{\Delta(\rho)^{\beta}\Delta(\sigma)} = (\log t)^{\alpha\beta-1},$$

from Theorem 5.2 it follows that our system (A) possesses mixed strongly monotone solutions belonging to class of regularly varying functions of index $(-1, 2)$ and that the asymptotic behavior of any such solution $(x(t), y(t))$ is governed by the formula

$$x(t) \sim t^{-1}(\log t), \quad y(t) \sim t^{2}(\log t)^{-1}, \quad t \to \infty.$$

If in particular

$$p(t) = 2t^{-(2\alpha+3)}(\log t)^{\alpha+1}\Big(1 - \frac{3}{2\log t}\Big), \quad q(t) = 2t^{\beta}(\log t)^{-(\beta+1)}\Big(1 - \frac{3}{2\log t} + \frac{1}{(\log t)^2}\Big),$$

then system (A) has an exact mixed strongly monotone solution $(t^{-1}\log t, t^{2}(\log t)^{-1})$ which is regularly varying of index $(-1, 2)$.

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