Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **16**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

ANTI-PERIODIC PROBLEMS FOR SEMILINEAR PARTIAL NEUTRAL EVOLUTION EQUATIONS†

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Abstract. We study the anti-periodic problem for the semilinear partial neutral evolution equation in the form

$$\frac{d}{dt}[u(t)+h(t,u(t))]+Au(t)=f(t,u(t)),\quad t\in\mathbb{R}$$

in a Banach space X, where h, f are given X-valued functions, and $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact analytic semigroup. Some new theorems concerning the existence of anti-periodic mild solutions for the problem are established. The theorems formulated are essential extensions of those given previously for the anti-periodic problems for evolution equations in Banach spaces. The main tools in our study are the analytic semigroup theory of linear operators, fractional powers of closed operators, and a fixed point theorem due to Krasnoselskii. Furthermore, we provide an illustrative example to justify the practical usefulness of the obtained abstract results.

1. Introduction

Anti-periodic problems arise from the mathematical models of various process and phenomena in physics. As in [1], the mathematical modelling of the electron beam focusing system in travelling-wave tube's theories can be cast into an anti-periodic problem. For a comprehensive exposition of further examples illustrating the outstanding importance of this class of problems in applications, we refer the readers to [2–4] and the references therein.

Anti-periodic problems represented by linear and nonlinear abstract evolution equations have been extensively studied up to now, especially since the work of Okochi [5] in 1988 (see also [6, 7]). Here, we mention some references, but not a list of all references is included. Following Okochi's work, Haraux [8] proved the existence of anti-periodic solutions for nonlinear first order evolution equations in Hilbert spaces by using Brouwer's or Schauder's fixed point theorem. Aftabizadeh et al. [9] and Aizicovici et al. [10] considered the anti-periodic solutions for second order evolution equations in Hilbert and Banach spaces by utilizing monotone and accretive operator theory (see also [11, 12] for nonmonotone cases). In particular, making use of the maximal monotone property of the derivative operator with anti-periodic conditions and the theory of pseudomonotone perturbations of maximal monotone mappings, Liu [13] recently studied the anti-periodic problem for nonlinear abstract differential equation with nonmonotone perturbation of form

$$\begin{cases} \dfrac{du(t)}{dt}+Au(t)+Gu(t)=f(t), & \text{a.e. } t\in(0,T),\\ u(T)=-u(0) \end{cases}$$

2000 *Mathematics Subject Classification.* Primary 34K13; Secondary 34K60.
Key words and phrases. Partial neutral evolution equation; Anti-periodicity; Mild solution; Analytic semigroup.
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†This work was supported by the NNSF of China (No. 11101202).

in a real reflexive Banach space V, where $T > 0$, and A is monotone and G is not; Q. Liu [14] recently dealed with the existence of the anti-periodic mild solutions to the semilinear abstract differential equation in the form

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \\ u(t+T) = -u(t), \quad t \in \mathbb{R}, \end{cases}$$

where $\mathbb{R}$ stands for the set of real numbers and A is the generator of a hyperbolic C_0-semigroup. From [2,15,16] and the references therein, one can find more results about anti-periodic problems for abstract evolution equations.

However, to the best of our knowledge, the existence of anti-periodic solutions for neutral evolution equation is still a untreated topic in the literature. Moreover, as indicated in [15], the existence of anti-periodic solutions plays a key role in characterizing the behavior of linear and nonlinear differential equations. Inspired by these, in the present paper we study the existence of T-anti-periodic mild solutions to the following semilinear partial neutral evolution equation

$$\frac{d}{dt}[u(t) + h(t, u(t))] + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

subject to anti-periodic condition

$$u(t+T) = -u(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t\geq 0}$ on a Banach space X and $h,\ f$ are given X-valued functions to be specified later. As can be seen, "$u(t+T) = -u(t)$ $(t \in \mathbb{R})$" constitutes an anti-periodic condition.

It is worth mentioning that neutral evolution equations arise in many areas of applied mathematics and have, in some cases, better effects in applications than evolution equations without neutral item (cf. [17–19]). For instance, Wu and Xia [18] proposed and studied a system of partial neutral evolution equations defined on the unit circle S, which models a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. That is why those equations have been objects of investigation with increasing interest during the past decades. The literature relative to neutral evolution equations is quite extensive; see, for instance, Hale [17] for ordinary neutral evolution equations, Adimy and Ezzinbi [20], Hernández [21], Hernández and Henriquez [22], Wu [23], and Wu and Xia [18, 19] for partial neutral evolution equations, and dos Santos and Cuevas [24] for fractional neutral evolution equations.

We would like to note that in the recent papers such as Agarwal et al. [25], Diagana et al. [26–28], Ezzinbi et al. [29], and N'Guerekata [30], the problem of the existence of almost periodic, asymptotically almost periodic, pseudo almost periodic, almost automorphic, and asymptotically almost automorphic solutions for partial neutral evolution equations has been investigated to a large extent. As to the study of related issues on neutral integral equations, we refer readers to Ait Dads and Ezzinbi [31], Burton and Furumochi [32], Ding et al. [33] and references therein.

Our object in this paper is to give some new results concerning the existence of anti-periodic mild solutions to the problem (1.1)-(1.2). The theorems formulated are essential extensions of those given previously for the anti-periodic problems for evolution equations in Banach spaces. As the reader will see, the hypotheses in our theorems are reasonably weak, the proofs provided are concise, and the methods used in this paper can also be applied to deal with the existence of periodic mild solutions for the semilinear partial neutral evolution equation with periodic condition (see Remark 3.3). The main tools in our study are the analytic semigroup theory, fractional powers of closed operators, and fixed point theorems due to Banach and Krasnoselskii. An application to partial differential equation with homogeneous Dirichlet boundary condition and anti-periodic condition is also presented.

Remark 1.1. *The constant function has the interesting property of being periodic with any period T and anti-periodic with any anti-period T for all nonzero real numbers T.*

Remark 1.2. *It can be easily shown that if u is anti-periodic with period T, then it is periodic with period $2T$. Hence, from the arguments of our paper, we can also obtain the existence results of periodic solutions of the problem (1.1) − (1.2). But if u is periodic with period $2T$, u may or may not be anti-periodic with period T.*

Remark 1.3. *As in* [34], *under certain conditions, the existence result is valid for the case of anti-periodic solutions, while there is no such a result in the periodic case. It is also noted that in dealing with the existence of certain problems, there is an essential difference between the periodic solutions and anti-periodic solutions* (*see* [35] *for more details*).

This work is organized as follows. In Section 2, we introduce some notions, definitions, hypotheses, and preliminary facts that are needed in the sequel. In Section 3, we present our main results and their proofs. An example in Section 4 is given to illustrate our abstract results.

2. Preliminaries and Notations

This section is devoted to some preliminary results needed in what follows.

Throughout this paper, X is assumed to be a Banach space with norm $\|\cdot\|$, $\mathscr{L}(X)$ stands for the Banach space of all bounded linear operators from X to X equipped with its natural topology, $C([0,T];X)$ stands for the Banach space of all continuous functions from $[0,T]$ into X with the uniform norm topology

$$|u|_0 = \sup\{\|u(t)\|, t \in [0,T]\}.$$

Let $-A : D(A) \subseteq X \to X$ be the infinitesimal generator of a compact analytic semigroup $\{S(t)\}_{t\geq 0}$ on X and $0 \in \rho(A)$ ($\rho(A)$ stands for the resolvent set of A), which implies that $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable, i.e., there exist constants $\omega > 0$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathscr{L}(X)} \leq Me^{-\omega t} \quad \text{for all } t \geq 0, \tag{2.1}$$

and allows us to define the fractional power A^α for $0 \leq \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$. Let X_α denote the Banach space $D(A^\alpha)$ endowed with the graph norm $\|u\|_\alpha = \|A^\alpha u\|$ for $u \in X_\alpha$.

The following are basic properties of A^α.

Proposition 2.1. ([36], pp. 69-75)).

(a) *$S(t) : X \to X_\alpha$ for each $t > 0$, and $A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in X_\alpha$ and $t \geq 0$.*

(b) *$A^\alpha S(t)$ is bounded on X for every $t > 0$ and there exists a $M_\alpha > 0$ such that*

$$\|A^\alpha S(t)\|_{\mathscr{L}(X)} \leq \frac{M_\alpha}{t^\alpha} e^{-\omega t}.$$

(c) *$A^{-\alpha}$ is a bounded linear operator in X with $D(A^\alpha) = Im(A^{-\alpha})$.*

(d) *If $0 < \alpha_1 \leq \alpha_2$, then $X_{\alpha_2} \hookrightarrow X_{\alpha_1}$.*

We denote by $C_b(\mathbb{R};X)$ the Banach space of all bounded, continuous functions from $\mathbb{R}$ to X equipped with the sup norm

$$\|u\|_{C_b(\mathbb{R};X)} = \sup\{\|u(t)\|; t \in \mathbb{R}\},$$

by $L(0,T;X)$ the Banach space of all Bocher integrable functions from $[0,T]$ to X equipped with the norm

$$\|u\|_{L(0,T;X)} = \int_0^T \|u(t)\|dt,$$

and by $L_{loc}(\mathbb{R};X)$ the set of all locally Bocher integrable functions from $\mathbb{R}$ to X.

A function $u \in C_b(\mathbb{R};X)$ is said to be $T-$anti-periodic if

$$u(t+T) = -u(t) \quad \text{for all } t \in \mathbb{R}.$$

In the rest of this section, by $P_{TA}(\mathbb{R};X)$, we denote the set of all $T-$anti-periodic functions from $\mathbb{R}$ to X. It is easy to see that $P_{TA}(\mathbb{R};X)$, equipped with the sup norm, is a Banach space. Additionally, similar definitions as above also apply to $C_b(\mathbb{R};X_\alpha)$ and $P_{TA}(\mathbb{R};X_\alpha)$.

Definition 2.1. *A function $u \in C_b(\mathbb{R};X)$ is said to be a mild solution of equation (1.1), if the function $\tau \to AS(t-\tau)h(\tau,u(\tau))$ is integrable on $[s,t)$ for all $t > s$ and it satisfies the following integral equation*

$$\begin{aligned} u(t) = S(t-s)\big[u(s) + h(s,u(s))\big] - h(t,u(t)) + \int_s^t AS(t-\tau)h(\tau,u(\tau))d\tau \\ + \int_s^t S(t-\tau)f(\tau,u(\tau))d\tau \end{aligned}$$

for all $t > s$.

To prove our main results, we introduce the following assumptions. For sake of brevity, put $B_r := \{x \in X; \|x\| \leq r\}$ for some $r > 0$.

(H_1) (i) There exists a $\beta \in (0,1)$ such that the function $h : \mathbb{R} \times X \to X_\beta$ is continuous and $h(t+T,-u) = -h(t,u)$ for all $t \in \mathbb{R},\ u \in X$.

(ii) There exist a constant L_h and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} &\|A^\beta h(t,u) - A^\beta h(t,v)\| \leq L_h\|u-v\|, \\ &\|A^\beta h(t,u)\| \leq \Psi(\|u\|) \end{aligned}$$

for all $t \in \mathbb{R},\ u,v \in X$ and

$$\liminf_{r\to+\infty} \frac{\Psi(r)}{r} = \sigma_1.$$

(H_2) The function $f : \mathbb{R} \times X \to X$ satisfies the following conditions.

(i) $f(\cdot,u)$ is measurable for each $u \in X$ and $f(t+T,-u) = -f(t,u)$ for all $t \in \mathbb{R}$, $u \in X$.

(ii) There exists a constant $L_f > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L_f\|u-v\|$$

for a.e. $t \in [0,T]$ and all $u,v \in X$.

(H_3) (i) The function $f : \mathbb{R} \times X \to X$ is a Carathéodory function, i.e., for every $u \in X$, $f(\cdot,u)$ is measurable and for a.e. $t \in \mathbb{R}$, $f(t,\cdot)$ is continuous, and $f(t+T,-u) = -f(t,u)$ for all $t \in \mathbb{R}$, $u \in X$.

(ii) There exists a function $\Phi_r(\cdot) \in L(0,T;\mathbb{R}^+)$ such that

$$\|f(t,u)\| \le \Phi_r(t)$$

for a.e. $t \in [0,T]$ and all $u \in B_r$, and

$$\liminf_{r\to+\infty} \frac{\int_0^T \Phi_r(s)ds}{r} = \sigma_2.$$

The considerations of this paper also need the following result.

Lemma 2.1 (Krasnoselskii′s Fixed Point Theorem)**.** *Let E be a Banach space and B be a bounded closed and convex subset of E, and let F_1, F_2 be maps of B into E such that $F_1u+F_2v \in B$ for every pair $u,v \in B$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1u + F_2u = u$ has a solution on B.*

Lemmas 2.1 is classical, which can be found in many books.

3. Main Results

Before stating the existence theorems of mild solutions, we first prove the following lemmas.

Lemma 3.1. *A set $D \subseteq P_{TA}(\mathbb{R};X)$ is relatively compact in $P_{TA}(\mathbb{R};X)$ if D is equicontinuous and the set $D(t) := \{u(t); u \in D\}$ is relatively compact in X for every $t \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n=1}^{\infty} \subseteq D$. One needs to show that $\{u_n\}_{n=1}^{\infty}$ is relatively compact in $P_{TA}(\mathbb{R};X)$.

To begin, put, for each $n \ge 1$,

$$u_{n,0}(t) := u_n(t) \quad \text{for } t \in [0,T] \text{ (the restriction of } u_n \text{ to the interval } [0,T]).$$

Therefore, from our hypotheses it follows that $\{u_{n,0}\}_{n=1}^{\infty}$ is equicontinuous in $C([0,T];X)$ and for every $t \in [0,T]$, the set $\{u_{n,0}(t)\}_{n=1}^{\infty}$ is relatively compact in X. From this, we conclude, using Ascoli-Arzelà theorem, that there exists a subsequence, denoted by $\{u_{n_k,0}\}_{k=1}^{\infty}$, of $\{u_{n,0}\}_{n=1}^{\infty}$ and $u_0 \in C([0,T];X)$ such that

$$u_{n_k,0} \to u_0 \quad \text{in } C([0,T];X) \quad (\text{as } k \to \infty). \tag{3.1}$$

Also, it is not difficult to see that $u_0(0) = -u_0(T)$.

Put

$$\overline{u}(t) := (-1)^m u_0(t-mT) \qquad \text{for } t \in [mT,(m+1)T],\ m \in \mathbb{Z}.$$

Therefore $\overline{u}$ belongs to $P_{TA}(\mathbb{R};X)$. Moreover, since u_{n_k} is T-anti-periodic, we have

$$u_{n_k}(t) - \overline{u}(t) = (-1)^m(u_{n_k}(t-mT) - u_0(t-mT))$$

for $t \in [mT,(m+1)T]$ $(m \in \mathbb{Z})$. Consequently, from (3.1) it follows that

$$\|u_{n_k} - \overline{u}\|_{C_b(\mathbb{R};X)} \le \sup_{m\in\mathbb{Z}} \|u_{n_k,0} - u_0\|_0 \to 0 \quad \text{as } k \to \infty,$$

which implies that $\{u_n\}_{n=1}^{\infty}$ is relatively compact in $P_{TA}(\mathbb{R};X)$. This completes the proof. □

Lemma 3.2. *Let $0 \le \mu < 1$. Suppose that $g_1 \in P_{TA}(\mathbb{R};X_\mu)$, $g_2 \in L_{loc}(\mathbb{R};X)$, and $g_2(t+T) = -g_2(t)$ for a.e. $t \in \mathbb{R}$. Define*

$$(\Phi_1 g_1)(t) := -g_1(t) + \int_{-\infty}^t AS(t-\tau)g_1(\tau)d\tau, \quad t \in \mathbb{R},$$
$$(\Phi_2 g_2)(t) := \int_{-\infty}^t S(t-\tau)g_2(\tau)d\tau, \quad t \in \mathbb{R}.$$

Then $\Phi_1 g_1, \Phi_2 g_2$ belong to $P_{TA}(\mathbb{R}; X)$.

Proof. Since $g_1 \in P_{TA}(\mathbb{R}; X_\mu)$, $g_2 \in L_{loc}(\mathbb{R}; X)$, we see, from (2.1) and Proposition 2.1 (b), that for any $t \in \mathbb{R}$,

$$\begin{aligned}
\|(\Phi_1 g_1)(t)\| &\leq \|g_1(t)\| + \left\| \int_{-\infty}^{t} AS(t-\tau) g_1(\tau) d\tau \right\| \\
&\leq \|A^{-\mu}\|_{\mathscr{L}(X)} \|A^{\mu} g_1(t)\| + \int_{-\infty}^{t} \|A^{1-\mu} S(t-\tau)\|_{\mathscr{L}(X)} \|A^{\mu} g_1(\tau)\| d\tau \\
&\leq \|A^{-\mu}\|_{\mathscr{L}(X)} \|g_1(t)\|_{\mu} + M_{1-\mu} \int_{-\infty}^{t} (t-\tau)^{\mu-1} e^{-\omega(t-\tau)} \|g_1(\tau)\|_{\mu} d\tau \\
&\leq \big(\|A^{-\mu}\|_{\mathscr{L}(X)} + M_{1-\mu} \omega^{-\mu} \Gamma(\mu) \big) \|g_1\|_{P_{TA}(\mathbb{R}; X_\mu)},
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function, and

$$\begin{aligned}
\|(\Phi_2 g_2)(t)\| &\leq \int_{-\infty}^{t} \|S(t-\tau) g_2(\tau)\| d\tau \\
&\leq M \sum_{k=0}^{\infty} e^{-k\omega T} \int_{t-T}^{t} e^{-\omega(t-\tau)} \|g_2(\tau)\| d\tau \\
&\leq \frac{M}{1 - e^{-\omega T}} \int_0^T \|g_2(\tau)\| d\tau,
\end{aligned}$$

which imply that Φ_1 and Φ_2 are well defined and $\Phi_1 g_1$ and $\Phi_2 g_2$ are bounded. Furthermore, we observe that for any $t, s \in \mathbb{R}$,

$$\begin{aligned}
\|(\Phi_1 g_1)(t) - (\Phi_1 g_1)(s)\| &\leq \|g_1(t) - g_1(s)\| + \int_{-\infty}^{s} \|AS(s-\tau)(g_1(\tau+t-s) - g_1(\tau))\| d\tau \\
&\leq \|A^{-\mu}\|_{\mathscr{L}(X)} \|A^{\mu} g_1(t) - A^{\mu} g_1(s)\| \\
&\quad + \int_{-\infty}^{s} \|A^{1-\mu} S(s-\tau)\|_{\mathscr{L}(X)} \|A^{\mu} g_1(\tau+t-s) - A^{\mu} g_1(\tau)\| d\tau \\
&\leq \|A^{-\mu}\|_{\mathscr{L}(X)} \|g_1(t) - g_1(s)\|_{\mu} \\
&\quad + M_{1-\mu} \int_{-\infty}^{s} (s-\tau)^{\mu-1} e^{-\omega(s-\tau)} \|g_1(\tau+t-s) - g_1(\tau)\|_{\mu} d\tau \\
&\leq \|A^{-\mu}\|_{\mathscr{L}(X)} \|g_1(t) - g_2(s)\|_{\mu} \\
&\quad + M_{1-\mu} \omega^{-\mu} \Gamma(\mu) \sup_{\tau \in \mathbb{R}} \|g_1(\tau+t-s) - g_1(\tau)\|_{\mu},
\end{aligned}$$

and

$$\begin{aligned}
\|(\Phi_2 g_2)(t) - (\Phi_2 g_2)(s)\| &\leq \int_{-\infty}^{s} \|S(t-\tau)(g_2(\tau+t-s) - g_2(\tau))\| d\tau \\
&\leq M \int_{-\infty}^{s} e^{-\omega(s-\tau)} \|g_2(\tau+t-s) - g_2(\tau)\| d\tau \\
&\leq \frac{M}{1 - e^{-\omega T}} \int_0^T \|g_2(\tau+t-s) - g_2(\tau)\| d\tau,
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Thus,

$$\begin{aligned}
&\|(\Phi_1 g_1)(t) - (\Phi_1 g_1)(s)\| \to 0 \quad \text{as } t - s \to 0, \\
&\|(\Phi_2 g_2)(t) - (\Phi_2 g_2)(s)\| \to 0 \quad \text{as } t - s \to 0,
\end{aligned}$$

which prove that $\Phi_1 g_1$ and $\Phi_2 g_2$ are continuous.

To complete the proof of the lemma, we have to show that $\Phi_1 g_1$ and $\Phi_2 g_2$ are T-anti-periodic. In fact, this can be seen from the observations that for any $t \in \mathbb{R}$,

$$\begin{aligned}(\Phi_1 g_1)(t+T) &= -g_1(t+T) + \int_{-\infty}^{t+T} AS(t+T-\tau)g_1(\tau)d\tau \\ &= g_1(t) + \int_{-\infty}^{t} AS(t-\tau)g_1(\tau+T)d\tau \\ &= g_1(t) - \int_{-\infty}^{t} AS(t-\tau)g_1(\tau)d\tau \\ &= -(\Phi_1 g_1)(t)\end{aligned}$$

and

$$\begin{aligned}(\Phi_2 g_2)(t+T) &= \int_{-\infty}^{t+T} S(t+T-\tau)g_2(\tau)d\tau \\ &= \int_{-\infty}^{t} S(t-\tau)g_2(\tau+T)d\tau \\ &= -(\Phi_2 g_2)(t)\end{aligned}$$

in view of the anti-periodicity of g_1 and g_2. Consequently, we obtain, by the arguments above, that $\Phi_1 g_1, \Phi_2 g_2$ belong to $P_{TA}(\mathbb{R}; X)$. The proof is complete. □

We now return to the problem (1.1)-(1.2). One of our main results in this paper is the following theorem.

Theorem 3.1. *Let* (H_1) *and* (H_2) *hold. Then, there exists a unique* T*-anti-periodic mild solution for the problem* (1.1)-(1.2)*, provided that*

$$L_h \|A^{-\beta}\|_{\mathscr{L}(X)} + M_{1-\beta} L_h \omega^{-\beta}\Gamma(\beta) + ML_f\omega^{-1} < 1. \tag{3.2}$$

Proof. Set, for $u \in P_{TA}(\mathbb{R}; X)$,

$$g_1(\cdot) := h(\cdot, u(\cdot)), \quad g_2(\cdot) := f(\cdot, u(\cdot)).$$

Then it follows from (H_1) (i) and (H_2) (i) that the functions g_1 and g_2 satisfy the conditions of Lemma 3.2 with $\mu = \beta$. This implies that the mapping Υ defined by

$$\begin{aligned}(\Upsilon u)(t) = &-h(t, u(t)) + \int_{-\infty}^{t} AS(t-\tau)h(\tau, u(\tau))d\tau \\ &+ \int_{-\infty}^{t} S(t-\tau)f(\tau, u(\tau))d\tau, \quad u \in P_{TA}(\mathbb{R}; X)\end{aligned}$$

is well defined and maps $P_{TA}(\mathbb{R}; X)$ into itself.

To prove the theorem, we first show that Υ has a unique fixed point in $P_{TA}(\mathbb{R};X)$. Let $u,v\in P_{TA}(\mathbb{R};X)$. Therefore, with the help of (H_1) (ii) and (H_2) (ii) one has

$$\begin{aligned}
\|(\Upsilon u)(t)-(\Upsilon v)(t)\| &\le \|h(t,u(t))-h(t,v(t))\|+\int_{-\infty}^{t}\|AS(t-\tau)(h(\tau,u(\tau))-h(\tau,v(\tau)))\|d\tau\\
&\quad+\int_{-\infty}^{t}\|S(t-\tau)(f(\tau,u(\tau))-f(\tau,v(\tau))\|d\tau\\
&\le \|A^{-\beta}\|_{\mathscr{L}(X)}\|A^{\beta}h(t,u(t))-A^{-\beta}h(t,v(t)\|\\
&\quad+\int_{-\infty}^{t}\|A^{1-\beta}S(t-\tau)\|_{\mathscr{L}(X)}\|A^{\beta}h(\tau,u(\tau))-A^{\beta}h(\tau,v(\tau))\|d\tau\\
&\quad+\int_{-\infty}^{t}\|S(t-\tau)\|_{\mathscr{L}(X)}\|f(\tau,u(\tau))-f(\tau,v(\tau))\|d\tau\\
&\le L_h\|A^{-\beta}\|_{\mathscr{L}(X)}\|u(t)-v(t)\|+L_hM_{1-\beta}\omega^{-\beta}\Gamma(\beta)\|u-v\|_{P_{TA}(\mathbb{R};X)}\\
&\quad+ML_f\omega^{-1}\|u-v\|_{P_{TA}(\mathbb{R};X)},
\end{aligned}$$

from which it follows that

$$\|\Upsilon u-\Upsilon v\|_{P_{TA}(\mathbb{R};X)}\le\big(L_h\|A^{-\beta}\|_{\mathscr{L}(X)}+M_{1-\beta}L_h\omega^{-\beta}\Gamma(\beta)+ML_f\omega^{-1}\big)\|u-v\|_{P_{TA}(\mathbb{R};X)},$$

which together with (3.2) yields that Υ is a contractive mapping on $P_{TA}(\mathbb{R};X)$. Thus, we conclude, using the Banach contraction principle, that Υ has a unique fixed point in $P_{TA}(\mathbb{R};X)$.

In the rest of the proof, we will prove that $u\in P_{TA}(\mathbb{R};X)$ is a mild solution of (1.1) if and only if it is a fixed point of Υ. To this end, we first let $u\in P_{TA}(\mathbb{R};X)$ be a mild solution of (1.1), that is, the function $\tau\to AS(t-\tau)h(\tau,u(\tau))$ is integrable on $[s,t)$ for all $t>s$ and u satisfies the integral equation

$$\begin{aligned}
u(t)=S(t-s)\big[u(s)+h(s,u(s))\big]-h(t,u(t))+\int_{s}^{t}AS(t-\tau)h(\tau,u(\tau))d\tau\\
+\int_{s}^{t}S(t-\tau)f(\tau,u(\tau))d\tau
\end{aligned}$$

for all $t>s$. Letting $t\in\mathbb{R}$ be fixed and $s\to-\infty$, it follows, noticing (2.1) and Proposition 2.1 (b), that

$$u(t)=-h(t,u(t))+\int_{-\infty}^{t}AS(t-\tau)h(\tau,u(\tau))d\tau+\int_{-\infty}^{t}S(t-\tau)f(\tau,u(\tau))d\tau, \tag{3.3}$$

which yields that u is a fixed point of Υ.

Conversely, if $u\in P_{TA}(\mathbb{R};X)$ is a fixed point of Υ, then u satisfies the integral equations (3.3) and

$$S(t-s)(u(s)+h(s,u(s)))=\int_{-\infty}^{s}AS(t-\tau)h(\tau,u(\tau))d\tau+\int_{-\infty}^{s}S(t-\tau)f(\tau,u(\tau))d\tau. \tag{3.4}$$

for all $t>s$. Therefore, it is not difficult to see, subtracting (3.3) from (3.4), that u is a mild solution of (1.1).

According to the discussion above we deduce that the problem (1.1)-(1.2) has a unique T-anti-periodic mild solution. The proof is completed. □

Now we are in a position to prove our second existence result of anti-periodic mild solutions for the problem (1.1)-(1.2). Below, set $L'=\max\{L_h,\sigma_1\}$.

Theorem 3.2. *Let* (H_1) *and* (H_3) *hold. Then the problem* (1.1)-(1.2) *has at least one* T*-anti-periodic mild solution provided that*

$$L'(\|A^{-\beta}\|_{\mathscr{L}(X)} + M_{1-\beta}\omega^{-\beta}\Gamma(\beta)) + \frac{M\sigma_2}{1-e^{-\omega T}} < 1. \tag{3.5}$$

Proof. Assume that the mapping Υ is defined the same as in Theorem 3.1. We first notice, thanks to assumptions (H_1) (i), (H_3) (i) and Lemma 3.2, that Υ is well defined and maps $P_{TA}(\mathbb{R};X)$ into itself.

Next, by applying a Krasnoselskii's fixed point theorem we show that Υ has at least one fixed point in $P_{TA}(\mathbb{R};X)$. To this end, let us decompose the mapping $\Upsilon = \Upsilon_h + \Upsilon_f$ as

$$(\Upsilon_h u)(t) = -h(t,u(t)) + \int_{-\infty}^{t} AS(t-\tau)h(\tau,u(\tau))d\tau, \quad u \in P_{TA}(\mathbb{R};X),\ t\in\mathbb{R},$$

and

$$(\Upsilon_f u)(t) = \int_{-\infty}^{t} S(t-\tau)f(\tau,u(\tau))d\tau, \quad u \in P_{TA}(\mathbb{R};X),\ t\in\mathbb{R}.$$

For any $r>0$, write

$$\Omega_r = \{u \in P_{TA}(\mathbb{R};X); \|u\|_{C_b(\mathbb{R};X)} \le r\}.$$

From (3.5), (H_1) (ii), and (H_3) (ii) it is easy to see that there exists a $k_0>0$ such that

$$(\|A^{-\beta}\| + M_{1-\beta}\omega^{-\beta}\Gamma(\beta))\Psi(k_0) + \frac{M}{1-e^{-\omega T}}\int_0^T \Phi_{k_0}(\tau)d\tau \le k_0.$$

Therefore, for every pair $u,v\in\Omega_{k_0}$, a direct calculation gives that

$$\begin{aligned}
&\|(\Upsilon_h u)(t) + (\Upsilon_f v)(t)\| \\
&\le \|h(t,u(t))\| + \int_{-\infty}^{t}\|A^{1-\beta}S(t-\tau)A^{\beta}h(\tau,u(\tau))\|d\tau + \int_{-\infty}^{t}\|S(t-\tau)f(\tau,v(\tau))\|d\tau \\
&\le \|A^{-\beta}\|\Psi(\|u(t)\|) + M_{1-\beta}\int_{-\infty}^{t}(t-\tau)^{\beta-1}e^{-\omega(t-\tau)}\Psi(\|u(\tau)\|)d\tau + M\int_{-\infty}^{t}e^{-\omega(t-\tau)}\Phi_r(\tau)d\tau \\
&\le (\|A^{-\beta}\| + M_{1-\beta}\omega^{-\beta}\Gamma(\beta))\Psi(k_0) + \frac{M}{1-e^{-\omega T}}\int_0^T\Phi_{k_0}(\tau)d\tau \\
&\le k_0,
\end{aligned}$$

from which we see that $\Upsilon_h u + \Upsilon_f v \in \Omega_{k_0}$ for every pair $u,v\in\Omega_{k_0}$.

What followed is to prove Υ_h and Υ_f satisfy the conditions of Lemma 2.1 with $\Upsilon_h = F_1$ and $\Upsilon_f = F_2$. Taking $u,v\in\Omega_{k_0}$, one can infer, thanks to (H_1) (ii), that

$$\begin{aligned}
&\|(\Upsilon_h u)(t) - (\Upsilon_h v)(t)\| \\
&\le \|h(t,u(t)) - h(t,v(t))\| + \int_{-\infty}^{t}\|AS(t-\tau)(h(\tau,u(\tau)) - h(\tau,v(\tau)))\|d\tau \\
&\le \|A^{-\beta}\|_{\mathscr{L}(X)}\|A^{\beta}h(t,u(t)) - A^{\beta}h(t,v(t))\| \\
&\quad + \int_{-\infty}^{t}\|A^{1-\beta}S(t-\tau)\|_{\mathscr{L}(X)}\|A^{\beta}h(\tau,u(\tau)) - A^{\beta}h(\tau,v(\tau))\|d\tau \\
&\le L_h\|A^{-\beta}\|_{\mathscr{L}(X)}\|u-v\|_{P_{TA}(\mathbb{R};X)} + L_h M_{1-\beta}\|u-v\|_{P_{TA}(\mathbb{R};X)}\int_{-\infty}^{t}(t-\tau)^{\beta-1}e^{-\omega(t-\tau)}d\tau \\
&\le L_h(\|A^{-\beta}\|_{\mathscr{L}(X)} + M_{1-\beta}\omega^{-\beta}\Gamma(\beta))\|u-v\|_{P_{TA}(\mathbb{R};X)}.
\end{aligned}$$

This together with (3.5) yields that Υ_h is a contractive mapping on Ω_{k_0}.

In the sequel, we show that Υ_f is completely continuous on Ω_{k_0}. The proof will be divided into two steps.

Step 1. Υ_f is continuous on Ω_{k_0}.

Take $u_1, u_2 \in \Omega_{k_0}$. It follows from (H_3) (ii) that

$$\begin{aligned}
&\int_{-\infty}^{t} \|S(t-\tau)(f(\tau,u_1(\tau)) - f(\tau,u_2(\tau)))\| d\tau \\
&\leq 2M \int_{-\infty}^{t} e^{-\omega(t-\tau)} \Phi_{k_0}(\tau) d\tau \\
&\leq \frac{2M}{1-e^{-\omega T}} \int_0^T \Phi_{k_0}(\tau) d\tau.
\end{aligned}$$

Then the Lebesgue dominated convergence theorem gives, noticing the continuity of f with respect to second variable, that

$$\begin{aligned}
&\|(\Upsilon_f u_1)(t) - (\Upsilon_f u_2)(t)\| \\
&\leq M \int_{-\infty}^{t} e^{-\omega(t-\tau)} \|(f(\tau,u_1(\tau)) - f(\tau,u_2(\tau)))\| d\tau \\
&\leq \frac{M}{1-e^{-\omega T}} \int_0^T \|f(\tau,u_1(\tau)) - f(\tau,u_2(\tau))\| d\tau \\
&\to 0, \quad as\ u_1 \to u_2\ in\ \Omega_{k_0},
\end{aligned}$$

which verifies the continuity of Υ_f.

Step 2. Υ_f is a compact operator on Ω_{k_0}.

Since $S(s)$ is compact for $s > 0$ in X,

$$\left\{ \int_{-\infty}^{t-\epsilon} S(t-\tau) f(\tau,u(\tau)) d\tau; u \in \Omega_{k_0} \right\} = \left\{ S(\varepsilon) \int_{-\infty}^{t} S(t-\tau) f(\tau,u(\tau)) d\tau; u \in \Omega_{k_0} \right\}$$

is relatively compact in X for each $t \in \mathbb{R}$ and $\varepsilon > 0$. Then, for every $u \in \Omega_{k_0}$, as

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} S(t-\tau) f(\tau,u(\tau)) d\tau - \int_{-\infty}^{t-\epsilon} S(t-\tau) f(\tau,u(\tau)) d\tau \right\| \\
&\leq \int_{t-\epsilon}^{t} \|S(t-\tau) f(\tau,u(\tau))\| d\tau \\
&\leq M \int_{t-\epsilon}^{t} e^{-\omega(t-\tau)} \Phi_{k_0}(\tau) d\tau \\
&\to 0 \quad as\ \varepsilon \to 0
\end{aligned}$$

in X, we conclude, in view of the total boundedness, that for each $t \in \mathbb{R}$, the set $\{(\Upsilon_f u)(t); u \in \Omega_{k_0}\}$ is relatively compact in X.

To prove that Υ_f is a compact operator, it remains to prove that the set $\{\Upsilon_f u; u \in \Omega_{k_0}\}$ is equicontinuous in view of Lemma 3.1.

Extend $\Phi_{k_0}(t)$ to $\mathbb{R}$ by defining $\Phi_{k_0}(t+T) = \Phi_{k_0}(t)$ for $t \in \mathbb{R}$ and again denote it by $\Phi_{k_0}(t)$. Note that $\Phi_{k_0} \in L_{loc}(\mathbb{R})$. Letting $u \in \Omega_{k_0}, t, s \in \mathbb{R}$ and $t > s$, we have

$$\begin{aligned}
&(\Upsilon_f u)(t) - (\Upsilon_f u)(s) \\
&= \int_s^t S(t-\tau) f(\tau,u(\tau)) d\tau + \int_{s-\epsilon}^{s} (S(t-\tau) - S(s-\tau)) f(\tau,u(\tau)) d\tau \\
&\quad + \int_{t-K}^{s-\epsilon} (S(t-\tau) - S(s-\tau)) f(\tau,u(\tau)) d\tau + \int_{-\infty}^{t-K} (S(t-\tau) - S(s-\tau)) f(\tau,u(\tau)) d\tau \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

where ϵ, K are positive constants yet to be determined.

Given $\eta > 0$. We first note that there exist $\delta, \epsilon > 0$ small enough such that

$$\|J_1\| \le \int_s^t \Phi_{k_0}(\tau)d\tau \le \frac{\eta}{4} \quad \text{when } t-s \le \delta,$$
$$\|J_2\| \le 2M \int_{s-\epsilon}^s \Phi_{k_0}(\tau)d\tau \le \frac{\eta}{4}.$$

For J_4, one can take a $K > 0$ big enough which is independent of t and s such that

$$\begin{aligned}
\|J_4\| &\le \int_{-\infty}^{t-K} \|S(t-\tau) - S(s-\tau)\|_{\mathscr{L}(X)} \|f(\tau, u(\tau))\| d\tau \\
&\le M \int_{-\infty}^{t-K} \big(e^{-\omega(t-\tau)} + e^{-\omega(s-\tau)}\big)\Phi_{k_0}(\tau)d\tau \\
&\le \frac{M(1+e^{\omega\delta})e^{-\omega K}}{1-e^{-\omega T}} \int_0^T \Phi_{k_0}(\tau)d\tau \\
&\le \frac{\eta}{4}.
\end{aligned}$$

For such ϵ, K fixed, it is easy to find that there exists a $d > 0$ big enough such that $|\epsilon - K| \le dT$, which together with the fact that $S(t)$ for $t > 0$ is continuous in uniform operator topology gives that

$$\begin{aligned}
\|J_3\| &\le \int_{t-K}^{s-\epsilon} \|S(t-\tau) - S(s-\tau)\|_{\mathscr{L}(X)} \|f(\tau, u(\tau))\| d\tau \\
&\le \int_{t-K}^{s-\epsilon} \|S(t-s+\epsilon) - S(\epsilon)\|_{\mathscr{L}(X)} \|S(s-\epsilon-\tau)\|_{\mathscr{L}(X)} \|f(\tau, u(\tau))\| d\tau \\
&\le M\|S(t-s+\epsilon) - S(\epsilon)\|_{\mathscr{L}(X)} \int_0^{dT} \Phi_{k_0}(\tau)d\tau \\
&\le \frac{\eta}{4} \quad \text{when } t-s \le \delta.
\end{aligned}$$

Thus, from the arguments above one can deduce that

$$\|(\Upsilon_f u)(t) - (\Upsilon_f u)(s)\| \le \eta,$$

when $t - s \le \delta$ and $u \in \Omega_{k_0}$, which implies that the set $\{\Upsilon_f u; u \in \Omega_{k_0}\}$ is equicontinuous. Consequently, by Lemma 3.1 we have that Υ_f is a compact operator on Ω_{k_0}.

Now, applying Lemma 2.1 we deduce that Υ has at least one fixed point $u \in P_{TA}(\mathbb{R}; X)$. Moreover, following from the same idea as the last part of the proof in Theorem 3.1, we obtain that u is a T-anti-periodic mild solution of the problem (1.1)-(1.2). This completes the proof of theorem. □

The following corollary gives a generalization of Theorem 3.2.

Corollary 3.1. *Let the hypotheses in Theorem* 3.2 *hold except that* (H_1) (ii) *is replaced by the following.*

(ii)$'$ *There exist constants* $L_h, \sigma_1' >$ *such that*

$$\|A^\beta h(t,u) - A^\beta h(t,v)\| \le L_h \|u - v\|,$$
$$\|A^\beta h(t,u)\| \le \sigma_1'(\|u\| + 1)$$

for all $t \in \mathbb{R}$ *and* $u, v \in X$.

Then the assertion in Theorem 3.2 *remains true provided that*

$$L''(\|A^{-\beta}\|_{\mathscr{L}(X)} + M_{1-\beta}\omega^{-\beta}\Gamma(\beta)) + \frac{M\sigma_2}{1-e^{-\omega T}} < 1,$$

where $L'' = \max\{L_h, \sigma_1'\}$.

Remark 3.1. *Theorems* 3.1 *and* 3.2 *cover recent results in* [14].

Remark 3.2. *Let* $0 < \alpha < 1$ *and let* X *be a separable Hilbert space. If* A *is a positive, self-adjoint liner operator on* X *with a discrete spectrum, and for each point spectrum of* A*, the corresponding eigenspace is finite dimentional, then* $-A$ *generates a compact analytic semigroup* $\{S(t)\}_{t\geq 0}$ *on* X *satisfying*

$$\begin{aligned} &\|S(t)\|_{\mathscr{L}(X)} \leq e^{-\lambda_1 t} \quad \text{for all } t \geq 0, \\ &\|A^\alpha S(t)\|_{\mathscr{L}(X)} \leq \left(\left(\tfrac{\alpha}{t}\right)^\alpha + \lambda_1^\alpha\right) e^{-\lambda_1 t} \quad \text{for all } t > 0, \end{aligned}$$

where $\lambda_1 > 0$ *is the minimal point of the spectrum of* A *(see* [37, 38] *for more details).*

By an obvious rescaling from the proof of Theorem 3.2 and Remark 3.2 we can obtain the following existence result.

Theorem 3.3. *Let* X *be a separable Hilbert space and* A *a positive, self-adjoint liner operator on* X *with a discrete spectrum. Suppose in addition that the hypotheses* (H_1) *and* (H_3) *are satisfied. Then the problem* (1.1)-(1.2) *has at least one* T*-anti-periodic mild solution provided that*

$$L'(\|A^{-\beta}\|_{\mathscr{L}(X)} + (1-\beta)^{1-\beta}\Gamma(\beta){\lambda_1}^{-\beta} + \lambda_1^{-\beta}) + \frac{\sigma_2}{1-e^{-\lambda_1 T}} < 1. \tag{3.6}$$

Remark 3.3. *We consider the following semilinear partial neutral functional differential equation with periodic condition*

$$\begin{cases} \dfrac{d}{dt}[u(t) + h(t, u(t))] + Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \\ u(t+T) = u(t), \quad t \in \mathbb{R}. \end{cases} \tag{3.7}$$

From the arguments of Theorems 3.1 *and Theorem* 3.2 *it is easy to see that if*

(1) *the hypotheses in Theorem* 3.1 *are satisfied except that the anti-periodic conditions on* h *and* f *are replaced by the following*

$$h(t+T, u) = h(t, u), \quad f(t+T, u) = f(t, u) \quad \text{for all } t \in \mathbb{R},\ u \in X, \tag{3.8}$$

then there exists a unique T*-periodic mild solution for the problem* (3.7).

(2) *the hypotheses in Theorem* 3.2 *are satisfied except that the anti-periodic conditions on* h *and* f *are replaced by* (3.8)*, then there exists at least a* T*-periodic mild solution for the problem* (3.7).

4. Application

In this section, we give an example to illustrate our abstract results, which does not aim at generality but indicate how our theorems can be applied to concrete problem.

Consider the anti-periodic problem for partial differential equation in the form

$$\begin{cases} \dfrac{\partial}{\partial t}[u(t,x) + \displaystyle\int_0^\pi a(t,x,y)u(t,y)dy] - \dfrac{\partial^2 u(t,x)}{\partial x^2} = g(t, u(t,x)), \quad t \in \mathbb{R},\ x \in [0,\pi], \\ u(t+T, x) = -u(t,x), \quad t \in \mathbb{R},\ x \in [0,\pi], \end{cases} \tag{4.1}$$

under homogeneous Dirichlet boundary conditions

$$u(t,0) = u(t,\pi) = 0 \quad t \in \mathbb{R},$$

where $a : \mathbb{R} \times [0,\pi] \times [0,\pi] \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions to be specified below. Here, our objective is to show the existence of T-anti-periodic solutions for the anti-periodic problem (4.1)

Let $X = L^2[0,\pi]$ with the inner product $(\cdot,\cdot)_2$ and the operator $A : D(A) \subset X \to X$ be defined by

$$\begin{cases} Aw = -\frac{\partial^2 w(x)}{\partial x^2}, \quad w \in D(A), \\ D(A) = \{w \in X; w, w' \text{ are absolutely continuous}, w'' \in X, \text{ and } w(0) = w(\pi) = 0\}. \end{cases}$$

Then, A has a discrete spectrum and its eigenvalues are $n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n \in \mathbb{N}$. Also, $-A$ generates a compact, analytic semigroup $\{S(t)\}_{t\geq 0}$ on X, and

(a) $S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, y_n)_2 y_n,\ \|S(t)\|_{\mathscr{L}(X)} \leq e^{-t}$ *for all* $t \geq 0$,

(b) $A^{-\frac{1}{2}}w = \sum_{n=1}^{\infty} \frac{1}{n}(w, y_n)_2 y_n$ for each $w \in X$. In particular, $\|A^{-\frac{1}{2}}\|_{\mathscr{L}(X)} = 1$, and

(c) $A^{\frac{1}{2}}w = \sum_{n=1}^{\infty} n(w, y_n)_2 y_n$ with the domain $D(A^{\frac{1}{2}}) = \{w \in X; \sum_{n=1}^{\infty} n^2 (w, y_n)_2^2 < +\infty\}$.

Moreover, as established in [38, Lemma 1.1 of Chapter 2], the estimate

$$\|A^{\frac{1}{2}}S(t)w\| \leq \left(\left(\frac{1}{2}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} + 1\right) e^{-t}\|w\|$$

holds for all $t > 0$ and $w \in X$. Hence, one finds that the estimates in Remark 3.2 are satisfied with $\alpha = \frac{1}{2}$ and $\lambda_1 = 1$.

Define

$$\begin{aligned} &u(t)(x) = u(t,x), \\ &h(t,w)(x) = \int_0^{\pi} a(t,x,y)w(y)dy, \\ &f(t,w)(x) = g(t,w(x)). \end{aligned}$$

Assume that the following conditions are verified:

(1) (i) $a : \mathbb{R} \times [0,\pi] \times [0,\pi] \to \mathbb{R}$ is continuously differential, and

$$a(t+T,x,y) = a(t,x,y) \quad \textit{for } t \in \mathbb{R},\ x, y \in [0,\pi].$$

(ii) $a(t,\pi,y) = a(t,0,y) = 0$ for $t \in \mathbb{R},\ y \in [0,\pi]$.

(iii) $c_0 := \sup_{t\in\mathbb{R}} \int_0^{\pi}\int_0^{\pi} \Big(\frac{\partial a(t,x,y)}{\partial x}\Big)^2 dxdy < \infty$.

(2) $b(t+T) = b(t)$ for a.e. $t \in \mathbb{R}$ and $b|_{[0,T]} \in L(0,T;\mathbb{R}^+)$.

From conditions (1)(i),(ii) we obtain that for all $t \in \mathbb{R}$ and $w \in X$,

$$\begin{aligned} (h(t,w), y_n)_2 &= \sqrt{\frac{2}{\pi}}\int_0^{\pi}\int_0^{\pi} a(t,x,y)w(y)\sin(nx)dydx \\ &= \frac{1}{n}\sqrt{\frac{2}{\pi}}\int_0^{\pi}\int_0^{\pi} \frac{\partial a(t,x,y)}{\partial x} w(y)\cos(nx)dydx \\ &= \frac{1}{n}\left(\int_0^{\pi} \frac{\partial a(t,x,y)}{\partial x} w(y)dy, \sqrt{\frac{2}{\pi}}\cos(nx)\right)_2. \end{aligned}$$

Therefore, noticing condition (1)(iii) and applying Bessel's inequality one has that for all $t \in \mathbb{R}$ and $w \in X$,

$$\begin{aligned}
\|A^{\frac{1}{2}} h(t,w)\| &= \left(\sum_{n=1}^{\infty} n^2 (h(t,w), y_n)_2^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{n=1}^{\infty} \left(\int_0^{\pi} \frac{\partial a(t,x,y)}{\partial x} w(y) dy, \sqrt{\frac{2}{\pi}} \cos(nx) \right)_2^2 \right)^{\frac{1}{2}} \\
&\leq \left\| \int_0^{\pi} \frac{\partial a(t,x,y)}{\partial x} w(y) dy \right\| \\
&\leq \sqrt{c_0} \|w\|,
\end{aligned}$$

which implies that $h(t,w) \in D(A^{\frac{1}{2}})$ for each $t \in \mathbb{R}$ and $w \in X$. Also, from condition (1)(i) note that $h(t+T,-w) = -h(t,w)$ in X for all $t \in \mathbb{R}$ and $w \in X$. Furthermore, for all $t \in \mathbb{R}$, $w_1, w_2 \in X$, a direct calculation yields that

$$\begin{aligned}
\|A^{\frac{1}{2}} h(t,w_1) - A^{\frac{1}{2}} h(t,w_2)\| &\leq \sqrt{c_0} \Big(\int_0^{\pi} (w_1(x) - w_2(x))^2 dx \Big)^{\frac{1}{2}} \\
&= \sqrt{c_0} \|w_1 - w_2\|.
\end{aligned}$$

On the other hand, taking $g(t,u(t,x)) = b(t) \sin u(t,x)$, one can find that $f : \mathbb{R} \times X \to X$ is a Carathéodory function and $f(t+T,-w) = -f(t,w)$ in X for all $t \in \mathbb{R}$ and $w \in X$. Furthermore,

$$\|f(t,w)\| \leq \sqrt{\pi} b(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } w \in X.$$

Therefore, the anti-periodic problem (4.1) can be transformed into the abstract problem (1.1)-(1.2) and assumptions (H_1) and (H_3) hold with

$$L_h = \sqrt{c_0}, \quad \Psi(r) = \sqrt{c_0} r, \quad \sigma_1 = \sqrt{c_0}, \quad \Phi_r(t) = \sqrt{\pi} b(t), \quad \sigma_2 = 0.$$

Thus, when $c_0 < \left(\frac{2}{4+\sqrt{2\pi}} \right)^2$ such that condition (3.6) is satisfied, the anti-periodic problem (4.1) has at least one T-anti-periodic mild solution due to Theorem 3.3.

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(Received June 20, 2012)

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