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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS AND GENERALIZED GUIDING FUNCTIONS

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Abstract

Let $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a continuous function and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly positive function. A sufficient condition such that the equation $\dot{x} = f(t, x)$ admits solutions $x : \mathbb{R} \to \mathbb{R}^N$ satisfying the inequality $|x(t)| \le k \cdot h(t)$, $t \in \mathbb{R}$, $k > 0$, where $|\cdot|$ is the euclidean norm in $\mathbb{R}^N$, is given. The proof of this result is based on the use of a special function of Lyapunov type, which is often called guiding function. In the particular case $h \equiv 1$, one obtains known results regarding the existence of bounded solutions.

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1 Introduction

Let $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a continuous function. Within the problem of the existence of bounded solutions (and in particularly of periodic solutions) for the equation

$$\dot{x} = f(t, x), \tag{1}$$

the method of guiding functions is very productive.

The guiding functions, which is fact are functions of Lyapunov type, have been introduced in [14] and then generalized and used in diverse ways (see e.g. [16], [17]). These cited works contain rich bibliographical informations in this field. The use of Lyapunov functions in the study of certain qualitative properties of solutions constitutes the object of numerous interesting works; in this direction we mention the ones of T.A. Burton (see e.g. [10], [11], [12]).

The classical guiding functions can not be used in general, in the study of some properties of solutions, more complicated than the ones of boundedness. Such a behavior of a solution $x(\cdot)$ of the equation (1) could be for example the existence of finite limits of this at $+\infty$ or $-\infty$., limits denoted $x(\pm\infty)$ (i.e. $x(\pm\infty) := \lim\limits_{t\to\pm\infty} x(t)$).

This type of behavior has been recently considered in the notes [1] − [8] and it is closely related to the existence of heteroclinic and homoclinic solutions. Indeed, in the case of an autonomous system $\dot{x} = f(x)$, each solution $x(\cdot)$ for which there exist $x(\pm\infty)$ is a **heteroclinic** solution and a solution $x(\cdot)$ for which $x(+\infty) = x(-\infty)$ is a **homoclinic** solution. In fact, some authors (see e.g. [1] , [9]) named the solutions $x(\cdot)$ for which $x(+\infty) = x(-\infty) = 0$ **homoclinic**. We shall call such a solution **evanescent**.

A way to establish the fact that a solution $x(\cdot)$ is evanescent is to prove that $x(\cdot)$ satisfies a inequality of type

$$|x(t)| \le k \cdot h(t), \ t \in \mathbb{R}, \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with $h(\pm\infty) = 0$.

The idea to use estimations of type (2) for qualitative informations for the solutions of the equation (1) , belongs to C. Corduneanu (see [13]), which has started from some classical results of Perron. Corduneanu organizes the set of continuous functions fulfilling (2) , for $t \ge 0$, as a Banach space. This manner to treat the qualitative problems has been used by many authors; through the interesting results obtained last years, we mention [10] .

In the present paper we give an existence theorem for the problem (1) , (2), by using a guiding function, adequate to this problem.

2 General hypothesis

We begin this section with the notations and general hypotheses.

Denote by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^N$ and by $|\cdot|$ the euclidean norm determined by this.

Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function and $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 with the property that

$$\inf \{g(t), \ t \in \mathbb{R}\} \ge 1.$$

Obviously, one can consider that the minimum of the function g on $\mathbb{R}$ is an arbitrary number $a > 0$, but the case $a = 1$ does not constitute a restriction.

Let us consider a continuous function $V : \mathbb{R}^N \to \mathbb{R}$ which satisfies the following conditions:

V$_1$) $\lim\limits_{|x|\to\infty} V(x) = \infty$;

V$_2$) V is of class C^1 on the set $\{x,\ \ |x| \geq r\}$, where $r > 0$ is a real number.

Set

$$V_0 := \sup\{V(x)\,,\ \ |x| \leq r\}\,.$$

By condition V$_1$) it results

$$(\exists)\ \ k \geq r,\ \ V(x) > V_0,\ \ |x| \geq k. \tag{3}$$

Denote by $(\mathrm{div}\ V)(x)$ the divergence of V in x; the divergence is defined for $|x| > r$.

Definition 1. *We call a guiding function for* (1) *along the function g, the following expression:*

$$V_g(x,t) := \langle(\mathrm{div}\ V)(g(t)x)\,,\dot{g}(t)x + gf(t,x)\rangle\,, \tag{4}$$

where $\dot{g}$ denoted the differential of g with respect to t.

An easy calculus shows us that if $x(\cdot)$ is solution for (1), then

$$V_g(x(t)\,,t) = \frac{d}{dt}V(g(t)x(t))\,, \tag{5}$$

for every t for which $|x(t)| \geq r$. Remark that if $|x(t)| \geq r$, then $|g(t)x(t)| \geq r$ and so the equality (4) has sense.

Consider the space

$$C_c := \left\{x : \mathbb{R} \to \mathbb{R}^N,\ \ x \text{ continuous}\right\}$$

endowed with the family of seminorms

$$|x|_n := \sup_{t\in[-n,n]}\{|x(t)|\}\,,\ \ n \geq 1.$$

The topology determined by this family of seminorms is the topology of uniform convergence on each compact of $\mathbb{R}$. Recall that the compactity in C_c is characterized by the Ascoli-Arzelà theorem; more precisely, a family of functions from C_c is relatively compact if and only if it is equi-continuous and uniformly bounded on each compact of $\mathbb{R}$.

3 The main result

The main result of this note is contained in the following theorem.

Theorem 1. *Suppose that*

$$V_g(t,x) \leq 0,\ t \in \mathbb{R},\ |x| \geq r. \tag{6}$$

Then, the equation (1) *admits at least one solution fulfilling the condition*

$$|x(t)| \leq k \cdot \frac{1}{g(t)},\ t \in \mathbb{R}. \tag{7}$$

Proof. The proof is partially inspired by the work [1].

Let $t_0 \in \mathbb{R}$ be arbitrary and let $x(\cdot)$ be a solution of the equation (1) satisfying

$$x(t_0) = 0. \tag{8}$$

The mapping $t \to |g(t)\, x(t)|$ being continuous, it follows that

$$(\exists)\ t_1 > t_0,\ |g(t)\, x(t)| < r \leq k,\ t \in [t_0, t_1). \tag{9}$$

If $t_1 = +\infty$, then

$$|x(t)| \leq k \cdot \frac{1}{g(t)},\ t \in [t_0, +\infty) \tag{10}$$

and the inequality (10) assures us that the solution $x(\cdot)$ is defined on the whole interval $[t_0, \infty)$.

If $t_1 < \infty$, then denoting by T the right extremity of the maximal interval of existence of the solution $x(\cdot)$, we have

$$\lim_{t \nearrow T} |x(t)| = +\infty$$

and therefore,

$$\lim_{t \nearrow T} |g(t)\, x(t)| = +\infty.$$

Hence, there exists $\tau \in [t_0, T)$, such that

$$|g(\tau)\, x(\tau)| \leq r.$$

Set

$$t_2 := \sup \{\tau \in [t_0, T),\ |g(t)\, x(t)| \leq r,\ t \in [t_0, \tau)\}.$$

It follows that

$$|g(t)\,x(t)| \leq r < k,\ t \in [t_0, t_2), \tag{11}$$

$$|g(t_2)\,x(t_2)| = r. \tag{12}$$

So,

$$V(g(t_2)\,x(t_2)) \leq V_0. \tag{13}$$

We want to prove that

$$|g(t)\,x(t)| \leq k,\ t \in [t_0, T). \tag{14}$$

Let us admit, by means of contradiction, that (14) does not hold. Then,

$$(\exists)\ \ t_3 > t_2,\ \ |g(t_3)\,x(t_3)| > k. \tag{15}$$

By (11) and (15) it results that there exists $t_4 > t_0$, such that

$$|g(t_4)\,x(t_4)| = k \tag{16}$$

and

$$r < |g(t)\,x(t)| < k,\ t \in [t_2, t_4]\,. \tag{17}$$

By hypothesis (6) we get

$$\frac{d}{dt}V(g(t)\,x(t)) \leq 0,\ t \in [t_2, t_4]$$

and therefore the function $V(g(t)\,x(t))$ is decreasing on $[t_2, t_4]$; from (3), (13), (16), it follows that

$$V_0 < V(g(t_4)\,x(t_4)) \leq V(g(t_2)\,x(t_2)) \leq V_0.$$

The obtained contradiction proves that the inequality (14) is true; but, then it follows that $T = +\infty$ since else we have

$$\lim_{t \nearrow T} |x(t)| = +\infty.$$

We obtain that for each $t_0 \in \mathbb{R}$, there exists a solution $x(\cdot)$ of the equation (1) which fulfills the initial condition $x(t_0) = 0$ and for which we have

$$|x(t)| \leq k \cdot \frac{1}{g(t)},\ t \geq t_0. \tag{18}$$

In particular, we take $t_0 = -n$ and denote by $x_n(\cdot)$ the solution of the equation (1), fulfilling the conditions

$$x_n(-n) = 0, \ \ |x_n(t)| \le k \cdot \frac{1}{g(t)}, \ t \ge -n. \tag{19}$$

Prolong at left of $-n$ the solution $x_n(\cdot)$, by setting

$$x_n(-t) = 0, \ t \le -n.$$

We get a sequence $(x_n)_n \subset C_c$, which is relatively compact.

Indeed, let $[-a, a] \subset \mathbb{R}$ be a compact arbitrary and let $n \ge a$; we have then

$$|x_n(t)| \le k \cdot \frac{1}{g(t)} \le k, t \in [-a, a], \ n \ge a. \tag{20}$$

Set

$$M(a) := \sup\{|f(t, x)|, \ t \in [-a, a], \ |x| \le k\}.$$

Since

$$\dot{x}_n(t) = f(t, x_n(t)), \ t \in [-a, a], \tag{21}$$

it results that

$$|x_n(t') - x_n(t'')| \le M(a)|t' - t''|, \ n \ge a, \ t', \ t'' \in [-a, a].$$

The sequence $x_n(\cdot)$ is relatively compact on $[-a, a]$ and since a is arbitrary, $x_n(\cdot)$ is relatively compact in C_c. One can suppose without loss of generality that $x_n(\cdot)$ converges in C_c at $x(\cdot)$. But then, by (24), it follows that $x(\cdot)$ is solution for (1) on every compact of $\mathbb{R}$, so on $\mathbb{R}$. On the other hand, from (20) it results that

$$|x(t)| \le k \cdot \frac{1}{g(t)}, \ t \in [-a, a] \tag{22}$$

and since a is arbitrary, it results that

$$|x(t)| \le k \cdot \frac{1}{g(t)}, \ t \in \mathbb{R}, \tag{23}$$

which ends the proof. $\square$

4 Final remarks

For $g \equiv 1$, the condition (6) becomes

$$\langle (\text{div } V)(x), f(t,x) \rangle \leq 0, \ t \in \mathbb{R}, |x| \geq r, \tag{24}$$

which deals us to a known result of Krasnoselskii, regarding the bounded solutions (see [14] or [17], Lemma 7).

One of the easiest choice for the function V is

$$V(x) = |x|;$$

in this case the condition (6) is satisfied if

$$|x|^2 \dot{g}(t) + g(t) \langle x, f(t,x) \rangle \leq 0, \ t \in \mathbb{R}, \ |x| \geq r. \tag{25}$$

Remark that the same condition is obtained if we take

$$V(x) = \sum_{i=1}^{N} x_i^2.$$

Setting $g(t) = 1 + t^2$, the condition (24) becomes

$$\langle x, f(t,x) \rangle t^2 + 2t|x|^2 + \langle x, f(t,x) \rangle \leq 0.$$

This last inequality will be fulfilled if

$$\langle f(t,x), x \rangle \leq -|x|^2. \tag{26}$$

For example, if $f = (f_i)_{i \in \overline{1,N}}$ and $f_i(t,x) = \varphi_i(t,x) x_i + \psi_i(t,x)$, where

$$\varphi_i(t,x) \leq -1, \ x_i \psi_i(t,x) \leq 0,$$

then (26) is fulfilled.

Remark that, by writing (26) under the form

$$\langle f(t,x) + x, x \rangle \leq 0,$$

we obtain (24), where $V(x) = |x|$ and instead of f is $f(t,x) + x$; in this way, the condition (26) ensures the existence of a bounded solution for the equation

$$\dot{x} = x + f(t,x).$$

Another possible choice for g is

$$g(t) = \exp\left(\frac{c^2 t^2}{2}\right),$$

when (25) becomes

$$\langle x, f(t,x) \rangle \leq -c^2 t |x|^2.$$

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