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REPRESENTATION AND STABILITY OF SOLUTIONS OF SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. This paper is devoted to the study of systems of nonlinear functional differential equations with time-dependent coefficients and multiple variable increasing delays represented by functions $g_i(t) < t$. The solution is found in terms of a piecewise-defined matrix function. Using our representation of the solution and Gronwall's, Bihari's and Pinto's integral inequalities, asymptotic stability results are proved for some classes of nonlinear functional differential equations with multiple variable delays and linear parts given by pairwise permutable constant matrices. The derived theory is illustrated on nontrivial examples.

1. INTRODUCTION

The classical method of steps [8] where the initial value problem

$$\dot{x}(t) = Bx(t-\tau), \qquad t \geq 0 \tag{1.1}$$

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0 \tag{1.2}$$

is solved by subsequent integrating of equation (1.1) on intervals $[0, \tau)$, $[\tau, 2\tau)$, $[2\tau, 3\tau)$,... was renown in 2003 by Khusainov and Shuklin [13]. Applying this method, they constructed so-called delayed matrix exponential e_τ^{Bt} defined as

$$e_\tau^{Bt} = \begin{cases} \Theta, & t < -\tau, \\ E, & -\tau \leq t < 0, \\ E + Bt + B^2\frac{(t-\tau)^2}{2} + \cdots + B^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leq t < k\tau, k \in \mathbb{N} \end{cases}$$

where Θ, E are the zero and the identity $N \times N$ matrix, respectively. Let us recall their result.

Theorem 1.1. *Let $\varphi \in C_\tau^1 := C^1([-\tau, 0], \mathbb{R}^N)$ and $AB = BA$. Then any solution $x(t)$ of the Cauchy problem consisting of equation*

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \qquad t \geq 0$$

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and initial condition (1.2) *has the form*

$$x(t) = \mathrm{e}^{A(t+\tau)}\mathrm{e}_{\tau}^{\widetilde{B}t}\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{e}^{A(t-\tau-s)}\mathrm{e}_{\tau}^{\widetilde{B}(t-\tau-s)}\mathrm{e}^{A\tau}(\varphi'(s) - A\varphi(s))ds,$$

where $\widetilde{B} = \mathrm{e}^{-A\tau}B$.

Using the variation of constants formula for retarded differential equations with constant delay [8] and e_{τ}^{Bt}, they stated the solution of nonhomogeneous equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + f(t)$$

with continuous function $f : [0,\infty) \to \mathbb{R}^N$, satisfying initial condition (1.2). Later, their result was used to establish sufficient conditions for the exponential stability of the trivial solution of the nonlinear equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + f(x(t), x(t-\tau))$$

with various functions f in [16]. The results from [13] were generalized to delay differential equations with multiple fixed delays and pairwise permutable matrices in [14] and analogical theory was developed for retarded oscillating systems and difference equations with one or more fixed delays (cf. [7, 11, 12, 15]). Recently, the matrix representation of solutions of systems of differential equations with a single fixed delay was applied to boundary-value problems in [4, 5, 6].

In this paper, we consider the functional differential equation (FDE) with one or multiple time-dependent delays. More precisely, we deal with equations of the form

$$\dot{x}(t) = B_1(t)x(g_1(t)) + \cdots + B_n(t)x(g_n(t)), \qquad t \geq 0$$

where $B_i \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_i \in G^0$ for $i = 1, \ldots, n$ and

$$G^s := \{g \in C([s,\infty), \mathbb{R}) \mid g(t) < t \text{ on } [s,\infty), g \text{ is increasing}\}.$$

In Section 2, we derive the solution of a corresponding nonhomogeneous equation. Later, in Section 3 we use the property of commutativity of matrices to transform the nonlinear FDE with multiple delays and linear term $Ax(t)$ to a nonlinear FDE with multiple delays but without a delay-independent linear term. After this transformation, we can apply the theory of Section 2, and so establish sufficient conditions for the exponential stability of the trivial solution of nonlinear FDE with multiple delays and linear term $Ax(t)$ added on the right-hand side, supposing that the linear parts are given by pairwise permutable constant matrices. So, in Section 3, we study the exponential stability of the trivial solution of systems of FDEs with linear parts given by pairwise permutable matrices (for stability criteria for scalar equations with variable coefficients see e.g. [1, 2, 10, 19]).

In the whole paper $\|E\| = 1$, $\mathbb{N}$ denotes the set of all positive integers and $g^k(t)$, $g^{-k}(t)$ for $k = 2, 3, \ldots$ denote the iterations of functions $g(t)$, $g^{-1}(t)$, respectively, e.g. if $k = 2$, then $g^2(t) = g(g(t))$ and $g^{-2}(t) = g^{-1}(g^{-1}(t))$. Moreover, $g^0(t) = t$. If $g : [s,\infty) \to [g(s),\infty)$ is not surjective, we define $g^{-1}(q) := \infty$ whenever q is such that $g(t) < q$ for any $t \in [s,\infty)$.

Definition 1.2. Given continuous function F, under the solution of a general FDE

$$\dot{x}(t) = F(x(g_1(t)),\dots,x(g_n(t)),t), \qquad t \geq 0 \tag{1.3}$$

satisfying initial condition (1.2) with $\tau = \min\{g_1(0),\dots,g_n(0)\}$ we understand function $x \in C([-\tau,\infty),\mathbb{R}^N) \cap C^1([0,\infty),\mathbb{R}^N)$ (at 0 we take the right-hand derivative) which solves equation (1.3) and satisfies (1.2).

2. Solutions of systems of FDEs

In this section we derive a representation of a solution of FDE with single variable delay using a piecewise-defined matrix function, which is analogical to delayed matrix exponential e_τ^{Bt} for equations with constant delay. Later, we find a solution of FDE with multiple delays as it was done in [14]. Throughout this part, we widely use the method of steps and variation of constants formula for FDEs (cf. [8, 9]). We note that the existence and uniqueness of solutions of problems of this section are obvious. First, we find the fundamental solution of linear FDE with one delay satisfying the below-stated initial condition (2.2).

Theorem 2.1. *Let $s \in \mathbb{R}$, $B \in C([s,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$, $g \in G^s$. Then the matrix solution of equation*

$$\dot{X}(t) = B(t)X(g(t)), \qquad t \geq s \tag{2.1}$$

satisfying initial condition

$$X(t) = \begin{cases} \Theta, & t < s, \\ E, & t = s \end{cases} \tag{2.2}$$

has the form $X(t) = X_g^B(t,s)$ where

$$X_g^B(t,s) := \begin{cases} \Theta, & t < s, \\ E, & s \leq t < g^{-1}(s), \\ E + \int_{g^{-1}(s)}^t B(q_1)dq_1 + \cdots & \\ \cdots + \int_{g^{-k}(s)}^t B(q_1) \int_{g^{-(k-1)}(s)}^{g(q_1)} B(q_2) \dots \int_{g^{-1}(s)}^{g(q_{k-1})} B(q_k) dq_k \dots dq_1, & \\ & g^{-k}(s) \leq t < g^{-(k+1)}(s), k \in \mathbb{N}. \end{cases} \tag{2.3}$$

Proof. From definition of $X_g^B(t,s)$, the initial condition is immediately verified. If $s \leq t < g^{-1}(s)$ then $g(s) \leq g(t) < s$ and

$$\dot{X}(t) = \dot{E} = \Theta = B(t)\Theta = B(t)X(g(t)).$$

Now, let $g^{-k}(s) \le t < g^{-(k+1)}(s)$ for some $k \in \mathbb{N}$. Then $g^{-(k-1)}(s) \le g(t) < g^{-k}(s)$ and we get

$$\dot{X}(t) = B(t) + B(t)\int_{g^{-1}(s)}^{g(t)} B(q_2)dq_2 + \cdots$$

$$\cdots + B(t)\int_{g^{-(k-1)}(s)}^{g(t)} B(q_2)\dots\int_{g^{-1}(s)}^{g(q_{k-1})} B(q_k)dq_k \dots dq_2 = B(t)X(g(t)).$$

Hence equation (2.1) is verified and the proof is finished. □

Now, when we have the fundamental solution, we can derive the solution of corresponding nonhomogeneous equation (see [9]). Without any loss of generality we assume the initial function to be given on $[g(0), 0]$.

Theorem 2.2. *Let $B \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g \in G^0$, $f \in C([0,\infty), \mathbb{R}^N)$, $\varphi \in C([g(0), 0], \mathbb{R}^N)$. Then the solution of the initial value problem*

$$\dot{x}(t) = B(t)x(g(t)) + f(t), \qquad t \ge 0 \tag{2.4}$$

$$x(t) = \varphi(t), \qquad g(0) \le t \le 0 \tag{2.5}$$

has the form

(2.6)

$$x(t) = \begin{cases} \varphi(t), & g(0) \le t < 0, \\ X_g^B(t,0)\varphi(0) + \int_0^t X_g^B(t,s)B(s)\psi(g(s))ds + \int_0^t X_g^B(t,s)f(s)ds, & 0 \le t \end{cases}$$

where

$$\psi(t) = \begin{cases} \varphi(t), & t \in [g(0), 0), \\ 0, & t \notin [g(0), 0). \end{cases}$$

Proof. Clearly, $x \in C([g(0),\infty), \mathbb{R}^N) \cap C^1([0, g^{-1}(0)) \cup (g^{-1}(0), \infty), \mathbb{R}^N)$ and it satisfies condition (2.5).

Let $0 \le t < g^{-1}(0)$. Then for $0 \le s \le t$ we get

$$g(0) \le g(s) \le g(t) < 0 \le s \le t < g^{-1}(0).$$

Consequently, $X_g^B(t,s) = E$ and $\psi(g(s)) = \varphi(g(s))$. Hence from formula (2.6)

$$x(t) = \varphi(0) + \int_0^t B(s)\varphi(g(s))ds + \int_0^t f(s)ds \tag{2.7}$$

what after differentiating with respect to t yields

$$\dot{x}(t) = B(t)\varphi(g(t)) + f(t).$$

Even though $X_g^B(t,s)$ is not C^1 at $t = g^{-1}(s)$ (left-hand derivative is Θ, right-hand derivative is $B(g^{-1}(s)))$, solution $x(t)$ is C^1 at $g^{-1}(0)$. To see this, we differentiate the solution (2.7) for $t \in [0, g^{-1}(0))$ and

$$x(t) = X_g^B(t,0)\varphi(0) + \int_0^t X_g^B(t,s)B(s)\psi(g(s))ds + \int_0^t X_g^B(t,s)f(s)ds$$

for $t \in [g^{-1}(0), g^{-2}(0))$, both at $t = g^{-1}(0)$. We obtain

$$\lim_{t\to(g^{-1}(0))^-} \dot{x}(t) = B(g^{-1}(0))\varphi(0) + f(g^{-1}(0)),$$
$$\lim_{t\to(g^{-1}(0))^+} \dot{x}(t) = B(g^{-1}(0))[\varphi(0) + \psi(0)] + f(g^{-1}(0)).$$

In fact, we get the equality since $\psi(0) = 0$.

Now if $g^{-k}(0) \le t < g^{-(k+1)}(0)$ for some $k \in \mathbb{N}$, then by differentiating formula (2.6) we obtain

$$\begin{aligned}\dot{x}(t) &= \dot{X}_g^B(t,0)\varphi(0) + B(t)\psi(g(t)) \\ &+ \int_0^t \dot{X}_g^B(t,s)B(s)\psi(g(s))ds + f(t) + \int_0^t \dot{X}_g^B(t,s)f(s)ds \\ &= B(t)X_g^B(g(t),0)\varphi(0) + B(t)\psi(g(t)) + B(t)\int_0^t X_g^B(g(t),s)B(s)\psi(g(s))ds \\ &+ f(t) + B(t)\int_0^t X_g^B(g(t),s)f(s)ds\end{aligned}$$

where we used the properties of $X_g^B(t,s)$ from Theorem 2.1. Next, we apply the identity

$$\int_0^t X_g^B(g(t),s)F(s)ds = \int_0^{g(t)} X_g^B(g(t),s)F(s)ds$$

with $F(s)$ standing for $B(s)\psi(g(s))$ or $f(s)$ to get

$$\begin{aligned}\dot{x}(t) = B(t)\Bigg[&X_g^B(g(t),0)\varphi(0) + \psi(g(t)) + \int_0^{g(t)} X_g^B(g(t),s)B(s)\psi(g(s))ds \\ &+ \int_0^{g(t)} X_g^B(g(t),s)f(s)ds\Bigg] + f(t).\end{aligned}$$

For $t \ge g^{-1}(0)$ it holds $\psi(g(t)) = 0$ and equation (2.4) is verified. □

Remark 2.3. In reality, if $t \ge g^{-1}(0)$ is fixed, two integrals in solution $x(t)$ of (2.6) are split into more integrals as s varies from 0 to t. Note that

$$\int_0^t X_g^B(t,s)B(s)\psi(g(s))ds = \int_0^{\min\{t,g^{-1}(0)\}} X_g^B(t,s)B(s)\varphi(g(s))ds.$$

If we denote

$$X_l(t,s) := E + \int_{g^{-1}(s)}^{t} B(q_1)dq_1 + \cdots$$
$$\cdots + \int_{g^{-l}(s)}^{t} B(q_1) \int_{g^{-(l-1)}(s)}^{g(q_1)} B(q_2) \ldots \int_{g^{-1}(s)}^{g(q_{l-1})} B(q_l)dq_l \ldots dq_1$$

for $l = 0, \ldots, k$, i.e. the lower index l denotes the number of integrals in the sum, then $x(t)$ can be written as

$$x(t) = X_k(t,0)\varphi(0) + \int_0^{g^k(t)} X_k(t,s)B(s)\varphi(g(s))ds$$
$$+ \int_{g^k(t)}^{g^{-1}(0)} X_{k-1}(t,s)B(s)\varphi(g(s))ds + \int_0^{g^k(t)} X_k(t,s)f(s)ds$$
$$+ \sum_{l=2}^{k} \int_{g^l(t)}^{g^{l-1}(t)} X_{l-1}(t,s)f(s)ds + \int_{g(t)}^{t} X_0(t,s)f(s)ds$$

for $g^{-k}(0) \le t < g^{-(k+1)}(0)$, $k \in \mathbb{N}$, i.e.

$$0 \le g^k(t) < g^{-1}(0) \le g^{k-1}(t) < \cdots$$
$$\cdots \le g^2(t) < g^{-(k-1)}(0) \le g(t) < g^{-k}(0) \le t < g^{-(k+1)}(0).$$

Here we used the form of $X_g^B(t,s)$ for fixed t and variable s (in (2.3) it was given for fixed s and variable t):
(2.8)

$$X_g^B(t,s) = \begin{cases} \Theta, & t < s, \\ E, & g(t) < s \le t, \\ E + \int_{g^{-1}(s)}^{t} B(q_1)dq_1 + \cdots & \\ \cdots + \int_{g^{-k}(s)}^{t} B(q_1) \int_{g^{-(k-1)}(s)}^{g(q_1)} B(q_2) \ldots \int_{g^{-1}(s)}^{g(q_{k-1})} B(q_k)dq_k \ldots dq_1, & \\ & g^{k+1}(t) < s \le g^k(t), k \in \mathbb{N}. \end{cases}$$

Now we provide an application of Theorem 2.2 on a problem with a bounded delay.

Example 2.4. Let us consider the following initial value problem

$$\dot{x}(t) = B(t)x\left(\frac{1-3\mathrm{e}^{-t}}{2}\right) + f(t), \qquad t \ge 0$$
$$x(t) = \varphi(t), \qquad -1 \le t \le 0.$$

Here we have $g : [0,\infty) \to [-1, 1/2)$, $g(t) = (1-3\mathrm{e}^{-t})/2$ and $g^{-1} : [-1,1/2) \to [0,\infty)$, $g^{-1}(s) = \ln \frac{3}{1-2s}$. Hence we set $g^{-1}(q) = \infty$ whenever

$q \geq 1/2$ and $g^{-2}(q) = \infty$ for all $q \geq \frac{\sqrt{\mathrm{e}}-3}{2\sqrt{\mathrm{e}}} < 0$. Since we can assume $s \geq 0$, from (2.3) we get

$$X_g^B(t,s) = \begin{cases} \Theta, & t < s, \\ E, & s \leq t < \ln \frac{3}{1-2s}, \\ E + \int_{\ln \frac{3}{1-2s}}^t B(q)dq, & \ln \frac{3}{1-2s} \leq t. \end{cases}$$

Accordingly, formula (2.6) gives the solution of Example 2.4.

Corollary 2.5. *The solution of Example 2.4 has the form*

$$x(t) = \begin{cases} \varphi(t), & -1 \leq t < 0, \\ \varphi(0) + \int_0^t B(s)\varphi\left(\frac{1-3\mathrm{e}^{-s}}{2}\right) ds + \int_0^t f(s)ds, & 0 \leq t < \ln 3, \\ F(t,0)\varphi(0) + \int_0^{\frac{1-3\mathrm{e}^{-t}}{2}} F(t,s)B(s)\varphi\left(\frac{1-3\mathrm{e}^{-s}}{2}\right) ds & \\ + \int_{\frac{1-3\mathrm{e}^{-t}}{2}}^{\ln 3} B(s)\varphi\left(\frac{1-3\mathrm{e}^{-s}}{2}\right) ds + \int_0^{\frac{1-3\mathrm{e}^{-t}}{2}} F(t,s)f(s)ds & \\ + \int_{\frac{1-3\mathrm{e}^{-t}}{2}}^t f(s)ds, & \ln 3 \leq t \end{cases}$$

where $F(t,s) = E + \int_{\ln \frac{3}{1-2s}}^t B(q)dq$.

In the next step, we shall use the solution of the nonhomogeneous initial value problem to construct the fundamental solution of FDE with two delays. Let us consider a matrix equation

$$\dot{X}(t) = B_1(t)X(g_1(t)) + B_2(t)X(g_2(t)), \qquad t \geq s \tag{2.9}$$

together with initial condition (2.2). Then formula (2.6) with $f(t) = B_2(t)X(g_2(t))$ and s instead of 0 yields

$$X(t) = \begin{cases} \Theta, & t < s, \\ X_{g_1}^{B_1}(t,s) + \int_s^t X_{g_1}^{B_1}(t,q)B_2(q)X(g_2(q))dq, & s \leq t. \end{cases}$$

From the initial condition, one can see that if $s \leq t < g_2^{-1}(s)$ then $X(g_2(q)) = \Theta$ for $s \leq q \leq t$, i.e. $X(t) = X_{g_1}^{B_1}(t,s)$. Next, for t such that $g_2^{-1}(s) \leq t < g_2^{-2}(s)$ it holds

$$X(g_2(q)) = \begin{cases} \Theta, & s \leq q < g_2^{-1}(s), \\ X_{g_1}^{B_1}(g_2(q),s), & g_2^{-1}(s) \leq q \leq t. \end{cases}$$

Hence for such t we get

$$X(t) = X_{g_1}^{B_1}(t,s) + \int_{g_2^{-1}(s)}^t X_{g_1}^{B_1}(t,q)B_2(q)X_{g_1}^{B_1}(g_2(q),s)dq.$$

Analogically we proceed on other intervals $[g_2^{-k}(s), g_2^{-(k+1)}(s))$ with $k = 2, 3, \dots$. By this process we obtain

(2.10)

$$X_{g_1,g_2}^{B_1,B_2}(t,s) := \begin{cases} \Theta, & t < s, \\ X_{g_1}^{B_1}(t,s), & s \le t < g_2^{-1}(s), \\ X_{g_1}^{B_1}(t,s) + \int_{g_2^{-1}(s)}^{t} X_{g_1}^{B_1}(t,q_1)B_2(q_1)X_{g_1}^{B_1}(g_2(q_1),s)dq_1 + \cdots & \\ \cdots + \int_{g_2^{-k}(s)}^{t} X_{g_1}^{B_1}(t,q_1)B_2(q_1)\int_{g_2^{-(k-1)}(s)}^{g_2(q_1)} X_{g_1}^{B_1}(g_2(q_1),q_2)B_2(q_2)\cdots & \\ \cdots \times \int_{g_2^{-1}(s)}^{g_2(q_{k-1})} X_{g_1}^{B_1}(g_2(q_{k-1}),q_k)B_2(q_k)X_{g_1}^{B_1}(g_2(q_k),s)dq_k \dots dq_1, & \\ & g_2^{-k}(s) \le t < g_2^{-(k+1)}(s), k \in \mathbb{N}. \end{cases}$$

Theorem 2.6. *Let $s \in \mathbb{R}$, $B_1, B_2 \in C([s,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_1, g_2 \in G^s$. Then $X(t) = X_{g_1,g_2}^{B_1,B_2}(t,s)$ is the matrix solution of equation* (2.9) *satisfying initial condition* (2.2).

Proof. At $t = s$ it holds

$$X(s) = X_{g_1}^{B_1}(s,s) = E.$$

Hence initial condition is immediately verified.

Let $s \le t < g_2^{-1}(s)$. Then

$$X(t) = X_{g_1}^{B_1}(t,s)$$

and $g_2(t) < s$. So we get

$$\begin{aligned}\dot X(t) &= \dot X_{g_1}^{B_1}(t,s) = B_1(t)X_{g_1}^{B_1}(g_1(t),s) \\ &= B_1(t)X(g_1(t)) + B_2(t)X(g_2(t)).\end{aligned}$$

Now, if $g_2^{-k}(s) \le t < g_2^{-(k+1)}(s)$ for some $k \in \mathbb{N}$, then $g_2^{-(k-1)}(s) \le g_2(t) < g_2^{-k}(s)$. Accordingly,

$$\begin{aligned}\dot X(t) &= B_1(t)X_{g_1}^{B_1}(g_1(t),s) + B_2(t)X_{g_1}^{B_1}(g_2(t),s) \\ &+ \int_{g_2^{-1}(s)}^{t} B_1(t)X_{g_1}^{B_1}(g_1(t),q_1)B_2(q_1)X_{g_1}^{B_1}(g_2(q_1),s)dq_1 + \cdots \\ &\cdots + B_2(t)\int_{g_2^{-(k-1)}(s)}^{g_2(t)} X_{g_1}^{B_1}(g_2(t),q_2)B_2(q_2)\dots\int_{g_2^{-1}(s)}^{g_2(q_{k-1})} X_{g_1}^{B_1}(g_2(q_{k-1}),q_k) \\ &\times B_2(q_k)X_{g_1}^{B_1}(g_2(q_k),s)dq_k\dots dq_2 + \int_{g_2^{-k}(s)}^{t} B_1(t)X_{g_1}^{B_1}(g_1(t),q_1)B_2(q_1) \\ &\times \int_{g_2^{-(k-1)}(s)}^{g_2(q_1)} X_{g_1}^{B_1}(g_2(q_1),q_2)B_2(q_2)\dots\int_{g_2^{-1}(s)}^{g_2(q_{k-1})} X_{g_1}^{B_1}(g_2(q_{k-1}),q_k) \\ &\times B_2(q_k)X_{g_1}^{B_1}(g_2(q_k),s)dq_k\dots dq_1 = B_1(t)X(g_1(t)) + B_2(t)X(g_2(t))\end{aligned}$$

since $X_{g_1}^{B_1}(g_1(t), q_1) = \Theta$ whenever $q_1 > g_1(t)$. In conclusion, we have proved that $X(t)$ solves equation (2.9) for any $t \geq s$. $\square$

Remark 2.7. Sometimes, it may be easier to use the "fixed t" form of $X_{g_1,g_2}^{B_1,B_2}(t,s)$ analogical to (2.8) instead of "fixed s" given by (2.10).

Matrix function $X_{g_1,g_2}^{B_1,B_2}(t,s)$ has some important properties which are concluded in the next lemma.

Lemma 2.8. *Let $s \in \mathbb{R}$, $B_1, B_2 \in C([s,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_1, g_2 \in G^s$. Then the following statements hold true for any $t \in \mathbb{R}$:*

1. *if $B_1 = \Theta$ then $X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_2}^{B_2}(t,s)$,*
2. *if $B_2 = \Theta$ then $X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_1}^{B_1}(t,s)$,*
3. *if $g_1(t) = g_2(t)$ for all $t \in [s,\infty)$, then $X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_1}^{B_1+B_2}(t,s)$,*
4. *$X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_2,g_1}^{B_2,B_1}(t,s)$.*

Proof. All statements of the lemma follow from the uniqueness of a solution of a corresponding initial value problem. For instance in 1., both sides of the identity solve equation

$$\dot{X}(t) = B_1(t)X(g_1(t)) + B_2(t)X(g_2(t)) = B_2(t)X(g_2(t))$$

together with initial condition (2.2). $\square$

As before, we obtain a result on the solution of nonhomogeneous equation, this time with two delays.

Theorem 2.9. *Let $B_1, B_2 \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_1, g_2 \in G^0$, $f \in C([0,\infty), \mathbb{R}^N)$, $\gamma := \min\{g_1(0), g_2(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$. Then the solution of the initial value problem*

$$\dot{x}(t) = B_1(t)x(g_1(t)) + B_2(t)x(g_2(t)) + f(t), \qquad t \geq 0 \tag{2.11}$$

$$x(t) = \varphi(t), \qquad \gamma \leq t \leq 0 \tag{2.12}$$

has the form

$$x(t) = \begin{cases} \varphi(t), & \gamma \leq t < 0, \\ X_{g_1,g_2}^{B_1,B_2}(t,0)\varphi(0) + \int_0^t X_{g_1,g_2}^{B_1,B_2}(t,s)\left[B_1(s)\psi(g_1(s)) \right. & \\ \left. + B_2(s)\psi(g_2(s))\right] ds + \int_0^t X_{g_1,g_2}^{B_1,B_2}(t,s)f(s)ds, & 0 \leq t \end{cases} \tag{2.13}$$

where

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\gamma, 0), \\ 0, & t \notin [\gamma, 0). \end{cases} \tag{2.14}$$

Proof. Clearly, the initial condition is satisfied. In verification of equation (2.11) we consider four cases with respect to t.

Let $0 \le t < \min\{g_1^{-1}(0), g_2^{-1}(0)\}$. Then $X_{g_1,g_2}^{B_1,B_2}(t,s) = E$ for $s \in [0,t]$ since

$$g_2(t) < 0 \le s \le t < g_2^{-1}(0), \qquad g_1(t) < 0 \le s \le t < g_1^{-1}(0).$$

Thus from (2.13) we get

$$x(t) = \varphi(0) + \int_0^t B_1(s)\varphi(g_1(s)) + B_2(s)\varphi(g_2(s))ds + \int_0^t f(s)ds$$

which is a solution of equation (2.11) since $x(g_i(t)) = \varphi(g_i(t))$ for $i = 1, 2$.

If $g_1^{-1}(0) \le t < g_2^{-1}(0)$ then $g_2(t) < 0 \le s \le t < g_2^{-1}(0)$ for $s \in [0,t]$. Therefore $x(g_2(t)) = \varphi(g_2(t))$, $X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_1}^{B_1}(t,s)$ and we obtain

$$x(t) = X_{g_1}^{B_1}(t,0)\varphi(0) + \int_0^t X_{g_1}^{B_1}(t,s)[B_1(s)\psi(g_1(s)) + B_2(s)\varphi(g_2(s))]ds$$
$$+ \int_0^t X_{g_1}^{B_1}(t,s)f(s)ds.$$

After differentiating

$$\dot{x}(t) = B_1(t)x(g_1(t)) + B_2(t)\varphi(g_2(t)) + f(t),$$

so one can see that $x(t)$ really solves (2.11).

The case $g_2^{-1}(0) \le t < g_1^{-1}(0)$ can be proved analogically to the previous one using the change $X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_2,g_1}^{B_2,B_1}(t,s)$ from Lemma 2.8.

Finally, if $\max\{g_1^{-1}(0), g_2^{-1}(0)\} \le t$ then $\psi(g_1(t)) = \psi(g_2(t)) = 0$ and direct differentiation of (2.13) gives the desired result. □

Now, we apply formula (2.13) on a problem with concrete unbounded delays.

Example 2.10. Let us consider the following initial value problem

$$\dot{x}(t) = B_1(t)x(t-1) + B_2(t)x(\sqrt{t} - 1), \qquad t \ge 0$$
$$x(t) = \varphi(t), \qquad -1 \le t \le 0.$$

In this case $g_1(t) = t - 1$, $g_1^{-1}(s) = s + 1$, $g_2(t) = \sqrt{t} - 1$, $g_2^{-1}(s) = (s+1)^2$ and we can assume $s \ge 0$. Hence by (2.3),

$$X_{g_1}^{B_1}(t,s) = \begin{cases} \Theta, & t < s, \\ E, & s \le t < s+1, \\ E + \int_{s+1}^t B_1(q_1)dq_1, & s+1 \le t < s+2, \\ E + \int_{s+1}^t B_1(q_1)dq_1 + \int_{s+2}^t B_1(q_1)\int_{s+1}^{q_1-1} B_1(q_2)dq_2dq_1, & \\ & s+2 \le t < s+3, \\ \dots, & s+3 \le t. \end{cases}$$

Next, from (2.10)
(2.15)

$$X^{B_1,B_2}_{g_1,g_2}(t,s)=\begin{cases}\Theta, & t<s,\\ E, & \text{on } M_1,\\ F_2(t,s):=E+\int_{s+1}^{t}B_1(q_1)dq_1, & \text{on } M_2,\\ F_3(t,s):=E+\int_{s+1}^{t}B_1(q_1)dq_1 & \\ +\int_{s+2}^{t}B_1(q_1)\int_{s+1}^{q_1-1}B_1(q_2)dq_2dq_1, & \text{on } M_3,\\ F_4(t,s):=E+\int_{s+1}^{t}B_1(q_1)dq_1+\int_{(s+1)^2}^{t}B_2(q_1)dq_1, & \text{on } M_4,\\ F_5(t,s):=E+\int_{s+1}^{t}B_1(q_1)dq_1+\int_{s+2}^{t}B_1(q_1)\int_{s+1}^{q_1-1}B_1(q_2)dq_2dq_1 & \\ +\int_{(s+1)^2}^{t}B_2(q_1)dq_1+\int_{(s+1)^2}^{t-1}\int_{q_1+1}^{t}B_1(q_2)dq_2B_2(q_1)dq_1, & \text{on } M_5,\\ F_6(t,s):=E+\int_{s+1}^{t}B_1(q_1)dq_1+\int_{s+2}^{t}B_1(q_1)\int_{s+1}^{q_1-1}B_1(q_2)dq_2dq_1 & \\ +\int_{(s+1)^2}^{t}B_2(q_1)dq_1, & \text{on } M_6,\\ \dots, & \text{otherwise}\end{cases}$$

where

$$M_1=\{(t,s)\in\mathbb{R}^2_+ \mid s\le t<(s+1)^2, s\le t<s+1\},$$
$$M_2=\{(t,s)\in\mathbb{R}^2_+ \mid s\le t<(s+1)^2, s+1\le t<s+2\},$$
$$M_3=\{(t,s)\in\mathbb{R}^2_+ \mid s\le t<(s+1)^2, s+2\le t<s+3\},$$
$$M_4=\{(t,s)\in\mathbb{R}^2_+ \mid (s+1)^2\le t<((s+1)^2+1)^2, s+1\le t<s+2\},$$
$$M_5=\{(t,s)\in\mathbb{R}^2_+ \mid (s+1)^2+1\le t<((s+1)^2+1)^2, s+2\le t<s+3\},$$
$$M_6=\{(t,s)\in\mathbb{R}^2_+ \mid (s+1)^2\le t<(s+1)^2+1, s+2\le t<s+3\}$$

with $\mathbb{R}^2_+=[0,\infty)\times[0,\infty)$. For the convenience, these sets are sketched in Figure 1.

Corollary 2.11. *The solution of Example 2.10 has the form*
(2.16)

$$x(t)=\begin{cases}\varphi(t), & -1\le t<0,\\ \varphi(0)+\int_0^t B(s)ds, & 0\le t<1,\\ F_4(t,0)\varphi(0)+\int_0^{\sqrt{t}-1}F_4(t,s)B(s)ds+\int_{\sqrt{t}-1}^{t-1}F_2(t,s)B(s)ds & \\ +\int_{t-1}^{1}B(s)ds, & 1\le t<2,\\ F_5(t,0)\varphi(0)+\int_0^{\sqrt{t-1}-1}F_5(t,s)B(s)ds+\int_{\sqrt{t-1}-1}^{t-2}F_6(t,s)B(s)ds & \\ +\int_{t-2}^{\sqrt{t}-1}F_4(t,s)B(s)ds+\int_{\sqrt{t}-1}^{1}F_2(t,s)B(s)ds, & 2\le t<(3+\sqrt{5})/2,\\ F_5(t,0)\varphi(0)+\int_0^{\sqrt{t-1}-1}F_5(t,s)B(s)ds+\int_{\sqrt{t-1}-1}^{\sqrt{t}-1}F_6(t,s)B(s)ds & \\ +\int_{\sqrt{t}-1}^{t-2}F_3(t,s)B(s)ds+\int_{t-2}^{1}F_2(t,s)B(s)ds, & (3+\sqrt{5})/2\le t<3,\\ \dots, & 3\le t\end{cases}$$

where $B(s) = B_1(s)\varphi(s-1) + B_2(s)\varphi(\sqrt{s}-1)$ *and* $F_2(t,s), \ldots, F_6(t,s)$ *are given in* (2.15).

For the graph of the solution with concrete functions $B_1, B_2 \in C([0,\infty), \mathbb{R})$, $\varphi \in C([-1,0], \mathbb{R})$ see Figure 1.

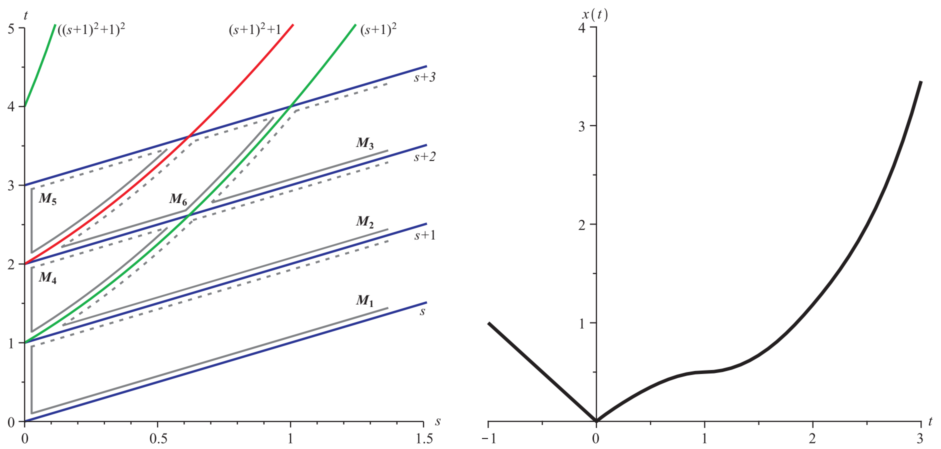

FIGURE 1. Sets $M_1, \ldots, M_6$ and solution (2.16) of Example 2.10 with $B_1(t) = t$, $B_2(t) = 1$ and $\varphi(t) = -t$.

One can proceed inductively from $X^{B_1,B_2}_{g_1,g_2}(t,s)$ and, with the aid of the latter theorem, construct the fundamental solution of FDE with any finite number $n \geq 3$ of variable delays. So one obtains

(2.17)

$$X^{B_1,\ldots,B_n}_{g_1,\ldots,g_n}(t,s) := \begin{cases} \Theta, & t < s, \\ Y(t,s), & s \leq t < g_n^{-1}(s), \\ Y(t,s) + \int_{g_n^{-1}(s)}^{t} Y(t,q_1)B_n(q_1)Y(g_n(q_1),s)dq_1 + \cdots \\ \cdots + \int_{g_n^{-k}(s)}^{t} Y(t,q_1)B_n(q_1)\int_{g_n^{-(k-1)}(s)}^{g_n(q_1)} Y(g_n(q_1),q_2)B_n(q_2)\cdots \\ \cdots \times \int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(g_n(q_{k-1}),q_k)B_n(q_k)Y(g_n(q_k),s)dq_k \ldots dq_1, \\ & g_n^{-k}(s) \leq t < g_n^{-(k+1)}(s), k \in \mathbb{N}. \end{cases}$$

where $Y(t,s) = X^{B_1,\ldots,B_{n-1}}_{g_1,\ldots,g_{n-1}}(t,s)$.

Theorem 2.12. *Let* $s \in \mathbb{R}$, $3 \leq n \in \mathbb{N}$, $B_i \in C([s,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_i \in G^s$ *for* $i = 1, \ldots, n$. *Then* $X(t) = X^{B_1,\ldots,B_n}_{g_1,\ldots,g_n}(t,s)$ *is the matrix solution of equation*

$$\dot{X}(t) = B_1(t)X(g_1(t)) + \cdots + B_n(t)X(g_n(t)), \qquad t \geq s \tag{2.18}$$

satisfying initial condition (2.2).

Proof. The case $n = 2$ was proved in Theorem 2.6. So here we suppose that the statement is true for $n-1$ and we show that it holds also for n.

Clearly, at $t = s$

$$X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(s,s) = X^{B_1,\dots,B_{n-1}}_{g_1,\dots,g_{n-1}}(s,s) = \dots = X^{B_1}_{g_1}(s,s) = E$$

and initial condition is verified. For the simplicity we denote $Y(t,s) = X^{B_1,\dots,B_{n-1}}_{g_1,\dots,g_{n-1}}(t,s)$ in the rest of the proof.

If $s \le t < g_n^{-1}(s)$ then $g_n(t) < s$ and

$$\begin{aligned}\dot{X}(t) &= \dot{Y}(t,s) = B_1(t)Y(g_1(t),s) + \dots + B_{n-1}(t)Y(g_{n-1}(t),s)\\ &= B_1(t)X(g_1(t)) + \dots + B_{n-1}(t)X(g_{n-1}(t)) + B_n(t)X(g_n(t))\end{aligned}$$

since $X(g_n(t)) = \Theta$.

Now, let $g_n^{-k}(s) \le t < g_n^{-(k+1)}(s)$ for some $k \in \mathbb{N}$. Then from (2.17) using the inductive hypothesis, we get for the derivative

$$\begin{aligned}&\dot{X}(t) = B_1(t)Y(g_1(t),s) + \dots + B_{n-1}(t)Y(g_{n-1}(t),s) + B_n(t)Y(g_n(t),s)\\ &+ \int_{g_n^{-1}(s)}^{t} \left[B_1(t)Y(g_1(t),q_1) + \dots + B_{n-1}(t)Y(g_{n-1}(t),q_1)\right] B_n(q_1)Y(g_n(q_1),s)dq_1\\ &+ \dots + B_n(t)\int_{g_n^{-(k-1)}(s)}^{g_n(t)} Y(g_n(t),q_2)B_n(q_2)\dots\int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(g_n(q_{k-1}),q_k)B_n(q_k)\\ &\times Y(g_n(q_k),s)dq_k\dots dq_2 + \int_{g_n^{-k}(s)}^{t} \left[B_1(t)Y(g_1(t),q_1) + \dots + B_{n-1}(t)Y(g_{n-1}(t),q_1)\right]\\ &\qquad \times B_n(q_1)\dots\int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(g_n(q_{k-1}),q_k)B_n(q_k)Y(g_n(q_k),s)dq_k\dots dq_1.\end{aligned}$$

By collecting terms beginning with $B_i(t)$ we obtain for each $i = 1,\dots,n-1$ exactly $B_i(t)X(g_i(t))$ since $Y(g_i(t),q_1) = \Theta$ for $g_i(t) < q_1$ (hence the upper boundary of integrals is changed from t to $g_i(t)$). Next, $g_n^{-(k-1)}(s) \le g_n(t) < g_n^{-k}(s)$, thus collecting terms beginning with $B_n(t)$ yields $B_n(t)X(g_n(t))$ (in comparison to $X(t)$, the number of integrals in $X(g_n(t))$ is decreased by one). In conclusion, the last identity is precisely the equation which $X(t)$ has to satisfy. □

Matrix function $X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s)$ has properties that are analogical to those of $X^{B_1,B_2}_{g_1,g_2}(t,s)$ provided in Lemma 2.8. We conclude them into a lemma without a proof.

Lemma 2.13. *Let $s \in \mathbb{R}$, $3 \le n \in \mathbb{N}$, $B_i \in C([s,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$, $g_i \in G^s$ for $i = 1,\dots,n$. Then the following statements hold true for any $t \in \mathbb{R}$:*

(1) *if $B_i = \Theta$ for some $i \in \{1,\dots,n\}$, then*

$$X^{B_1,\dots,B_{i-1},B_i,B_{i+1},\dots,B_n}_{g_1,\dots,g_{i-1},g_i,g_{i+1},\dots,g_n}(t,s) = X^{B_1,\dots,B_{i-1},B_{i+1},\dots,B_n}_{g_1,\dots,g_{i-1},g_{i+1},\dots,g_n}(t,s),$$

(2) *if* $g_i(t) = g_j(t)$ *for any* $t \in [s, \infty)$ *and some* $i < j$, $i, j \in \{1, \dots, n\}$, *then*

$$X^{B_1,\dots,B_{i-1},B_i,B_{i+1},\dots,B_{j-1},B_j,B_{j+1},\dots,B_n}_{g_1,\dots,g_{i-1},g_i,g_{i+1},\dots,g_{j-1},g_j,g_{j+1},\dots,g_n}(t,s)$$
$$= X^{B_1,\dots,B_{i-1},B_i+B_j,B_{i+1},\dots,B_{j-1},B_{j+1},\dots,B_n}_{g_1,\dots,g_{i-1},g_i,g_{i+1},\dots,g_{j-1},g_{j+1},\dots,g_n}(t,s),$$

(3) *for any bijective mapping* $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$

$$X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s) = X^{B_{\sigma(1)},\dots,B_{\sigma(n)}}_{g_{\sigma(1)},\dots,g_{\sigma(n)}}(t,s).$$

The statement on the solution of the nonhomogeneous initial value problem with n delays follows (cf. [9]).

Theorem 2.14. *Let* $3 \le n \in \mathbb{N}$, $B_i \in C([0,\infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g_i \in G^0$ *for* $i = 1, \dots, n$, $f \in C([0,\infty), \mathbb{R}^N)$, $\gamma := \min\{g_1(0), \dots, g_n(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$. *Then the solution of the equation*

$$\dot{x}(t) = B_1(t)x(g_1(t)) + \dots + B_n(t)x(g_n(t)) + f(t), \qquad t \ge 0 \tag{2.19}$$

satisfying initial condition (2.12) *has the form*

$$x(t) = \begin{cases} \varphi(t), & \gamma \le t < 0, \\ X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,0)\varphi(0) + \int_0^t X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s)\,[B_1(s)\psi(g_1(s)) & \\ + \dots + B_n(s)\psi(g_n(s))]\,ds + \int_0^t X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s)f(s)ds, & 0 \le t \end{cases} \tag{2.20}$$

where $\psi(t)$ *is given by* (2.14).

Proof. The proof is similar to the proof of Theorem 2.9, so we omit some details. Note that

$$\psi(g_i(t)) = \begin{cases} \varphi(g_i(t)), & t < g_i^{-1}(0), \\ 0, & g_i^{-1}(0) \le t \end{cases}$$

for each $i = 1, \dots, n$.

If $0 \le t < \min\{g_1^{-1}(0), \dots, g_n^{-1}(0)\}$ then $X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s) = E$ for $s \in [0,t]$ and $x(t)$ has the form

$$x(t) = \varphi(0) + \int_0^t [B_1(s)\varphi(g_1(s)) + \dots + B_n(s)\varphi(g_n(s))]\,ds + \int_0^t f(s)ds$$

what solves equation (2.19).

If there are the nonempty sets

$$M_1 = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}, \qquad M_2 = \{1, \dots, n\} \backslash M_1$$

such that $g_i^{-1}(0) \le t < g_j^{-1}(0)$ for each $i \in M_1$, $j \in M_2$, then applying Lemma 2.13 we obtain $X^{B_1,\dots,B_n}_{g_1,\dots,g_n}(t,s) = X^{B_{i_1},\dots,B_{i_k}}_{g_{i_1},\dots,g_{i_k}}(t,s)$ for $s \in [0,t]$ and

$$\psi(g_i(t)) = \begin{cases} 0, & i \in M_1, \\ \varphi(g_i(t)), & i \in M_2. \end{cases}$$

Consequently, we rewrite $x(t)$ as

$$x(t) = X(t,0)\varphi(0) + \int_0^t X(t,s)\left[\sum_{i\in M_1} B_i(s)\psi(g_i(s)) + \sum_{j\in M_2} B_j(s)\varphi(g_j(s))\right] ds$$
$$+\int_0^t X(t,s)f(s)ds$$

where $X(t,s) = X_{g_{i_1},\ldots,g_{i_k}}^{B_{i_1},\ldots,B_{i_k}}(t,s)$. Then for the derivative it holds

$$\dot{x}(t) = \sum_{i\in M_1} B_i(t)X(g_i(t),0)\varphi(0) + \sum_{j\in M_2} B_j(t)\varphi(g_j(t))$$
$$+\int_0^t \sum_{i\in M_1} B_i(t)X(g_i(t),s)\left[\sum_{i\in M_1} B_i(s)\psi(g_i(s)) + \sum_{j\in M_2} B_j(s)\varphi(g_j(s))\right] ds$$
$$+f(t) + \int_0^t \sum_{i\in M_1} B_i(t)X(g_i(t),s)f(s)ds.$$

After collecting the terms by $B_i(t)$ one gets

$$\dot{x}(t) = \sum_{i\in M_1} B_i(t)x(g_i(t)) + \sum_{j\in M_2} B_j(t)\varphi(g_j(t)) + f(t)$$

what is exactly equation (2.19) since $x(g_j(t)) = \varphi(g_j(t))$ for each $j \in M_2$.

Finally, if $\max\{g_1^{-1}(0),\ldots,g_n^{-1}(0)\} \le t$ then $\psi(g_i(t)) = 0$ for each $i \in \{1,\ldots,n\}$ and direct differentiation of $x(t)$ given by (2.20) verifies equation (2.19). □

In Section 3 we shall seek conditions for the exponential stability of the trivial solution of FDE with constant coefficients at linear terms. Here we find the solution of such an equation.

Theorem 2.15. *Let $n \in \mathbb{N}$, $A, B_1,\ldots,B_n$ be pairwise permutable $N\times N$ constant matrices, i.e. $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i,j \in \{1,\ldots,n\}$, $g_i \in G^0$ for $i = 1,\ldots,n$, $f \in C([0,\infty),\mathbb{R}^N)$, $\gamma := \min\{g_1(0),\ldots,g_n(0)\}$, $\varphi \in C([\gamma,0],\mathbb{R}^N)$. Then the solution of the equation*

$$\dot{x}(t) = Ax(t) + B_1x(g_1(t)) + \cdots + B_nx(g_n(t)) + f(t), \qquad t \ge 0 \tag{2.21}$$

satisfying initial condition (2.12) *has the form*

(2.22)
$$x(t) = \begin{cases} \varphi(t), & \gamma \le t < 0,\\ X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,0)\mathrm{e}^{At}\varphi(0) + \int_0^t X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\mathrm{e}^{A(t-s)}\left[B_1\psi(g_1(s))\right. & \\ \left. + \cdots + B_n\psi(g_n(s))\right]ds + \int_0^t X_{g_1,\ldots,g_n}^{\widetilde{B}_1,\ldots,\widetilde{B}_n}(t,s)\mathrm{e}^{A(t-s)}f(s)ds, & 0 \le t \end{cases}$$

where $\widetilde{B}_i(t) = \mathrm{e}^{-A(t-g_i(t))}B_i$ for $i = 1,\ldots,n$ and $\psi(t)$ is given by (2.14).

Proof. Denote $y(t) = \mathrm{e}^{-At}x(t)$. Then from (2.21), (2.12)

$$\begin{aligned} &\dot{y}(t) = \widetilde{B}_1(t)y(g_1(t)) + \cdots + \widetilde{B}_n(t)y(g_n(t)) + \widetilde{f}(t), \qquad t \geq 0 \\ &y(t) = \widetilde{\varphi}(t), \qquad \gamma \leq t \leq 0 \end{aligned}$$

where $\widetilde{f}(t) = \mathrm{e}^{-At}f(t)$, $\widetilde{\varphi}(t) = \mathrm{e}^{-At}\varphi(t)$. Applying Theorem 2.14 to this problem yields

$$y(t) = \begin{cases} \widetilde{\varphi}(t), & \gamma \leq t < 0, \\ X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,0)\varphi(0) + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,s)\Big[\widetilde{B}_1(s)\widetilde{\psi}(g_1(s)) & \\ + \cdots + \widetilde{B}_n(s)\widetilde{\psi}(g_n(s))\Big]\,ds + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,s)\widetilde{f}(s)ds, & 0 \leq t \end{cases}$$

where

$$\widetilde{\psi}(t) = \begin{cases} \widetilde{\varphi}(t), & t \in [\gamma, 0), \\ 0, & t \notin [\gamma, 0). \end{cases}$$

Note that $\widetilde{\psi}(t) = \mathrm{e}^{-At}\psi(t)$ for any $t \in \mathbb{R}$ and $\widetilde{B}_i(s)\widetilde{\psi}(g_i(s)) = \mathrm{e}^{-As}B_i\psi(g_i(s))$. When one returns to $x(t)$, the formula (2.22) is obtained. □

3. Exponential stability of nonlinear FDEs

In this section, we apply the theory derived in the preceding section to establish criteria for the exponential stability of the trivial solution of nonlinear FDE with multiple variable delays where the linear parts are given by pairwise permutable constant matrices. First, we estimate the fundamental solutions $X_g^B(t,s)$ and $X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s)$ with the aid of the next lemma.

Lemma 3.1. *If $s \in \mathbb{R}$ and $f \in C([s,\infty),\mathbb{R})$, then*

$$\int_s^t f(q_1)\int_s^{q_1} f(q_2)\dots\int_s^{q_{k-1}} f(q_k)dq_k\dots dq_1 = \frac{1}{k!}\left(\int_s^t f(q)dq\right)^k$$

for each $k \in \mathbb{N}$, $t \in \mathbb{R}$.

Proof. We prove the lemma via induction with respect to k. Denote

$$F_k(t) = \int_s^t f(q_1)\int_s^{q_1} f(q_2)\dots\int_s^{q_{k-1}} f(q_k)dq_k\dots dq_1$$

$$G_k(t) = \frac{1}{k!}\left(\int_s^t f(q)dq\right)^k.$$

Clearly $F_1(t) = G_1(t)$. Let $F_{k-1}(t) = G_{k-1}(t)$. Then $F_k(s) = 0 = G_k(s)$ and

$$F_k'(t) = f(t)F_{k-1}(t) = f(t)G_{k-1}(t) = \frac{f(t)}{(k-1)!}\left(\int_s^t f(q)dq\right)^{k-1} = G_k'(t).$$

□

Lemma 3.2. *Let $s \in \mathbb{R}$, $B \in C([s,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$, $g \in G^s$. Then*

$$\|X_g^B(t,s)\| \le \exp\left\{\int_s^t \|B(q)\|dq\right\}$$

for any $t \ge s$.

Proof. It is sufficient to prove the statement for $g : [s,\infty) \to [g(s),\infty)$ surjective. By this, the other case is also covered.

Let $t \ge g^{-1}(s)$ be arbitrary and fixed, $k \in \mathbb{N}$ be such that $g^{-k}(s) \le t < g^{-(k+1)}(s)$. Then from (2.3) we know that

$$X_g^B(t,s) = E + X_1(t,s) + \cdots + X_k(t,s)$$

where

$$X_j(t,s) = \int_{g^{-j}(s)}^t B(q_1) \int_{g^{-(j-1)}(s)}^{g(q_1)} B(q_2) \dots \int_{g^{-1}(s)}^{g(q_{j-1})} B(q_j)dq_j \dots dq_1$$

for $j = 1,\dots,k$. Since g is increasing and according to Lemma 3.1 we derive

$$\|X_j(t,s)\| \le \int_{g^{-j}(s)}^t \|B(q_1)\| \int_{g^{-(j-1)}(s)}^{g(q_1)} \|B(q_2)\| \dots \int_{g^{-1}(s)}^{g(q_{j-1})} \|B(q_j)\|dq_j \dots dq_1$$

$$\le \int_s^t \|B(q_1)\| \int_s^{q_1} \|B(q_2)\| \dots \int_s^{q_{j-1}} \|B(q_j)\|dq_j \dots dq_1 = \frac{1}{j!}\left(\int_s^t \|B(q)\|dq\right)^j.$$

As a consequence,

$$\|X_g^B(t,s)\| \le 1 + \|X_1(t,s)\| + \cdots + \|X_k(t,s)\|$$

$$\le \sum_{j=0}^k \frac{1}{j!}\left(\int_s^t \|B(q)\|dq\right)^j \le \sum_{j=0}^\infty \frac{1}{j!}\left(\int_s^t \|B(q)\|dq\right)^j = \exp\left\{\int_s^t \|B(q)\|dq\right\}.$$

Obviously, the last estimate holds for each $k \in \mathbb{N}$ and hence for any $t \ge g^{-1}(s)$.

If $s \le t < g^{-1}(s)$ then

$$\|X_g^B(t,s)\| = \|E\| = 1 \le \exp\left\{\int_s^t \|B(q)\|dq\right\}$$

so it remains true for such t. □

Lemma 3.3. *Let $s \in \mathbb{R}$, $2 \le n \in \mathbb{N}$, $B_i \in C([s,\infty), L(\mathbb{R}^N,\mathbb{R}^N))$, $g_i \in G^s$ for $i = 1,\dots,n$. Then*

$$\|X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s)\| \le \exp\left\{\int_s^t \sum_{i=1}^n \|B_i(q)\|dq\right\}$$

for any $t \ge s$.

Proof. As before, it is enough to prove the lemma for $g_i : [s, \infty) \to [g(s), \infty)$ surjective for each $i = 1, \dots, n$.

We show that if the statement holds for $n-1$ delays, then it is true for n. Let $k \in \mathbb{N}$ be such that $g_n^{-k}(s) \le t < g_n^{-(k+1)}(s)$ for arbitrary and fixed $t \ge g_n^{-1}(s)$. Then from (2.17) we know that

$$X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s) = Y(t,s) + X_1(t,s) + \dots + X_k(t,s)$$

where

$$X_j(t,s) = \int_{g_n^{-j}(s)}^{t} Y(t,q_1)B_n(q_1) \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} Y(g_n(q_1),q_2)B_n(q_2)\cdots$$
$$\cdots \times \int_{g_n^{-1}(s)}^{g_n(q_{j-1})} Y(g_n(q_{j-1}),q_j)B_n(q_j)Y(g_n(q_j),s)dq_j \dots dq_1$$

for $j = 1, \dots, k$ and $Y(t,s) = X_{g_1,\dots,g_{n-1}}^{B_1,\dots,B_{n-1}}(t,s)$. For $X_j(t,s)$ we get

(3.1)
$$\|X_j(t,s)\| \le \int_{g_n^{-j}(s)}^{t} \|Y(t,q_1)\|\|B_n(q_1)\| \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} \|Y(g_n(q_1),q_2)\|\|B_n(q_2)\|\cdots$$
$$\cdots \times \int_{g_n^{-1}(s)}^{g_n(q_{j-1})} \|Y(g_n(q_{j-1}),q_j)\|\|B_n(q_j)\|\|Y(g_n(q_j),s)\|dq_j \dots dq_1.$$

Applying the inductive hypothesis, we know that

$$\|Y(t,s)\| \le \exp\left\{\int_s^t \sum_{i=1}^{n-1} \|B_i(q)\|dq\right\}.$$

Thus we estimate the right-hand side of inequality (3.1) by

$$\int_{g_n^{-j}(s)}^{t} \|B_n(q_1)\| \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} \|B_n(q_2)\| \dots \int_{g_n^{-1}(s)}^{g_n(q_{j-1})} \|B_n(q_j)\|$$
$$\times Z(t,q_1,\dots,q_j,s)dq_j \dots dq_1$$

for each $j = 1, \dots, k$, where for $\beta(q) = \sum_{i=1}^{n-1} \|B_i(q)\|$

$$Z(t,q_1,\dots,q_j,s) = \exp\left\{\int_{q_1}^{t} \beta(q)dq + \int_{q_2}^{g_n(q_1)} \beta(q)dq + \cdots\right.$$
$$\left.\cdots + \int_{q_j}^{g_n(q_{j-1})} \beta(q)dq + \int_{s}^{g_n(q_j)} \beta(q)dq\right\} \le \exp\left\{\int_s^t \beta(q)dq\right\}.$$

Therefore we get

$$\|X_j(t,s)\| \le \exp\left\{\int_s^t \beta(q)dq\right\} \int_{g_n^{-j}(s)}^{t} \|B_n(q_1)\| \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} \|B_n(q_2)\| \cdots$$
$$\cdots \times \int_{g_n^{-1}(s)}^{g_n(q_{j-1})} \|B_n(q_j)\| dq_j \dots dq_1$$

for each $j = 1, \dots, k$. In conclusion,

$$\|X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s)\| \le \exp\left\{\int_s^t \beta(q)dq\right\} \left(1 + \int_{g_n^{-1}(s)}^{t} \|B_n(q_1)\|dq_1 + \cdots\right.$$
$$\left.\cdots + \int_{g_n^{-k}(s)}^{t} \|B_n(q_1)\| \int_{g_n^{-(k-1)}(s)}^{g_n(q_1)} \|B_n(q_2)\| \dots \int_{g_n^{-1}(s)}^{g_n(q_{k-1})} \|B_n(q_k)\| dq_k \dots dq_1\right).$$

Finally, applying Lemma 3.1 as in the proof of Lemma 3.2 we obtain

$$\|X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s)\| \le \exp\left\{\int_s^t \beta(q) + \|B_n(q)\|dq\right\}.$$

Thus the statement is proved for $t \ge g_n^{-1}(s)$. Analogically, one can prove it for $t \ge g_i^{-1}(s)$ for any $i = 1, \dots, n-1$ by the change of order described in Lemma 2.13. If $s \le t < \min\{g_1^{-1}(s), \dots, g_n^{-1}(s)\}$, $X_{g_1,\dots,g_n}^{B_1,\dots,B_n}(t,s) = E$, hence the statement holds. □

Now, we define what exactly we shall understand under the notion of exponential stability. Then we use the estimations of fundamental solutions to derive the sufficient conditions for the exponential stability of FDEs with different types of nonlinearities (see [14, 15, 16, 17] for analogical criteria for delay differential and difference equations with constant delays).

Definition 3.4. Let $n \in \mathbb{N}$, $A, B_1, \dots, B_n$ be pairwise permutable $N \times N$ constant matrices, i.e. $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i, j \in \{1, \dots, n\}$, $g_i \in G^0$ for $i = 1, \dots, n$, $\gamma := \min\{g_1(0), \dots, g_n(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$ and

$$f : \underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{n+1} \to \mathbb{R}^N$$

be a given continuous function. A solution $x_\varphi : [\gamma, \infty) \to \mathbb{R}^N$ of the equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1x(g_1(t)) + \cdots + B_nx(g_n(t)) \\ &\quad + f(x(t), x(g_1(t)), \dots, x(g_n(t))), \qquad t \ge 0 \end{aligned} \tag{3.2}$$

with initial condition (2.12) is called exponentially stable, if there exist positive constants c_1, c_2, δ, depending on A, $B_1, \dots, B_n$, f and $\|\varphi\| = \max_{t\in[\gamma,0]} \|\varphi(t)\|$, such that

$$\|x_\eta(t) - x_\varphi(t)\| \le c_1 e^{-c_2 t}, \qquad t \ge 0$$

for any solution $x_\eta(t)$ of the equation (3.2) satisfying the initial condition

$$x_\eta(t) = \eta(t), \qquad \gamma \le t \le 0$$

with $\eta \in C([\gamma, 0], \mathbb{R}^N)$, $\|\eta - \varphi\| < \delta$.

Theorem 3.5. *Let $n \in \mathbb{N}$, $A, B_1, \dots, B_n$ be pairwise permutable $N \times N$ constant matrices, i.e. $AB_i = B_i A$, $B_i B_j = B_j B_i$ for each $i, j \in \{1, \dots, n\}$, A have eigenvalues $\lambda_1, \dots, \lambda_N$ such that*

$$\operatorname{Re} \lambda_1 \le \dots \le \operatorname{Re} \lambda_N \le -k < 0,$$

$g_i \in G^0$ for $i = 1, \dots, n$ and there be $k_1 < k$ such that

$$\int_0^t \sum_{i=1}^n \|\widetilde{B}_i(q)\| dq \le k_1 t$$

for all $t \ge 0$, where $\widetilde{B}_i(t) = \mathrm{e}^{-A(t-g_i(t))} B_i$ for $i = 1, \dots, n$. Then if $f(x) = o(\|x\|)$, the trivial solution of equation

$$\dot{x}(t) = Ax(t) + B_1 x(g_1(t)) + \dots + B_n x(g_n(t)) + f(x(t)) \tag{3.3}$$

is exponentially stable.

Proof. Let $\gamma := \min\{g_1(0), \dots, g_n(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$. According to Theorem 2.15 the solution of equation (3.3) satisfying condition (2.12) has the form

$$x(t) = X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t, 0)\mathrm{e}^{At}\varphi(0) + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t, s)\mathrm{e}^{A(t-s)} \left[B_1 \psi(g_1(s)) \right.$$
$$\left. + \dots + B_n \psi(g_n(s))\right] ds + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t, s)\mathrm{e}^{A(t-s)} f(x(s)) ds$$

for $t \ge 0$, where $\psi(t)$ is given by (2.14). From the property of eigenvalues of A it follows that there are positive constants k, K such that $\|\mathrm{e}^{At}\| \le K\mathrm{e}^{-kt}$ for all $t \ge 0$. Next, since $f(x) = o(\|x\|)$, for any $P > 0$ there is $\delta > 0$ such that if $\|x\| < \delta$, then $\|f(x)\| < P\|x\|$. Applying these two estimations, Lemma 3.3 and assuming that $\|x(s)\|$ is sufficiently small for $s \in [0, t]$, $t \ge 0$ we derive

$$\|x(t)\| \le K\mathrm{e}^{\int_0^t \beta(q)dq - kt}\|\varphi(0)\| + K \sum_{i=1}^n \|B_i\| \int_0^t \mathrm{e}^{\int_s^t \beta(q)dq - k(t-s)} \|\psi(g_i(s))\| ds$$
$$+ KP \int_0^t \mathrm{e}^{\int_s^t \beta(q)dq - k(t-s)} \|x(s)\| ds$$

with $\beta(t) = \sum_{i=1}^n \|\widetilde{B}_i(t)\|$. Denoting $u(t) = \exp\{kt - \int_0^t \beta(q)dq\}\|x(t)\|$ we get the estimate for $u(t)$

$$u(t) \le K\|\varphi(0)\| + K \sum_{i=1}^n \|B_i\| \int_0^t \mathrm{e}^{ks - \int_0^s \beta(q)dq} \|\psi(g_i(s))\| ds + KP \int_0^t u(s) ds. \tag{3.4}$$

Now, the property of k_1 implies that for each $i \in \{1,\dots,n\}$ function $g_i : [0,\infty) \to [g_i(0),\infty)$ is surjective and, especially, $g_i^{-1}(0) < \infty$. Indeed, suppose in contrary that there exists $Q \in \mathbb{R}$ such that $g_i(t) < Q$ for all $t \geq 0$ and some $i \in \{1,\dots,n\}$. The property of eigenvalues of A yields the existence of a positive constant L_i such that $L_i \mathrm{e}^{kt} \leq \|\mathrm{e}^{-At} B_i\|$ for all $t \geq 0$ (assuming $B_i \neq \Theta$). Consequently,

$$k_1 t \geq \int_0^t \sum_{j=1}^n \|\widetilde{B}_j(q)\| dq \geq \int_0^t \|\widetilde{B}_i(q)\| dq$$

$$\geq L_i \int_0^t \mathrm{e}^{k(q-g_i(q))} dq \geq L_i \mathrm{e}^{-kQ} \int_0^t \mathrm{e}^{kq} dq = L_i \mathrm{e}^{-kQ} \frac{\mathrm{e}^{kt}-1}{k}$$

for all $t \geq 0$, a contradiction results. So using the definition of $\psi(t)$ we can estimate

$$\int_0^t \mathrm{e}^{ks-\int_0^s \beta(q)dq} \|\psi(g_i(s))\| ds \leq \int_0^{g_i^{-1}(0)} \mathrm{e}^{ks-\int_0^s \beta(q)dq} \|\varphi(g_i(s))\| ds$$

for all $t \geq 0$ and $i = 1,\dots,n$, where the right-hand side is constant. Next, from (3.4) we get $u(t) \leq M + KP \int_0^t u(s)ds$ where

(3.5)

$$0 \leq M = M(\varphi) = K\|\varphi(0)\| + K \sum_{i=1}^n \|B_i\| \int_0^{g_i^{-1}(0)} \mathrm{e}^{ks-\int_0^s \beta(q)dq} \|\varphi(g_i(s))\| ds.$$

Finally, applying Gronwall's inequality, $u(t) \leq M\mathrm{e}^{KPt}$ which for $x(t)$ means

$$\|x(t)\| \leq M \exp\left\{KPt - kt + \int_0^t \beta(q)dq\right\} \leq M\mathrm{e}^{(KP-k+k_1)t}.$$

Therefore, if $P < \frac{k-k_1}{K}$ then for $\max\{\|\varphi(0)\|, M\} < \delta$ it holds $\|x(t)\| \leq M\mathrm{e}^{-\eta t}$ for all $t \geq 0$ with $\eta = k - k_1 - KP > 0$, i.e. the trivial solution of (3.3) is exponentially stable. □

Theorem 3.6. *Let the assumptions of Theorem 3.5 be fulfilled. Moreover, let $S_i := \sup_{t\geq 0} t - g_i(t) < \infty$ for each $i = 1,\dots,n$. If*

$$f(x, y_1, \dots, y_n) = o(\|x\| + \|y_1\| + \dots + \|y_n\|),$$

i.e. for any $P > 0$ there exists $\delta > 0$ such that

$$\|x\|, \|y_1\|, \dots, \|y_n\| < \delta \quad \Rightarrow \quad \|f(x, y_1, \dots, y_n)\| < P(\|x\|+\|y_1\|+\dots+\|y_n\|),$$

then the trivial solution of equation (3.2) *is exponentially stable.*

Proof. Let $\gamma := \min\{g_1(0),\dots,g_n(0)\}$, $\varphi \in C([\gamma,0],\mathbb{R}^N)$. Denote

(3.6)

$$E(t) = \begin{cases} 0, & t < 0, \\ kt - \int_0^t \beta(q)dq, & t \geq 0, \end{cases} \quad \beta(t) = \sum_{i=1}^n \|\widetilde{B}_i(t)\|, \quad u(t) = \mathrm{e}^{E(t)}\|x(t)\|$$

with $x(t)$ being the solution of equation (3.2) together with condition (2.12), which is known to have a form (see Theorem 2.15)

$$x(t) = X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,0)\mathrm{e}^{At}\varphi(0) + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,s)\mathrm{e}^{A(t-s)}\left[B_1\psi(g_1(s)) + \cdots\right.$$

$$\left.\cdots + B_n\psi(g_n(s))\right] ds + \int_0^t X_{g_1,\dots,g_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n}(t,s)\mathrm{e}^{A(t-s)} f(x(s), x(g_1(s)), \dots, x(g_n(s)))ds$$

for $t \geq 0$, where $\psi(t)$ is given by (2.14). Note that $E(t)$ is a continuous function defined on the whole $\mathbb{R}$. Analogically to the proof of Theorem 3.5, we derive estimation

$$u(t) \leq M + KP\int_0^t u(s)ds + KP\sum_{i=1}^n \int_0^t \mathrm{e}^{E(s)-E(g_i(s))}u(g_i(s))ds$$

with M given by (3.5), assuming $\|\varphi\|$ and $\|x(s)\|$ to be sufficiently small for all $s \in [0,t]$, $t \geq 0$. Denoting $h(t)$ the nondecreasing continuous function defined on $[0,\infty)$ and given by

$$h(t) := c + KP\int_0^t u(s)ds + KP\sum_{i=1}^n \int_0^t \mathrm{e}^{E(s)-E(g_i(s))}u(g_i(s))ds$$

where $c = \max\{M, \|\varphi\|\}$, we get the inequality $u(t) \leq h(t)$ on $[0,\infty)$. Let us estimate for $s \in [0,t]$:

$$\begin{aligned} u(g_i(s)) &\leq \sup_{0\leq\sigma\leq s} u(g_i(\sigma)) \leq \sup_{0\leq\sigma\leq g_i^{-1}(0)} u(g_i(\sigma)) + \sup_{g_i^{-1}(0)\leq\sigma\leq s} u(g_i(\sigma)) \\ &\leq \|\varphi\| + \sup_{0\leq\sigma\leq s} u(\sigma) \leq 2h(s), \end{aligned} \tag{3.7}$$

by the property of a nondecreasing function h. Thus we obtain

$$h(t) \leq c + KP\int_0^t h(s)ds + 2KP\sum_{i=1}^n \int_0^t \mathrm{e}^{E(s)-E(g_i(s))}h(s)ds$$

for all $t \geq 0$. Furthermore, for each $i \in \{1,\dots,n\}$, if $0 \leq s < g_i^{-1}(0)$, then

$$E(s) - E(g_i(s)) = ks - \int_0^s \beta(q)dq \leq kg_i^{-1}(0) = k(t-g_i(t))|_{t=g_i^{-1}(0)} \leq kS_i \tag{3.8}$$

and if $s \geq g_i^{-1}(0)$,

$$E(s) - E(g_i(s)) = k(s-g_i(s)) - \int_{g_i(s)}^s \beta(q)dq \leq k(s-g_i(s)) \leq kS_i. \tag{3.9}$$

Therefore

$$h(t) \leq c + KP\left(1 + 2\sum_{i=1}^n \mathrm{e}^{kS_i}\right)\int_0^t h(s)ds$$

whenever $t \geq 0$. Finally, from Gronwall's inequality

$$u(t) \leq h(t) \leq c \exp\left\{KP\left(1+2\sum_{i=1}^{n} \mathrm{e}^{kS_i}\right)t\right\}$$

for all $t \geq 0$. Hence

$$\begin{aligned}\|x(t)\| &\leq c \exp\left\{KP\left(1+2\sum_{i=1}^{n} \mathrm{e}^{kS_i}\right)t - kt + \int_0^t \beta(q)dq\right\} \\ &\leq c \exp\left\{\left(KP\left(1+2\sum_{i=1}^{n} \mathrm{e}^{kS_i}\right) - k + k_1\right)t\right\}.\end{aligned}$$

So, if $P < \frac{k-k_1}{K(1+2\sum_{i=1}^{n} \mathrm{e}^{kS_i})}$, then for $c < \delta$ (that is for $\|\varphi\|$ sufficiently small) the solution $x(t)$ satisfies $\|x(t)\| \leq c\mathrm{e}^{-\eta t}$ with $\eta = k - k_1 - KP(1 + 2\sum_{i=1}^{n} \mathrm{e}^{kS_i}) > 0$, i.e. the trivial solution of (3.2) is exponentially stable. $\square$

In further work we shall write $\omega_1 \propto \omega_2$ for functions $\omega_1, \omega_2 : A \to \mathbb{R}\backslash\{0\}$, $A \subset \mathbb{R}$, if the function $\frac{\omega_2}{\omega_1}$ is nondecreasing on A.

Theorem 3.7. *Let the assumptions of Theorem 3.5 be fulfilled and* $S_i := \sup_{t\geq 0} t - g_i(t) < \infty$ *for each* $i = 1, \dots, n$. *If*

$$f(x, y_1, \dots, y_n) = o(\|x\|^{\gamma_0} + \|y_1\|^{\gamma_1} + \dots + \|y_n\|^{\gamma_n})$$

for given constants $1 < \gamma_0, \gamma_1, \dots, \gamma_n$, *i.e. for any* $P > 0$ *there exists* $\delta > 0$ *such that if* $\|x\|, \|y_1\|, \dots, \|y_n\| < \delta$ *then*

$$\|f(x, y_1, \dots, y_n)\| < P\left(\|x\|^{\gamma_0} + \|y_1\|^{\gamma_1} + \dots + \|y_n\|^{\gamma_n}\right),$$

then the trivial solution of equation (3.2) *is exponentially stable.*

Proof. Let $\gamma := \min\{g_1(0), \dots, g_n(0)\}$, $\varphi \in C([\gamma, 0], \mathbb{R}^N)$ and $x(t)$ be a solution of equation (3.2) satisfying initial condition (2.12). As before, assuming $\|\varphi\|$, $\|x(s)\|$ to be sufficiently small for all $s \geq 0$ and using notations (3.5), (3.6) we obtain inequality

$$u(t) \leq M + KP\int_0^t \mathrm{e}^{E(s)(1-\gamma_0)} u(s)^{\gamma_0} ds + KP\sum_{i=1}^{n}\int_0^t \mathrm{e}^{E(s)-\gamma_i E(g_i(s))} u(g_i(s))^{\gamma_i} ds.$$

Denote

$$h(t) := c + KP\int_0^t \mathrm{e}^{E(s)(1-\gamma_0)} u(s)^{\gamma_0} ds + KP\sum_{i=1}^{n}\int_0^t \mathrm{e}^{E(s)-\gamma_i E(g_i(s))} u(g_i(s))^{\gamma_i} ds$$

with $c = \max\{\|\varphi\|, M\}$. Clearly, $u(t) \leq h(t)$ for all $t \geq 0$ and, arguing like in (3.7), also $u(g_i(t)) \leq 2h(t)$ for all $t \geq 0$ and each $i = 1, \dots, n$. Next,

$E(t)(1-\gamma_0) \le (k-k_1)(1-\gamma_0)t$ for all $t \ge 0$ and by (3.8), (3.9)

$$E(t) - \gamma_i E(g_i(t)) = \begin{cases} E(t) \le kS_i, & 0 \le t < g_i^{-1}(0), \\ E(t) - E(g_i(t)) + E(g_i(t))(1-\gamma_i) & \\ \le kS_i + (k-k_1)(1-\gamma_i)g_i(t), & t \ge g_i^{-1}(0). \end{cases}$$

Therefore

$$h(t) \le c + \sum_{i=0}^{n} \int_0^t \lambda_i(s)\omega_i(u(s))ds$$

where $\lambda_0(t) = KP\mathrm{e}^{(k-k_1)(1-\gamma_0)t}$,

$$\lambda_i(t) = \begin{cases} 2^{\gamma_i} KP\mathrm{e}^{kS_i}, & 0 \le t < g_i^{-1}(0), \\ 2^{\gamma_i} KP\mathrm{e}^{kS_i+(k-k_1)(1-\gamma_i)g_i(t)}, & t \ge g_i^{-1}(0), \end{cases} \qquad i = 1, \dots, n$$

are continuous and positive functions on $[0, \infty)$ and $\omega_i(z) = z^{\gamma_i}$ for $i = 0, \dots, n$. Accordingly,

$$h(t) \le c + \sum_{i=0}^{n} \int_0^t \lambda_i(s)\omega_i(h(s))ds$$

for all $t \ge 0$. Without any loss of generality we can assume that $\gamma_0, \dots, \gamma_n$ form a nondecreasing sequence, i.e. $1 < \gamma_0 \le \gamma_1 \le \dots \le \gamma_n$ (in the other case we change the notation for them and also for corresponding coefficients $\lambda_i(s)$).

If $\gamma_0 = \gamma_n$, the statement follows from Bihari's lemma [3, 18]. Indeed, in this case

$$h(t) \le c + \int_0^t \lambda(s)\omega(h(s))ds =: H(t)$$

with $\lambda(t) = \sum_{i=0}^{n} \lambda_i(t)$, $\omega(z) = \omega_0(z) = \dots = \omega_n(z)$. Then

$$W(H(t)) \le \int_0^t \lambda(s)ds, \qquad W(u) = \int_c^u \frac{dz}{\omega(z)}$$

since $W(H(0)) = W(c) = 0$. Let P be such small that

$$\|\lambda\| := \int_0^\infty \lambda(s)ds < \int_c^\infty \frac{dz}{\omega(z)} \quad (< \infty).$$

Consequently, $W(H(t)) < \|\lambda\|$ for all $t \ge 0$ and

$$u(t) \le h(t) \le H(t) < W^{-1}(\|\lambda\|) =: C < \infty.$$

Hence we have

$$\|x(t)\| \le C \exp\left\{ \int_0^t \beta(q)dq - kt \right\} \le C\mathrm{e}^{-(k-k_1)t}$$

whenever $t \geq 0$. Now, if $\|\varphi\|$ is sufficiently small, then c is small and

$$W(C) = \|\lambda\| < \int_c^\delta \frac{dz}{\omega(z)} = W(\delta).$$

Thus $C < \delta$ and the exponential stability of the trivial solution follows.

In the other case, when $\gamma_0 < \gamma_n$, Pinto's inequality [18] is applied. Note that $\omega_0 \propto \cdots \propto \omega_n$. Let P be such small that

$$\|\lambda_i\| := \int_0^\infty \lambda_i(s)ds < \int_{c_{i-1}}^\infty \frac{dz}{\omega_i(z)}, \qquad i = 0, \ldots, n$$

where

$$c_{-1} = c, \qquad c_i = W_i^{-1}(W_i(c_{i-1}) + \|\lambda_i\|), \qquad i = 0, \ldots, n-1,$$
$$W_i(u) = \int_{u_i}^u \frac{dz}{\omega_i(z)}, \qquad u, u_i > 0, \qquad i = 0, \ldots, n.$$

Then Pinto's inequality yields

$$u(t) \leq h(t) \leq W_n^{-1}\left(W_n(c_{n-1}) + \int_0^t \lambda_n(s)ds\right) \leq W_n^{-1}(W_n(c_{n-1}) + \|\lambda_n\|).$$

Here the right-hand side is constant for all $t \geq 0$ and we denote it by C. The trivial solution of equation (3.2) is exponentially stable if $C < \delta$. So it remains to verify, if this inequality can be assured by making $\|\varphi\|$ sufficiently small. From definition of C we know that $C < \delta$ if

$$\|\lambda_n\| < \int_{c_{n-1}}^\delta \frac{dz}{\omega_n(z)},$$

i.e. if $c_{n-1} < \delta_{n-1} \leq \delta$ for $\delta_{n-1} > 0$ sufficiently small. Analogically, this is satisfied if $c_{n-2} < \delta_{n-2} \leq \delta_{n-1}$ with $\delta_{n-2} > 0$ small. Finally, we obtain that $C < \delta$ if $c = c_{-1} < \delta_{-1} \leq \delta_0 \leq \cdots \leq \delta_{n-1} \leq \delta$ with $\delta_{-1} > 0$ sufficiently small. So if $\|\varphi\|$ is sufficiently small, the trivial solution is really exponentially stable. This completes the proof. □

We have also a result for nonautonomous nonlinear FDEs:

Theorem 3.8. *Let the assumptions of Theorem 3.5 be fulfilled and* $m_i \in \mathbb{N}$, $\gamma_{ij} > 1$ *for* $i = 0, \ldots, n$, $j = 1, \ldots, m_i$ *be given constants. Assume that there* *exists a nonnegative function* $r \in C([0,\infty),\mathbb{R})$ *satisfying*

$$\int_0^\infty r(s)\exp\left\{\left(ks - \sum_{j=1}^n \int_0^s \|\widetilde{B}_j(q)\|dq\right)(1-\gamma_0)\right\}ds < \infty,$$

$$\int_{g_i^{-1}(0)}^\infty r(s)\exp\left\{k(s-\gamma_i g_i(s)) - \sum_{j=1}^n \left(\int_0^s \|\widetilde{B}_j(q)\|dq \right.\right.$$
$$\left.\left. -\gamma_i \int_0^{g_i(s)} \|\widetilde{B}_j(q)\|dq\right)\right\} ds < \infty$$

for $i = 1,\dots,n$ *where* $\gamma_i = \min\{\gamma_{i1},\dots,\gamma_{im_i}\}$, $i = 0,\dots,n$, *such that for any positive constants* a_{ij}, $i = 0,\dots,n$, $j = 1,\dots,m_i$ *there is* $\delta > 0$ *such that if* $\|x\|, \|y_1\|,\dots,\|y_n\| < \delta$ *then*

$$\|f(t,x,y_1,\dots,y_n)\| \le r(t)\left[\sum_{j=1}^{m_0} a_{0j}\|x\|^{\gamma_{0j}} + \sum_{i=1}^n\sum_{j=1}^{m_i} a_{ij}\|y_i\|^{\gamma_{ij}}\right].$$

Then the trivial solution of equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1x(g_1(t)) + \cdots + B_nx(g_n(t)) \\ &+f(t,x(t),x(g_1(t)),\dots,x(g_n(t))), \qquad t \ge 0 \end{aligned} \tag{3.10}$$

is exponentially stable.

Proof. Let $\gamma := \min\{g_1(0),\dots,g_n(0)\}$, $\varphi \in C([\gamma,0],\mathbb{R}^N)$. For the solution $x(t)$ of equation (3.10) satisfying initial condition (2.12) we assume that $\|\varphi\|$ and $\|x(s)\|$ are sufficiently small for all $s \ge 0$. In the notation of (3.5), (3.6) we can write

$$u(t) \le M + K\sum_{j=1}^{m_0} a_{0j}\int_0^t r(s)\mathrm{e}^{E(s)(1-\gamma_{0j})}u(s)^{\gamma_{0j}}ds$$
$$+K\sum_{i=1}^n\sum_{j=1}^{m_i} a_{ij}\int_0^t r(s)\mathrm{e}^{E(s)-\gamma_{ij}E(g_i(s))}u(g_i(s))^{\gamma_{ij}}ds.$$

Next, we denote

$$\lambda_{0j}(t) = Ka_{0j}r(t)\mathrm{e}^{E(t)(1-\gamma_{0j})}, \qquad j = 1,\dots,m_0,$$
$$\lambda_{ij}(t) = Ka_{ij}r(t)\mathrm{e}^{E(t)-\gamma_{ij}E(g_i(t))}, \qquad i = 1,\dots,n,\ j = 1,\dots,m_i$$

to arrive at

$$u(t) \le M + \sum_{j=1}^{m_0}\int_0^t \lambda_{0j}(s)u(s)^{\gamma_{0j}}ds + \sum_{i=1}^n\sum_{j=1}^{m_i}\int_0^t \lambda_{ij}(s)u(g_i(s))^{\gamma_{ij}}ds$$

for all $t \geq 0$. If

$$h(t) := c + \sum_{j=1}^{m_0} \int_0^t \lambda_{0j}(s)u(s)^{\gamma_{0j}}ds + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \int_0^t \lambda_{ij}(s)u(g_i(s))^{\gamma_{ij}}ds$$

with $c = \max\{M, \|\varphi\|\}$, then $u(t) \leq h(t)$ for all $t \geq 0$. Analogically to (3.7), $u(g_i(t)) \leq 2h(t)$ whenever $t \geq 0$ and $i = 1, \dots, n$. Therefore

$$h(t) \leq c + \sum_{i=0}^{n}\sum_{j=1}^{m_i} \int_0^t \mu_{ij}(s)h(s)^{\gamma_{ij}}ds$$

with $\mu_{0j}(t) = \lambda_{0j}(t)$ for $j = 1, \dots, m_0$ and $\mu_{ij}(t) = 2^{\gamma_{ij}}\lambda_{ij}(t)$ for $i = 1, \dots, n$, $j = 1, \dots, m_i$.

Now we collect the coefficients $\mu_{ij}(s)$ by the same exponents and create an increasing sequence of exponents. So we get exponents $1 < \delta_1 < \cdots < \delta_p$ such that $1 \leq p \leq m_0 + \cdots + m_n$ and $\{\delta_i\}_{i=1}^p = \{\gamma_{i1}, \dots, \gamma_{im_i}\}_{i=0}^n$. Moreover, denoting $\Omega := \{(i,1), \dots, (i,m_i)\}_{i=0}^n$ the set of all indices, for each $k \in \{1, \dots, p\}$: if $\delta_k = \gamma_{i_1 j_1} = \cdots = \gamma_{i_{L_k} j_{L_k}}$, $\{(i_l, j_l)\}_{l=1}^{L_k} \subset \Omega$ and $\delta_k \neq \gamma_{ij}$ for all $(i,j) \in \Omega \backslash \{(i_l, j_l)\}_{l=1}^{L_k}$ (i.e. the set $\{(i_l, j_l)\}_{l=1}^{L_k}$ is the maximal subset of index set Ω such that $\delta_k = \gamma_{ij}$ for each (i,j) from this subset), then we define $\nu_k(t) := \sum_{l=1}^{L_k} \mu_{i_l j_l}(t)$. Now, for $\omega_i(z) := z^{\delta_i}$, $i = 1, \dots, p$ we get the sequence $\omega_1 \propto \cdots \propto \omega_p$ and $h(t)$ fulfils

$$h(t) \leq c + \sum_{i=1}^{p} \int_0^t \nu_i(s)\omega_i(h(s))ds.$$

The proof can be finished exactly as the previous one using Bihari's inequality if $p = 1$, or Pinto's inequality if $p > 1$. In addition, the assumptions of the theorem establish the convergence of $\int_0^\infty \nu_i(s)ds$, $i = 1, \dots, p$, which is important for the mentioned inequalities (see proof of Theorem 3.7 or [18]). $\square$

Finally, we apply one of the derived stability criteria on a simple biological model with delayed birthrates, concerning two species whose predator-prey-position is periodically changed in time.

Example 3.9. Let us consider the following system

$$\begin{aligned} \dot{x}_1(t) &= -\alpha_1 x_1(t) + \beta_1 x_1(t - \mathrm{e}^{-t}) - \gamma x_1(t)x_2(t)\sin t \\ \dot{x}_2(t) &= -\alpha_2 x_2(t) + \beta_2 x_2(t - \mathrm{e}^{-2t}) + \gamma x_1(t)x_2(t)\sin t \end{aligned} \tag{3.11}$$

with parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma > 0$ such that $\alpha_1 \leq \alpha_2$.

This time, we have delay functions $g_1(t) = t - \mathrm{e}^{-t}$, $g_2(t) = t - \mathrm{e}^{-2t}$, matrices

$$A = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

and nonlinear function $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(t, x) = (-\gamma x_1 x_2 \sin t, \gamma x_1 x_2 \sin t)$$

for vector $x = (x_1, x_2)$. It is easy to see that A, B_1, B_2 are pairwise permutable and

$$\|f(t, x)\| \leq \frac{\gamma |\sin t|}{\sqrt{2}} \|x\|^2$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}^2$ and vector norm $\|x\| = \sqrt{x_1^2 + x_2^2}$. We have the next lemma.

Lemma 3.10. *Let $1 < \gamma_0 < 2$ be arbitrary and fixed. Then for any $a > 0$ there exists $\delta > 0$ such that if $\|x\| < \delta$, then*

$$\|f(t, x)\| \leq \frac{a\gamma |\sin t|}{\sqrt{2}} \|x\|^{\gamma_0}.$$

Proof. Clearly, if $\|x\| = 0$, then $\|f(t, x)\| = 0$. Now, let $a > 0$ be fixed and $0 < \|x\| < \delta$. Accordingly,

$$\frac{\|f(t, x)\|}{\|x\|^{\gamma_0}} \leq \frac{\gamma |\sin t|}{\sqrt{2}} \|x\|^{2-\gamma_0} < \frac{\gamma |\sin t|}{\sqrt{2}} \delta^{2-\gamma_0}.$$

Hence, it is sufficient to set $\delta = a^{\frac{1}{2-\gamma_0}}$ to obtain the statement of the lemma. □

Using the matrix norm $\|C\| = \frac{1}{\sqrt{2}} \sqrt{\sum |c_{ij}|^2}$ (in order to satisfy the basic assumption $\|E\| = 1$), we obtain $\|\widetilde{B}_1(t)\| = \beta_1 \mathrm{e}^{\alpha_1 \mathrm{e}^{-t}} / \sqrt{2}$ and $\|\widetilde{B}_2(t)\| = \beta_2 \mathrm{e}^{\alpha_2 \mathrm{e}^{-2t}} / \sqrt{2}$. Consequently,

$$\beta(t) = \|\widetilde{B}_1(t)\| + \|\widetilde{B}_2(t)\| \leq \frac{(\beta_1 + \beta_2)\mathrm{e}^{\alpha_2}}{\sqrt{2}}$$

for all $t \geq 0$. So we can take $k_1 = (\beta_1 + \beta_2)\mathrm{e}^{\alpha_2}/\sqrt{2}$ and condition $\int_0^t \beta(q)dq \leq k_1 t$ is satisfied.

Corollary 3.11. *If $(\beta_1 + \beta_2)\mathrm{e}^{\alpha_2} < \sqrt{2}\alpha_1$, then system* (3.11) *has exponentially stable trivial solution.*

Proof. The corollary follows from Theorem 3.8. We show that all assumptions of this theorem are fulfilled. Let $1 < \gamma_0 < 2$ be arbitrary and fixed, and $r(t) := \frac{\gamma |\sin t|}{\sqrt{2}}$. Then, by the assumption of the corollary, $k_1 < \alpha_1$ and

$$\int_0^\infty r(s) \exp\left\{ \left(\alpha_1 s - \int_0^s \beta(q)dq \right) (1 - \gamma_0) \right\} ds$$

$$\leq \int_0^\infty r(s) \exp\left\{ (k_1 - \alpha_1)(\gamma_0 - 1)s \right\} ds$$

$$\leq \frac{\gamma}{\sqrt{2}} \int_0^\infty \mathrm{e}^{(k_1 - \alpha_1)(\gamma_0 - 1)s} ds = \frac{\gamma}{\sqrt{2}(\gamma_0 - 1)(\alpha_1 - k_1)} < \infty.$$

Next, since $s-1 \le g_i(s) < s$ and $\beta(s) \ge \frac{\beta_1+\beta_2}{\sqrt{2}}$ for $i=1,2$, $s \ge 0$, we have

$$\int_{g_i^{-1}(0)}^{\infty} r(s) \exp\left\{\alpha_1(s-\gamma_i g_i(s)) - \int_0^s \beta(q)dq + \gamma_i \int_0^{g_i(s)} \beta(q)dq\right\} ds$$
$$\le \frac{\gamma}{\sqrt{2}} \int_{g_i^{-1}(0)}^{\infty} \exp\left\{(k_1-\alpha_1)\gamma_i g_i(s) + \alpha_1 s - \int_0^s \beta(q)dq\right\} ds$$
$$\le \frac{\gamma}{\sqrt{2}} \int_{g_i^{-1}(0)}^{\infty} \exp\left\{(k_1-\alpha_1)\gamma_i g_i(s) + \left(\alpha_1 - \frac{\beta_1+\beta_2}{\sqrt{2}}\right) s\right\} ds$$
$$\le \frac{\gamma e^{(\alpha_1-k_1)\gamma_i}}{\sqrt{2}} \int_{g_i^{-1}(0)}^{\infty} \exp\left\{\left[(k_1-\alpha_1)\gamma_i + \alpha_1 - \frac{\beta_1+\beta_2}{\sqrt{2}}\right] s\right\} ds$$

for $i=1,2$. The right-hand side of the latter inequality has the form

$$\frac{\gamma e^{(\alpha_1-k_1)\gamma_i}}{\sqrt{2}} \int_{g_i^{-1}(0)}^{\infty} e^{-\eta s} ds < \infty$$

with $\eta = (\alpha_1-k_1)\gamma_i + \frac{\beta_1+\beta_2}{\sqrt{2}} - \alpha_1 > 0$ whenever

$$\gamma_i > \frac{\sqrt{2}\alpha_1 - (\beta_1+\beta_2)}{\sqrt{2}\alpha_1 - (\beta_1+\beta_2)e^{\alpha_2}} \quad (>1).$$

Clearly, when we set $f(t,x,y_1,y_2) := f(t,x)$, then by Lemma 3.10

$$\|f(t,x,y_1,y_2)\| \le ar(t)\left(\|x\|^{\gamma_0} + \|y_1\|^{\gamma_1} + \|y_2\|^{\gamma_2}\right).$$

By that, the proof is finished. □

References

[1] L. Berezansky, E. Braverman, On stability of some linear and nonlinear delay differential equations, J. Math. Anal. Appl. 314 (2006) 391–411.

[2] L. Berezansky, E. Braverman, On exponential stability of linear differential equations with several delays, J. Math. Anal. Appl. 324 (2006) 1336–1355.

[3] I.A. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation, Acta Math. Acad. Sci. H. 7 (1956) 81–94.

[4] A. Boichuk, J. Diblík, D. Khusainov, and M. Růžičková, Boundary value problems for delay differential systems, Adv. Differ. Equ. (2010), Article ID 593834, 20 pages.

[5] A. Boichuk, J. Diblík, D. Khusainov, and M. Růžičková, Fredholm's boundary-value problems for differential systems with a single delay, Nonlinear Anal.-Theor. 72 (2010) 2251–2258.

[6] A. Boichuk, J. Diblík, D. Khusainov, and M. Růžičková, Boundary-value problems for weakly nonlinear delay differential systems, Abstr. Appl. Anal. (2011), Article ID 631412, 19 pages.

[7] J. Diblík, D.Ya. Khusainov, Representation of solutions of discrete delayed system $x(k+1) = Ax(k) + Bx(k-m) + f(k)$ with commutative matrices, J. Math. Anal. Appl. 318 (2006) 63–76.

[8] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[9] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[10] T. Krisztin, On stability properties for one-dimensional functional differential equations, Funkc. Ekvac. 34 (1991) 241–256.
[11] D.Ya. Khusainov, J. Diblík, Representation of solutions of linear discrete systems with constant coefficients and pure delay, Adv. Differ. Equ. (2006), Article ID 80825, 1–13.
[12] D.Ya. Khusainov, J. Diblík, M. Růžičková, J. Lukáčová, Representation of a solution of the Cauchy problem for an oscillating system with pure delay, Nonlinear Oscil. 11, No. 2 (2008) 276–285.
[13] D.Ya. Khusainov, G. V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, Stud. Univ. Žilina, Math. Ser. 17 (2003) 101–108.
[14] M. Medved', M. Pospíšil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, Nonlinear Anal.-Theor. 75 (2012) 3348–3363.
[15] M. Medved', M. Pospíšil, Representation and stability of solutions of systems of difference equations with multiple delays and linear parts defined by pairwise permutable matrices, Commun. Appl. Anal., 2012, accepted.
[16] M. Medved', M. Pospíšil, L. Škripková, Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices, Nonlinear Anal.-Theor. 74 (2011) 3903–3911.
[17] M. Medved', L. Škripková, Sufficient conditions for the exponential stability of delay difference equations with linear parts defined by permutable matrices, Electron. J. Qual. Theo. 22 (2012) 1–13.
[18] M. Pinto, Integral inequalities of Bihari-type and applications. Funkc. Ekvac. 33 (1990) 387–403.
[19] T. Wang, Inequalities and stability for a linear scalar functional differential equation, J. Math. Anal. Appl. 298 (2004) 33–44.

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