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EXISTENCE THEORY FOR NONLINEAR FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Abstract

In this paper the existence of a solution of a general nonlinear functional two point boundary value problem is proved under mixed generalized Lipschitz and Carathéodory conditions. An existence theorem for extremal solutions is also proved under certain monotonicity and weaker continuity conditions. Examples are provided to illustrate the theory developed in this paper.

Key Words and Phrases: Functional differential equation and existence theorem.
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1 Statement of Problem

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space with a norm $|\cdot|$ defined by

$$|x| = |x_1| + \cdots + |x_n|$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $a, r \in \mathbb{R}$ be such that $a > 0$, $r > 0$ and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $C = C(I_0, \mathbb{R}^n)$ denote a Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$. For every continuous $x : I \to \mathbb{R}$, and every $t \in I$ we define a continuous function $x_t : I_0 \to \mathbb{R}$ by $x_t(\theta) = x(t + \theta)$ for each $\theta \in I_0$. Let $J = [-r, a]$ and let $BM(J, \mathbb{R}^n)$ denote the space of bounded and measurable $\mathbb{R}^n$-valued functions on J. Define a maximum norm $\|\cdot\|$ in $BM(J, \mathbb{R}^n)$ by $\|x\| = \max_{t \in J} |x(t)|$. Given a bounded operator $G : X \subset BM(J, \mathbb{R}^n) \to Y \subset BM(J, \mathbb{R}^n)$, consider the perturbed functional boundary value problem (in short FBVP)

$$\left.\begin{aligned} -x''(t) &= f(t, x(t), Sx) \text{ a.e. } t \in I \\ Gx(0) &= x(0) = 0 = x(a) \\ x(t) &= Gx(t),\ t \in I_0 \end{aligned}\right\} \tag{1.1}$$

where $f : I \times \mathbb{R}^n \times BM(J, \mathbb{R}^n) \to \mathbb{R}^n$ and $S : X \subset BM(J, \mathbb{R}^n) \to Y \subset BM(J, \mathbb{R}^n)$.

By a solution of FBVP (1.1) we mean a function $x \in AC^1(J, \mathbb{R}^n)$ that satisfies the equations in (1.1), where $AC^1(J, \mathbb{R}^n)$ is the space of all continuous $\mathbb{R}^n$-valued functions whose first derivative exist and is absolutely continuous on J with $J = I_0 \bigcup I$.

The FBVP (1.1) seems to be new, yet special cases of it have been discussed in the literature at length. These special cases of FBVP (1.1) can be obtained by defining the operators G and S appropriately. The operators B and S are called the **functional operators** of the functional boundary value problem (1.1) on J. As far as the authors are aware there is no previous work on the existence theory for the FBVP (1.1) in the framework of Caratheódory as well as monotonicity conditions. Now take $X = \{x \in BM(J, \mathbb{R}) \mid x \in AC(I, \mathbb{R})\}$. Let $G : X \to BM(I_0, \mathbb{R})$ and define the operator $S : X \to X$ by $Sx = x, \quad t \in J$. Then the FBVP (1.1) takes the form

$$\left.\begin{aligned} -x''(t) &= f(t, x(t), x) \text{ a.e. } t \in I \\ Gx(0) &= x(0) = 0 = x(a) \\ x(t) &= Gx(t),\ t \in I_0 \end{aligned}\right\} \tag{1.2}$$

which is the functional differential equation discussed in Xu and Liz [15] for the existence of solutions in the framework of upper and lower solutions. Again the FBVP (1.2) includes several important classes of functional differential equations as special cases. See Henderson and Hudson [8], Henderson [7] and the references therein. Again when $S, G : X \to C(I_0, \mathbb{R})$ are two operators defined by $Sx(t) = x_t,\ t \in I$ and $Gx(t) = \phi(t),\ t \in I_0$, the FBVP (1.1) reduces to the FBVP

$$\left.\begin{aligned} -x''(t) &= f(t, x(t), x_t) \text{ a.e. } t \in I \\ \phi(0) &= x(0) = 0 = x(a) \\ x(t) &= \phi(t),\ t \in I_0 \end{aligned}\right\} \tag{1.3}$$

where $f : I \times \mathbb{R}^n \times C(I_0, \mathbb{R}^n) \to \mathbb{R}^n$ and $\phi \in C(I_0, \mathbb{R}^n)$.

We note that the FBVP (1.3) again covers several important classes of functional differential equations; see Henderson [7], Ntouyas [12] and the references therein.

We shall apply fixed point theorems for proving existence theorems for the FBVP (1.1) under generalized Lipschitz and monotonicity conditions.

2 Existence Theorem

An operator $T : X \to X$ is called **compact** if $\overline{T(X)}$ is a compact subset of X. Similarly $T : X \to X$ is called **totally bounded** if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. However the two notions are equivalent on bounded subsets of X.

In this paper we shall establish the existence of solutions for the FBVP (1.1) via the following local version of the nonlinear alternative proved by Leray and Schauder [3]. See also Dhage [1].

Theorem 2.1 *Let $B(0, r)$ and $B[0, r]$ denote the open and closed balls in a Banach space X and let $T : B[0, r] \to X$ be a completely continuous operator. Then either*

(i) the equation $\lambda Ax = x$ has a solution in $B[0, r]$ for $\lambda = 1$, or

(ii) there exists an element $u \in X$ with $\|u\| = r$ satisfying $\lambda Au = u$, for some $0 < \lambda < 1$.

Let $M(J, \mathbb{R}^n)$ and $B(J, \mathbb{R}^n)$ respectively denote the spaces of measurable and bounded real-valued functions on J. We shall seek a solution of FBVP (1.1) in the space $AC(J, \mathbb{R}^n)$, of all bounded and measurable real-valued functions on J. Define a norm $\| \cdot \|$ in $AC(J, \mathbb{R}^n)$ by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly $AC(J, \mathbb{R}^n)$ becomes a Banach space with this norm. We need the following definition in the sequel.

Definition 2.1 *A mapping $\beta : J \times \mathbb{R}^n \times C \to \mathbb{R}^n$ is said to be L^1-Carathéodory, if*

(i) $t \to \beta(t, x, y)$ is measurable for each $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$,

(ii) $(x, y) \to \beta(t, x, y)$ is continuous almost everywhere for $t \in J$, and

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq h_r(t), \quad a.e. \ t \in J$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$ with $|x| \leq r, \|y\| \leq r$.

We will need the following hypotheses:

(A_1) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is continuous.

(A_2) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ is completely continuous.

(A_3) The function $f(t, x, y)$ is L^1-Carathéodory.

(A_4) There exists a nondecreasing function $\phi : [0, \infty) \to (0, \infty)$ and a function $\gamma \in L^1(J, \mathbb{R}^n)$ such that $\gamma(t) > 0$, a.e. $t \in J$ and

$$|f(t, x, y)| \leq \gamma(t)\phi(|x|), \quad a.e. \ t \in I,$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$.

Theorem 2.2 *Assume that the hypotheses (A_1)-(A_4) hold. Suppose that there exists a real number $r > 0$ such that*

$$r > N + \left(\frac{a}{4}\right)\|\gamma\|_{L^1}\phi(r) \tag{2.1}$$

where $N = \sup_{\|x\| \leq r} \|Gx\|$. Then the FBVP (1.1) has a solution on J.

Proof. In the space $AC(J, \mathbb{R})$, let $B[0, r]$ be a closed ball centered at the origin of radius r, where r satisfies the inequality (2.1). Now the FBVP (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = \begin{cases} \displaystyle\int_0^a k(t, s)f(s, x(s), Sx)\, ds, & t \in I, \\ Gx(t), & t \in I_0, \end{cases} \tag{2.2}$$

where $k(t, s)$ is the Green's function associated with the homogeneous linear BVP

$$\left.\begin{aligned} -x''(t) &= 0, \text{ a.e. } t \in I, \\ x(0) &= 0 = x(a). \end{aligned}\right\} \tag{2.3}$$

It is known that the Green's function $k(t, s)$ is continuous and nonnegative on $I \times I$ and satisfies the inequality

$$|k(t, s)| = k(t, s) \leq \frac{a}{4}$$

for all $t, s \in I$.

Let $X = AC(J, \mathbb{R}^n)$. Define a mapping T on X by

$$Tx(t) = \begin{cases} \displaystyle\int_0^a k(t, s)f(s, x(s), Sx)\, ds, & t \in I, \\ Gx(t), & t \in I_0. \end{cases} \tag{2.4}$$

Obviously T satisfies $T : B[0,r] \to X$. We show that T is completely continuous on $B[0,r]$. Using the dominated convergence theorem and standard arguments as in Granas, *et al.* [4], it is shown that T is a continuous operator on X, with respect to the norm $\|\cdot\|$. We shall show that $T(B[0,r])$ is a uniformly bounded and equi-continuous set in X. First, for any $x \in B[0,r]$, we have by (A_1),

$$\begin{aligned}
|Tx(t)| &\leq N + \int_0^a |k(t,s)||f(s,x(s),Sx)|\,ds \\
&\leq N + \int_0^a (\frac{a}{4})h_r(s)\,ds \\
&\leq N + \|h_r\|_{L^1},
\end{aligned}$$

i.e. $\|Tx\| \leq M$ for all $x \in B[0,r]$, where $M = N + (\frac{a}{4})\|h_r\|_{L^1}$. This shows that $T(B[0,r])$ is a uniformly bounded set in X. Now we show that $T(B[0,r])$ is an equi-continuous set. Let $t, \tau \in I$. Then for any $x \in B[0,r]$ we have by (2.4),

$$\begin{aligned}
|Tx(t) - Tx(\tau)| &\leq \left| \int_0^a k(t,s)f(s,x(s),Sx)\,ds - \int_0^\tau k(\tau,s)f(s,x(s),Sx)\,ds \right| \\
&\leq \int_0^a |k(t,s) - k(\tau,s)|\,|f(s,x(s),Sx)|\,ds \\
&\leq \int_0^a |k(t,s) - k(\tau,s)|\,h_r(s)\,ds \\
&\to 0 \quad \text{as } |t-\tau| \to 0.
\end{aligned}$$

Similarly if $\tau, t \in I_0$, then we obtain

$$|Tx(t) - Tx(\tau)| = |Gx(t) - Gx(\tau)|.$$

Since G is completely continuous on X, $G(B[0,r])$ is a totally bounded set in $C(I_0, \mathbb{R}^n)$. Consequently $G(B[0,r])$ is a equi-continuous set in $C(I_0, \mathbb{R}^n)$.

Finally, if $\tau \in I_0$ and $t \in I$, then

$$\begin{aligned}
|Tx(t) - Tx(\tau)| &\leq |Gx(\tau) - Gx(0)| + \left| \int_0^a k(t,s)g(s,x(s),Sx)\,ds \right| \\
&\leq |Gx(\tau) - Gx(0)| + |Tx(t) - Tx(0)| \\
&\leq |Gx(\tau) - Gx(0)| + \int_0^a |k(t,s) - k(0,s)|h_r(s)\,ds.
\end{aligned}$$

Note that $|t-\tau| \to 0$ implies that $t \to 0$ and $\tau \to 0$, and so independent of x in all three cases,

$$|Tx(t) - Tx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Hence $T(B[0,r])$ is an equi-continuous set and consequently $T([0,r])$ is compact by the Arzelá-Ascoli theorem. Consequently T is a completely continuous operator on X. Thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either conclusion (i) or conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be any solution to FBVP (1.1). Then we have, for any $\lambda \in (0,1)$,

$$\begin{aligned} u(t) &= \lambda T u(t) \\ &= \lambda \int_0^a k(t,s)\, f(s,u(s),Sx)\, ds \end{aligned}$$

for $t \in I$, and

$$u(t) = \lambda T u(t) = \lambda G u(t)$$

for all $t \in I_0$. Then we have

$$\begin{aligned} |u(t)| &\leq N + \left| \int_0^a k(t,s) f(s,u(s),Su)\, ds \right| \\ &\leq N + \int_0^a |k(t,s)|\, |f(s,u(s),Su)|\, ds \\ &\leq N + \int_0^a k(t,s)\gamma(s)\phi(|u(s)|)\, ds \\ &\leq N + \int_0^a k(t,s)\gamma(s)\phi(|u(s)|)\, ds \\ &\leq N + \int_0^a (\frac{a}{4})\gamma(s)\phi(\|u(s)\|)\, ds \\ &\leq N + (\frac{a}{4})\|\gamma\|_{L^1}\phi(\|u(s)\|). \end{aligned}$$

Taking the supremum in the above inequality yields that

$$\|u\| \leq N + (\frac{a}{4})\|\gamma\|_{L^1}\phi(\|u\|).$$

Substituting $\|u\| = r$ in the above inequality,

$$r \leq N + (\frac{a}{4})\|\gamma\|_{L^1}\phi(r).$$

which is a contradiction to (2.1). Hence the conclusion (i) of Theorem 2.1 holds. Therefore the operator equation $Tx = x$ has a solution in $B[0,r]$. This further implies that the FBVP (1.1) has a solution on J. This completes the proof. $\square$

Example 2.1 Let $I_0 = [-\pi/2, 0]$ and $I = [0,1]$ be two closed and bounded intervals in $\mathbb{R}$. For a given function $x \in AC(J,\mathbb{R})$, consider the FBVP

$$\left. \begin{aligned} & -x''(t) = p(t)\frac{|x(t)|}{1+\|x_t\|} \ \text{ a.e. } \ t \in I \\ & x(t) = \sin t, \quad t \in I_0 \end{aligned} \right\} \tag{2.5}$$

where $p \in L^1(I, \mathbb{R}^+)$ with $\|p\|_{L^1} \leq 1$ and $x_t \in C(I_0, \mathbb{R})$ with $x_t(\theta) = x(t+\theta), \theta \in I_0$.

Define the functional operator S and the boundary operator G on $BM(J, \mathbb{R})$ by $Sx(t) = x_t \in C(I_0, \mathbb{R})$ for $t \in I$ and $Gx(t) = \sin t$ for all $t \in I_0$. Obviously S is continuous and G is bounded with $N = \max\{\|Gx\| : x \in BM(J, \mathbb{R})\} = 1$.

Define a function $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ by

$$f(t, x, y) = p(t)\frac{|x|}{1 + \|y_t\|}.$$

It is very easy to prove that the function $f(t, x, y)$ is L^1-Carathéodory. Again we have

$$\begin{aligned} |f(t, x, y)| &= \left| p(t)\frac{|x|}{1 + \|y_t\|} \right| \\ &\leq p(t)(1 + |x(t)|) \end{aligned}$$

and so the hypothesis (A_4) is satisfied with $\phi(r) = 1 + r$. Now there exists a real number $r = 2$ satisfying the condition (2.1). Hence we apply Theorem 2.1 to yield that the FBVP (1.1) has a solution on $J = I_0 \bigcup I$.

3 Uniqueness Theorem

Let X be a Banach space with norm $\|\cdot\|$. A mapping $T : X \to X$ is called **$\mathcal{D}$-Lipschitzian** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Tx - Ty\| \leq \psi(\|x - y\|) \tag{3.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. Sometimes we call the function ψ a **$\mathcal{D}$-function** of T on X. In the special case when $\psi(r) = \alpha r,\ \ \alpha > 0$, T is called Lipchitzician with a Lipschitz constant α. In particular if $\alpha < 1, T$ is called a contraction with a contraction constant α. Further if $\psi(r) < r$ for $r > 0$, then T is called a nonlinear contraction on X. Finally if $\psi(r) = r$, then T is called a nonexpansive operator on X.

The following fixed point theorem for the nonlinear contraction is well-known and useful for proving existence and uniqueness theorems for the nonlinear differential and integral equations.

Theorem 3.1 (Browder [16]) *Let X be a Banach space and let $T : X \to X$ be a nonlinear contraction. Then T has a unique fixed point.*

We will need the following hypotheses:

(B_1) The function $f : I \times \mathbb{R}^n \times BM(J, \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and satisfies

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \max\left\{\frac{|x_1 - x_2|}{a^2 + |x_1 - x_2|}, \frac{\|y_1 - y_2\|}{a^2 + \|y_1 - y_2\|}\right\}, \quad \text{a.e.} \quad t \in I$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $y_1, y_2 \in BM(J, \mathbb{R}^n)$.

(B_2) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is nonexpansive.

(B_3) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ satisfies

$$|Gx(t) - Gy(t)| \leq \frac{|x(t) - y(t)|}{a + |x(t) - y(t)|}, \text{ a.e. } t \in I_0$$

for all $x, y \in BM(J, \mathbb{R}^n)$.

Theorem 3.2 *Assume that the hypotheses $(B_1) - (B_3)$ hold. Then the FBVP (1.1) has a unique solution on J.*

Proof : Let $X = AC(J, \mathbb{R})$ and define an operator T on X by (2.2). We show that T is a nonlinear contraction on X. Let $x, y \in X$. By hypothesis (H_1),

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^a k(t,s)|f(s, x(s), Sx) - f(s, y(s), Sy)|\, ds \\
&\leq \int_0^a \left(\max\left\{\frac{|x(s) - y(s)|}{a^2 + |x(s) - y(s)|}, \frac{\|Sx - Sy\|}{a^2 + \|Sx - Sy\|}\right\}\right) ds \\
&\leq \frac{a^2\|x - y\|}{a^2 + \|x - y\|}.
\end{aligned}$$

for all $t \in I$. Again

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq |Gx(t) - Gy(t)| \\
&\leq \frac{a|x(t) - y(t)|}{a + |x(t) - y(t)|} \\
&\leq \frac{a\|x - y\|}{a + \|x - y\|}
\end{aligned}$$

for all $t \in J$. Taking supremum over t we obtain

$$\|Tx - Ty\| \leq \psi(\|x - y\|),$$

for all $x, y \in X$ where $\psi(r) = \max\left\{\frac{ar}{a+r}, \frac{a^2 r}{a^2+r}\right\} < r$, which shows that T is a nonlinear contraction on X. We now apply Theorem 3.1 to yield that the operator T has a unique fixed point. This further implies that the FBVP (1.1) has a unique solution on J. This completes the proof. □

Example 3.1 Let $I_0 = [-\pi/2, 0]$ and $I = [0, 1]$ be two closed and bounded intervals in $\mathbb{R}$. For a given function $x \in AC(J, \mathbb{R})$, consider the functional differential equation (FBVP)

$$\left.\begin{aligned} -x''(t) &= \frac{1}{2}\left[\frac{|x(t)|}{1+|x(t)|} + \frac{\|x_t\|}{1+\|x_t\|}\right] \quad \text{a.e. } t \in I \\ x(0) &= 0 = x(a) \\ x(t) &= \sin t, \quad t \in I_0 \end{aligned}\right\} \tag{3.2}$$

where $x_t \in C(I_0, \mathbb{R})$ with $x_t(\theta) = x(t+\theta), \theta \in I_0$.

Define the functional operator S and the boundary operator G on $BM(J, \mathbb{R})$ by $Sx = x_t \in C(I_0, \mathbb{R})$ for $t \in I$ and $Gx(t) = \sin t$ for all $t \in I_0$. Obviously S is continuous and G is bounded with $C = \max\{\|Gx\| : x \in BM(J, \mathbb{R})\} = 1$. Clearly S is nonexpansive on $BM(J, \mathbb{R})$.

Define a function $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ by

$$f(t, x, y) = \frac{1}{2}\left[\frac{|x|}{1+|x|} + \frac{\|y_t\|}{1+\|y_t\|}\right].$$

It is very easy to prove that the function f is continuous on $I \times \mathbb{R} \times BM(J, \mathbb{R})$. Finally we show that the function f satisfies the inequality given in (B_1). Let $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 \in BM(J, \mathbb{R})$ be arbitrary. Then we have

$$\begin{aligned} |f(t, x_1, y_1) \quad &- \quad f(t, x_2, y_2)| \\ &\leq \frac{1}{2}\left(\left|\frac{|x_1|}{1+|x_1|} - \frac{|x_2|}{1+|x_2|}\right|\right) + \frac{1}{2}\left(\left|\frac{\|y_1\|}{1+\|y_1\|} - \frac{\|y_2\|}{1+\|y_2\|}\right|\right) \\ &\leq \frac{1}{2}\left(\frac{|\,|x_1| - |x_2|\,|}{(1+|x_1|)(1+|x_2|)}\right) + \frac{1}{2}\left(\frac{|\|y_1\| - \|y_2\||}{(1+\|y_1\|)(1+\|y_2\|)}\right) \\ &\leq \frac{1}{2}\left(\frac{|x_1 - x_2|}{1+|x_1 - x_2|}\right) + \frac{1}{2}\left(\frac{\|y_1 - y_2\|}{1+\|y_1 - y_2\|}\right) \\ &\leq \max\left\{\frac{|x_1 - x_2|}{1+|x_1 - x_2|}, \frac{\|y_1 - y_2\||}{1+\|y_1 - y_2\|}\right\} \end{aligned}$$

for all $t \in J$. Hence the hypothesis (B_1) of Theorem 3.1 is satisfied. Therefore an application of Theorem 3.1 yields that the FBVP (3.2) has a unique solution on $[-\pi/2, 1]$.

4 Existence of Extremal Solutions

Let $x, y \in \mathbb{R}^n$ be such that $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. We define the coordinate-wise order relation in $\mathbb{R}^n$, that is, $x \leq y$ if and only if $x_i \leq y_i$, for all $i = 1, \dots, n$. We equip the Banach space $AC(J, \mathbb{R}^n)$ with the order relation "$\leq$" by $\xi_1 \leq \xi_2$ if and only if $\xi_1(t) \leq \xi_2(t)$, for all $t \in J$. By the order interval $[a, b]$ in $AC(J, \mathbb{R}^n)$ we mean

$$[a, b] = \{x \in AC(J, \mathbb{R}^n) \mid a \leq x \leq b\}.$$

We use the following fixed point theorem of Heikkila and Lakshmikantham [6] in the sequel.

Theorem 4.1 *Let $[a, b]$ be an order interval in an order Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subseteq Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover*

$$x_* = \min\{y \in [a, b] \mid y \geq Qy\} \quad \textit{and} \quad x^* = \max\{y \in [a, b] \mid y \leq Qy\}.$$

We need the following definitions in the sequel.

Definition 4.1 *A mapping $\beta : J \times \mathbb{R}^n \times C \to \mathbb{R}^n$ is said to satisfy Chandrabhan's conditions or simply is called L^1-**Chandrabhan** if*

(i) $t \to \beta(t, x, y)$ is measurable for each $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$,

(ii) The function $\beta(t, x, y)$ is nondecreasing in x and y almost everywhere for $t \in J$, and

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq h_r(t), \quad \textit{a.e. } t \in J$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$ with $|x| \leq r$, $\|y\| \leq r$.

Definition 4.2 *A function $u \in AC^1(J, \mathbb{R}^n)$ is called a lower solution of the FBVP (1.1) on J if*

$$-u''(t) \leq f(t, u(t), Su) \quad a.e \quad t \in I$$

$$Gu(0) = u(0) = 0 = u(a),$$

and

$$u(t) \leq Gu(t) \ \text{ for all } t \in I_0.$$

Again a function $v \in AC(J, \mathbb{R}^n)$ is called an upper solution of the BVP (1.1) on J if

$$-v''(t) \geq f(t, v(t), Sv) \quad a.e \quad t \in I$$

$$Gv(0) = v(0) = 0 = v(a)$$

and

$$v(t) \geq Gv(t) \quad \text{for all } t \in I_0.$$

Definition 4.3 *A solution x_M of the FBVP (1.1) is said to be maximal if for any other solution x to FBVP(1.1) one has $x(t) \leq x_M(t)$, for all $t \in J$. Again a solution x_m of the FBVP (1.1) is said to be minimal if $x_m(t) \leq x(t)$, for all $t \in J$, where x is any solution of the FBVP (1.1) on J.*

We consider the following set of assumptions:

(C_1) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is nondecreasing.

(C_2) The functions $f(t, x, y)$ is Chandrabhan.

(C_3) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ is nondecreasing.

(C_4) The FBVP (1.1) has a lower solution u and an upper solution v on J with $u \leq v$.

Remark 4.1 *Assume that hypotheses $(C_1) - (C_4)$ hold. Define a function $h : J \to \mathbb{R}^+$ by*

$$h(t) = |f(t, u(t), Su)| + |f(t, v(t), Sv)|, \quad t \in I.$$

Then h is Lebesgue integrable and

$$|f(t, x(t), Sx)| \leq h(t), \quad a.e. \ t \in I, \ x(t) \in [u, v].$$

Theorem 4.2 *Suppose that the assumptions (C_1)-(C_4) hold. Then FBVP (1.1) has a minimal and a maximal solution on J.*

Proof. Now FBVP (1.1) is equivalent to FIE (2.2) on J. Let $X = AC(J, \mathbb{R}^n)$. Define the operators T on $[a, b]$ by (2.3). Then FIE (1.1) is transformed into an operator equation $Tx(t) = x(t)$ in a Banach space X. Now the hypothesis (B_2) implies that T is nondecreasing on $[u, v]$. To see this, let $x, y \in [u, v]$ be such that $x \le y$. Then by (B_2),

$$\begin{aligned} Tx(t) &= \int_0^a k(t,s) f(s, x(s), Sx)\, ds \\ &\le \int_0^a k(t,s) f(s, y(s), Sy)\, ds \\ &= Ty(t),\ \ t \in I, \end{aligned}$$

and

$$Tx(t) = Gx(t) \le Gy(t) = Ty(t) \quad \text{for all} \quad t \in I_0.$$

So T is a nondecreasing operator on $[u, v]$. Finally we show that T defines a mapping $T : [u, v] \to [u, v]$. Let $x \in [u, v]$ be an arbitrary element. Then for any $t \in I$, we have

$$\begin{aligned} u(t) &\le \int_0^a k(t,s) f(s, u(s), Su)\, ds \\ &\le \int_0^a k(t,s) f(s, x(s), Sx)\, ds \\ &\le \int_0^a k(t,s) f(s, v(s), Sv)\, ds \\ &\le v(t), \end{aligned}$$

for all $t \in I$. Again from (B_2) it follows that

$$u(t) \le Tu(t) = Gu(t) \le Gx(t) \le Tx(t) \le Gv(t) = Tv(t) \le v(t)$$

for all $t \in I_0$. As a result $u(t) \le Tx(t) \le v(t)$, for all $t \in J$. Hence $Tx \in [u, v]$, for all $x \in [u, v]$.

Finally let $\{x_n\}$ be a monotone sequence in $[u, v]$. We shall show that the sequence $\{Tx_n\}$ converges in $T([u, v])$. Obviously the sequence $\{Tx_n\}$ is monotone in $T([u, v])$. Now it can be shown as in the proof of Theorem 2.2 that the sequence $\{Tx_n\}$ is uniformly bounded and equicontinuous in $T([u, v])$ with the function h playing the role of h_r. Hence an application of the Arzela-Ascoli theorem yields that the sequence $\{Tx_n\}$ converges in $T([u, v])$. Thus all the conditions of Theorem 4.1 are satisfied and hence the operator T has a least and a greatest fixed point in $[u, v]$. This further implies that the the FBVP (1.1) has maximal and minimal solutions on J. This completes the proof. □

Example 4.1 Given two closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, 1]$ in $\mathbb{R}$

for some $0 < r < 1$, consider the functional differential equation

$$\left.\begin{aligned} -x''(t) = {} & \frac{\tanh\left(\left[\max\limits_{s\in[-r,t]} x(s)\right]\right)}{\sqrt{t}} + \operatorname{sgn}(x(t)) \text{ a.e. } t \in I \\ x(0) = {} & 0 = x(a) \\ x_0 = {} & \sin t \text{ for } t \in I_0. \end{aligned}\right\} \tag{4.1}$$

where tanh is the hyperbolic tangent, square bracket means the integer part and

$$\operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

Define the operators $S, B : BM(J, \mathbb{R}) \to BM(J, \mathbb{R})$ by

$$Sx(t) = \begin{cases} \left[\max\limits_{s\in[-r,t]} x(s)\right], \text{ if } t \in I \\ 0, \text{otherwise.} \end{cases}$$

and

$$Gx(t) = \begin{cases} \sin t, \text{ if } t \in I_0 \\ 0, \text{ otherwise.} \end{cases}$$

Consider the mapping $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ defined by

$$f(t, x, y) = \frac{\tanh y}{\sqrt{t}} + \operatorname{sgn}(x)$$

for $t \neq 0$. Obviously the operators S and B are nondecreasing on $BM(J, \mathbb{R})$. It is not difficult to verify that the function $f(t, x, y)$ is L^1-Chandrabhan. Again note that

$$-1 - \frac{1}{\sqrt{t}} < f(t, x, y) < 1 + \frac{1}{\sqrt{t}}$$

for all $t \in J, x \in \mathbb{R}$ and $y \in BM(J, \mathbb{R})$. Therefore if we define the functions α and β by

$$-\alpha''(t) = -1 - \frac{1}{\sqrt{t}}\ ,\ \alpha(0) = 0 = \alpha(a)$$

and

$$-\beta''(t) = 1 + \frac{1}{\sqrt{t}}\ ,\ \beta(0) = 0 = \beta(a)$$

for all $t \in I$ with

$$\alpha(t) = \sin t = \beta(t)\ t \in I_0,$$

then α and β are respectively the lower and upper solutions of FBVP (4.1) on J with $\alpha \leq \beta$. Thus all the conditions of Theorem 3.1 are satisfied and hence the FBVP (4.1) has a maximal and minimal solution on J.

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