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On Some Elliptic Problems with Nonlocal Boundary Coefficient-operator Conditions in the Framework of Hölderian Spaces

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Abstract

In this paper we give some new results on second order differential-operator equations of elliptic type with nonregular boundary conditions with coefficient-operator. The study is developped in Hölder spaces and uses the reduction method of S. G. Krein. Necessary and sufficient conditions of compatibility are established to obtain different types of solutions. Maximal regularity properties are also studied.

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1 Introduction and hypotheses

Many authors have studied nonlocal boundary value problems: we can first refer to the pioneering works by T. Carleman [4] and J. D. Tamarkin [19], see also A. V. Bitsadze and A. A Samarskii [3] who introduce some nonlocal boundary conditions, to study elliptic problems coming from plasma theory. The case of a non linear elliptic equation with a nonlocal boundary condition has been treated by Y. Wang [22]. More bibliographic details on nonlocal elliptic problems can be found in the monograph of A. L. Skubachevskii [18]. Such nonlocal problems have been also considered in the framework of elliptic differential-operator equations, studying coerciveness and Fredholmness, see S. Yakubov [20] and also more recently A. Favini and Y. Yakubov [10],[11], B. A. Aliev and S. Yakubov [1].

In this work we consider the following second order differential-operator problem:

$$\begin{cases} u''(x) + Au(x) = f(x), \quad x \in [0,1[\\ u(0) = u_0 \\ u(1) + Hu'(0) = u_{1,0}, \end{cases} \tag{1}$$

where X is a complex Banach space, $f \in C^\theta([0,1];X)$ with $0 < \theta < 1$, $u_0, u_{1,0}$ are given elements of X, A is a closed linear operator with domain $D(A)$ not necessarily dense in X and H is a closed linear operator with domain $D(H)$. Recall that, for any interval J

$$C^\theta(J;X) = \left\{ h : J \longrightarrow X, \sup_{x,y\in J, x\neq y} \frac{\|h(x)-h(y)\|}{|x-y|^\theta} < +\infty \right\}.$$

Our main assumptions on the two operators A and H are

$$[0,+\infty[\subset \rho(A) \text{ and } \sup_{\lambda\geq 0} \left\|\lambda (A-\lambda I)^{-1}\right\|_{\mathcal{L}(X)} < +\infty; \tag{2}$$

this assumption implies that $Q = -(-A)^{\frac{1}{2}}$, is the infinitesimal generator of a generalized analytic semigroup on X, see for instance Balakrishnan [2]

for densely defined operators and C. Martinez and M. Sanz [16] otherwise.

$$D(Q) \subset D(H), \tag{3}$$

$$\forall \zeta \in D(H) : A^{-1}H\zeta = HA^{-1}\zeta, \tag{4}$$

$$0 \in \rho\left(\Lambda\right), \tag{5}$$

where $\Lambda = -2HQe^{Q} + I - e^{2Q}$ which is well defined on X and belongs to $\mathcal{L}(X)$, due to *(2)-(3)*. We will see that this operator Λ is in some sense the "determinant" of Problem (1).

Remark 1

1. *Under (2)∼(4) one has, for any* $\zeta \in D(H), \lambda \in \rho(A), \mu \in \rho(Q)$ *and* $x \geq 0$
$$\begin{cases} \left(\lambda I - A\right)^{-1} H\zeta = H\left(\lambda I - A\right)^{-1}\zeta \\ \left(\mu I - Q\right)^{-1} H\zeta = H\left(\mu I - Q\right)^{-1}\zeta \\ He^{xQ}\xi = e^{xQ}H\xi. \end{cases}$$

2. *Due to (2), there exists* $\varepsilon_A > 0$, $\beta_A \in \left]0, \frac{\pi}{2}\right[$ *such that* $\rho\left(A\right)$ *contains a sectorial domain*
$$S_{\varepsilon_A,\beta_A} = \left\{z \in \mathbb{C}\setminus\left\{0\right\} : \left|\arg\left(z\right)\right| < \beta_A\right\} \cup B\left(0, \varepsilon_A\right),$$
satisfying
$$\exists M_{\beta_A} > 0 : \forall z \in S_{\varepsilon_A,\beta_A}, \quad \left\|\left(A - zI\right)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M_{\beta_A}}{1 + |z|}.$$
Moreover $\rho\left(-A\right) \supset \left\{z \in \mathbb{C}\setminus\left\{0\right\} : \left|\arg\left(z\right)\right| > \pi - \beta_A\right\}$, *thus, we obtain*
$$\rho\left(\left(-A\right)^{\frac{1}{2}}\right) \supset \left\{z \in \mathbb{C}\setminus\left\{0\right\} : \left|\arg\left(z\right)\right| > \frac{\pi - \beta_A}{2}\right\},$$
and setting $\beta_Q = \dfrac{\pi + \beta_A}{2}$ *we get*
$$\rho\left(Q\right) \supset \left\{z \in \mathbb{C}\setminus\left\{0\right\} : \left|\arg\left(z\right)\right| < \beta_Q\right\}.$$

We adapt to our situation the definitions of strict and classical solutions given by E. Sinestrari in [17], Section 2, p. 34:

We first notice that here $D(A)$ is endowed with the graph norm that is

$$\left\|\phi\right\|_{D(A)} = \left\|\phi\right\|_X + \left\|A\phi\right\|_X, \ \phi \in D(A),$$

and then for an interval J, we define $C(J; D(A))$ in the following manner

$$h \in C(J; D(A)) \Longleftrightarrow \left(\begin{array}{l} h \in C(J; X), \\ h(x) \in D(A) \text{ for any } x \in J \\ Ah \in C(J; X), \end{array} \right).$$

For example if $\phi \in \overline{D(Q)} \backslash D(Q)$ then $h = e^{\cdot}\phi$ defined from $[0, 1]$ to X then

$$h \in C([0, 1]; X) \cap C^{\infty}([0, 1[; X) \cap C([0, 1[; D(A)),$$

(see Proposition 2 below).

- a strict solution u of problem (1) is a function u such that

$$C^2([0, 1]; X) \cap C([0, 1]; D(A)),$$

and which satisfies (1). This strict solution satisfies the maximal regularity property if

$$u'', Au \in C^{\theta}([0, 1]; X). \tag{6}$$

When $H = 0$, which means that we consider Dirichlet boundary conditions, it is known that, under assumption (2), problem (1) has a strict solution u if and only if $u_0, u_{1,0} \in D(A)$ and

$$f(0) - Au_0, \ f(1) - Au_{1,0} \in \overline{D(A)}.$$

Moreover u has the maximal regularity property if and only if $u_0, u_{1,0} \in D(A)$ and $f(0) - Au_0, \ f(1) - Au_{1,0} \in D_A(\theta/2, +\infty)$, see R. Labbas [13].

When $H \neq 0$, the nonregular boundary condition $u(1) + Hu'(0) = u_{1,0}$, involves in general, a loss of regularity for the solution u at point 1, but we must also take into account the fact that this nonregular boundary condition make sense if u is continuous at 1, with u' continuous at 0. This leads us to introduce new types of solutions of problem (1):

- a semiclassical solution of problem (1) is a function u such that

$$u \in C([0, 1]; X) \cap C^2([0, 1[; X) \cap C([0, 1[; D(A)),$$

and which satisfies (1); moreover we say that this semiclassical solution satisfies the maximal regularity property if

$$\left\{ \begin{array}{l} u \in C^{\theta}([0, 1]; X) \text{ and} \\ u'', Au \in C^{\theta}([0, b]; X) \text{ for any } b \in]0, 1[. \end{array} \right. \tag{7}$$

It is well known that any $h \in C^{\theta}([0,1[;X)$ can be extended in a function $\widetilde{h} \in C^{\theta}([0,1];X)$, so $C^{\theta}([0,1[;X) = C^{\theta}([0,1];X)$, this explain the introduction of the spaces $C^{\theta}([0,b];X)$, $b \in]0,1[$.

- a semistrict solution of problem (1) is a semiclassical solution of problem (1) satisfying moreover $u \in C^1([0,1],X) \cap C\left([0,1], D((-A)^{\frac{1}{2}})\right)$. We will say this semistrict solution satisfies the maximal regularity property if it satisfies (7) together with

$$u', (-A)^{\frac{1}{2}} u \in C^{\theta}([0,1],X). \tag{8}$$

Note that a particular case of Problem (1), that is $H = \alpha I$, has been studied by Labbas-Maingot (see [14]). These authors used a direct method based on the techniques of Dunford integrals to build a representation formula of the solution.

In this work, a representation formula of problem (1) is found by using analytic semigroups and fractional operators theory.

This work is organized as follows:

Section 2 is devoted to Problem (1) and contains our main result (Theorem 13): we first recall classical results on generalized analytic semigroup, then, under assumptions (2)$\sim$(5), we build a representation formula for the solution of (1) and study the regularity of this representation. Finally we consider some particular cases in which our invertibility assumption (5) is satisfied.

In Section 3 we introduce a spectral parameter $\omega \geq 0$ which allows us to apply the results of section 2.

In section 4, a concrete problem is considered to illustrate our results.

2 Study of Problem (1)

2.1 Generalized analytic semigroup

As in [9], section 2 pp. 975-977, we recall here the definition of a generalized analytic semigroup (see E. Sinestrari [17], A. Lunardi [15]) and some classical results (see [6], [7] and [17]).

Let L be a linear operator in X such that

$$\left\{\begin{array}{l}\rho(L) \supset S_{\mu,\delta} = \left\{\lambda \in \mathbb{C}\backslash \left\{\mu\right\} \ / \ |\arg(\lambda-\mu)| < \frac{\pi}{2}+\delta\right\} \text{ and} \\ \sup\limits_{\lambda \in S_{\mu,\delta}} \|(\lambda-\mu)\left(\lambda I - L\right)^{-1}\|_{\mathcal{L}(X)} < +\infty,\end{array}\right.$$

for some given $\mu \in \mathbb{R}$ and $\delta \in \left]0, \frac{\pi}{2}\right[$. This says exactly that L is the infinitesimal generator of a generalized analytic semigroup $\left(e^{xL}\right)_{x\geq 0}$, "generalized" in the sense that L is not supposed to be densely defined.

Proposition 2 *Let L is the infinitesimal generator of a generalized analytic semigroup $\left(e^{xL}\right)_{x\geq 0}$.*

1. *Let $\varphi \in X$. Then the two following assertions are equivalent*
   (a) $e^{\cdot L}\varphi \in C\left([0,1];X\right)$.
   (b) $\varphi \in \overline{D\left(L\right)}$.
2. *Let $\theta \in \left]0,1\right[, g \in C^{\theta}\left([0,1];X\right), \varphi \in X$. Set*

   $$S(x) = e^{xL}\varphi + \int_0^x e^{(x-s)L} g\left(s\right) ds, \quad x \in \left[0,1\right].$$

   Then the two following assertions are equivalent
   (a) $S \in C^1\left([0,1];X\right) \cap C\left([0,1];D\left(L\right)\right)$.
   (b) $\varphi \in D\left(L\right)$ and $g\left(0\right) + L\varphi \in \overline{D\left(L\right)}$.

Let us recall that for an operator P in X satisfying $\rho(P) \supset \left]0,+\infty\right[$ and

$$\exists C > 0, \forall \lambda > 0, \quad \left\|\left(P - \lambda I\right)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\lambda},$$

we define the interpolation space $D_P\left(\theta,+\infty\right)$ by

$$D_P\left(\theta,+\infty\right) = \left\{x \in X : \sup_{t>0}\left\|t^{\theta}P\left(P - tI\right)^{-1}x\right\| < +\infty\right\}.$$

Proposition 3 *Let $\theta \in \left]0,1\right[$ and L be the infinitesimal generator of a generalized analytic semigroup $\left(e^{xL}\right)_{x\geq 0}$.*

1. *Then the two following assertions are equivalent*
   (a) $e^{\cdot L}\varphi \in C^{\theta}([0,1];X)$.
   (b) $\varphi \in D_L(\theta,+\infty)$.
2. *Let* $g \in C([0,1];X)$ *and* $\varphi \in X$. *Set*

$$S(x) = e^{xL}\varphi + \int_0^x e^{(x-s)L} g(s)\,ds, \quad x \in [0,1].$$

   Then the two following assertions are equivalent
   (a) $S \in C^{1,\theta}([0,1];X) \cap C^{\theta}([0,1];D(L))$.
   (b) $g \in C^{\theta}([0,1];X)$, $\varphi \in D(L)$ *and* $g(0) + L\varphi \in D_L(\theta,+\infty)$.
3. *Let* $g \in C^{\theta}([0,1];X)$. *Then*

$$L\int_0^1 e^{sL}(g(s) - g(0))\,ds \in D_L(\theta,+\infty).$$

For these two propositions see, for instance, E. Sinestrari [17].

Notation 4 *Let* g *and* h *be two given* X*-valued functions defined on* $[0,1]$ *and* $\theta \in]0,1[$. *We write* $g \simeq_\theta h$ *if* $g - h \in C^{\theta}([0,1];X)$.

As a consequence of Proposition 3 we get (see [9] Proposition 8, p. 976):

Proposition 5 *Let* $g \in C^{\theta}([0,1];X), \varphi \in D(L)$ *and set*

$$S(x) = e^{xL}\varphi + \int_0^x e^{(x-s)L} g(s)\,ds, \; x \in [0,1];$$

then

$$LS(\cdot) \simeq_\theta e^{\cdot L}(L\varphi + g(0)).$$

2.2 Representation of the solution

We assume (2)∼(5) and suppose that u is a semiclassical solution of problem (1). Note that, since $u \in C\left([0,1[;D(A)\right)$ we have $u_0 = u(0) \in D(A)$. In the following we assume moreover that $u_{1,0} \in D(A)$.

Lemma 6 *One has*

$$\begin{aligned} u\left(x\right) &= e^{xQ}e^{Q}\varphi_0 + e^{(1-x)Q}e^{Q}\varphi_1 \\ &\quad + e^{xQ}\left(u_0 - J_0\right) + I_x \\ &\quad + e^{(1-x)Q}\Lambda^{-1}\left(u_{1,0} - HQu_0 + 2HQJ_0 - I_1\right) + J_x, \end{aligned} \tag{9}$$

where

$$I_x = \frac{1}{2}Q^{-1}\int_0^x e^{(x-s)Q}f\left(s\right)ds \text{ and } J_x = \frac{1}{2}Q^{-1}\int_x^1 e^{(s-x)Q}f\left(s\right)ds, \tag{10}$$

and

$$\left\{\begin{array}{l} \varphi_0 = \Lambda^{-1}HQu_0 - \Lambda^{-1}u_{1,0} - 2\Lambda^{-1}HQJ_0 + e^{Q}\Lambda^{-1}u_0 - e^{Q}\Lambda^{-1}J_0 + \Lambda^{-1}I_1 \\ \varphi_1 = \Lambda^{-1}J_0 - \Lambda^{-1}u_0. \end{array}\right.$$

Proof. As in [5] (see also S. Yakubov and Y Yakubov [21]), we immediately deduce that u has the representation

$$u(x) = e^{xQ}\xi_0 + e^{(1-x)Q}\xi_1 + I_x + J_x, \quad x \in [0,1] \tag{11}$$

where $\xi_0, \xi_1 \in X$ and I_x, J_x satisfy (10).

To obtain the final representation of u, it is enough to find ξ_0 and ξ_1 by taking into account the data $u_0, u_{1,0}$, f and A. A formal computation gives

$$\begin{aligned} \xi_0 &= u_0 - J_0 - e^{Q}\Lambda^{-1}u_{1,0} + e^{Q}\Lambda^{-1}HQu_0 - 2e^{Q}\Lambda^{-1}HQJ_0 \\ &\quad + e^{Q}\Lambda^{-1}e^{Q}u_0 - e^{Q}\Lambda^{-1}e^{Q}J_0 + e^{Q}\Lambda^{-1}I_1, \end{aligned}$$

and

$$\xi_1 = \Lambda^{-1}u_{1,0} - \Lambda^{-1}HQu_0 + 2\Lambda^{-1}HQJ_0 - \Lambda^{-1}e^{Q}u_0 + \Lambda^{-1}e^{Q}J_0 - \Lambda^{-1}I_1,$$

from which we deduce (9) by using $e^{Q}\Lambda^{-1} = \Lambda^{-1}e^{Q}$ (which is a consequence of (4)). We need to justify the terms HQu_0, HQJ_0 in (9) : $u_0 \in D\left(A\right) = D\left(Q^2\right)$

so $Qu_0 \in D(Q) \subset D(H)$, moreover, using Proposition 3, assertion 3, we can write

$$\begin{aligned} QJ_0 &= \frac{1}{2}\int_0^1 e^{sQ}\left(f(s)-f(0)\right)ds + \frac{1}{2}\int_0^1 e^{sQ} f(0)\,ds \\ &= \frac{1}{2}Q^{-1}\left(Q\int_0^1 e^{sQ}\left(f(s)-f(0)\right)ds + e^{Q}f(0) - f(0)\right), \end{aligned}$$

and thus $QJ_0 \in D(Q) \subset D(H)$. ■

In order to simplify representation (9) we first show the following Lemma.

Lemma 7

1. *There exists $W \in \mathcal{L}(X)$ such that $WQ^{-1} = Q^{-1}W$ and*

$$\Lambda^{-1} = I - W \text{ with } W(X) \subset \bigcap_{k=1}^{+\infty} D\left(Q^k\right).$$

2. *We have*

$$\begin{cases} J_0 = \frac{1}{2}Q^{-1}\int_0^1 e^{sQ}\left(f(s)-f(0)\right)ds + \frac{1}{2}Q^{-2}e^{Q}f(0) - \frac{1}{2}Q^{-2}f(0) \\ I_1 = \frac{1}{2}Q^{-1}\int_0^1 e^{sQ}\left(f(1-s)-f(1)\right)ds + \frac{1}{2}Q^{-2}e^{Q}f(1) - \frac{1}{2}Q^{-2}f(1). \end{cases}$$

Proof. For statement 1 we write

$$\Lambda = -2HQe^{Q} + I - e^{2Q} = I + V,$$

where $V = -2HQe^{Q} - e^{2Q} \in \mathcal{L}(X)$. It is clear that $VQ^{-1} = Q^{-1}V$, moreover since Q generates a generalized analytic semigroup, we have for all $m \in \mathbb{N}$

$$e^{Q} \in L\left(X, D\left(Q^m\right)\right),$$

so

$$V(X) \subset \bigcap_{k=1}^{+\infty} D\left(Q^k\right),$$

thus $W := \Lambda^{-1}V \in \mathcal{L}(X)$, $WQ^{-1} = Q^{-1}W$ and $W(X) \subset \bigcap_{k=1}^{+\infty} D\left(Q^k\right)$.

We conclude by noting that

$$(I-W)\,\Lambda=\Lambda\,(I-W)=\Lambda-V=I.$$

For statement 2, it is enough to remark that for any $g\in C^{\theta}\left([0,1]\,;X\right)$

$$\begin{aligned}\int_0^1 e^{sQ}g\left(s\right)ds &= \int_0^1 e^{sQ}\left(g\left(s\right)-g\left(0\right)\right)ds+\int_0^1 e^{sQ}g\left(0\right)ds\\ &= \int_0^1 e^{sQ}\left(g\left(s\right)-g\left(0\right)\right)ds+e^{Q}Q^{-1}g\left(0\right)-Q^{-1}g\left(0\right).\end{aligned}$$

■ Now, using (9) and Lemma 7, we can write

$$\begin{aligned}u\left(x\right) &= e^{xQ}e^{Q}\varphi_0+e^{(1-x)Q}e^{Q}\varphi_1-\frac{1}{2}e^{xQ}Q^{-2}e^{Q}f\left(0\right)\\ &\quad -\frac{1}{2}e^{(1-x)Q}\Lambda^{-1}Q^{-2}e^{Q}f\left(1\right)\\ &\quad -e^{(1-x)Q}W\left(u_{1,0}-HQu_0+2HQJ_0-I_1\right)\\ &\quad -\frac{1}{2}e^{(1-x)Q}Q^{-2}e^{Q}f\left(1\right)+e^{(1-x)Q}HQ^{-1}e^{Q}f\left(0\right)\\ &\quad +e^{xQ}\left(u_0+\frac{1}{2}Q^{-2}f\left(0\right)\right)+I_x-\frac{1}{2}e^{xQ}Q^{-1}\int_0^1 e^{sQ}\left(f\left(s\right)-f\left(0\right)\right)ds\\ &\quad +e^{(1-x)Q}\left(-HQu_0-HQ^{-1}f\left(0\right)+u_{1,0}+\frac{1}{2}Q^{-2}f\left(1\right)\right)+J_x\\ &\quad +e^{(1-x)Q}\left(H\int_0^1 e^{sQ}\left(f\left(s\right)-f\left(0\right)\right)ds\right)\\ &\quad -\frac{1}{2}e^{(1-x)Q}\left(Q^{-1}\int_0^1 e^{sQ}\left(f\left(1-s\right)-f\left(1\right)\right)ds\right).\end{aligned}$$

Setting for $\psi\in X$ and $g\in C^{\theta}\left([0,1]\,;X\right)$

$$S\left(x,\psi,g\right)=e^{xQ}\psi+\int_0^x e^{(x-s)Q}g\left(s\right)ds,$$

we can rearrange the terms of u to obtain the decomposition

$$u=u_R+v+w, \tag{12}$$

with the regular part u_R in $[0,1]$ given by

$$\begin{aligned} u_R(x) &= e^{xQ}e^{Q}\varphi_0 + e^{(1-x)Q}e^{Q}\varphi_1 - \frac{1}{2}e^{xQ}e^{Q}Q^{-2}f(0) \\ &\quad -\frac{1}{2}e^{(1-x)Q}e^{Q}\Lambda^{-1}Q^{-2}f(1) \\ &\quad -e^{(1-x)Q}W(u_{1,0} - HQu_0 + 2HQJ_0 - I_1) \\ &\quad -\frac{1}{2}e^{(1-x)Q}e^{Q}Q^{-2}f(1) + e^{(1-x)Q}e^{Q}HQ^{-1}f(0), \end{aligned} \tag{13}$$

the terms which gives the behavior near 0

$$\begin{aligned} v(x) &= S\left(x, u_0 + \frac{1}{2}Q^{-2}f(0), \frac{1}{2}Q^{-1}f\right) \\ &\quad -\frac{1}{2}e^{xQ}Q^{-1}\int_0^1 e^{sQ}\left(f(s) - f(0)\right)ds, \end{aligned} \tag{14}$$

and the one concerning the nonlocal behavior in 0 and 1

$$\begin{aligned} & w(x) \\ = \; & S\left(1-x, -HQu_0 - HQ^{-1}f(0) + u_{1,0} + \frac{1}{2}Q^{-2}f(1), \frac{1}{2}Q^{-1}f(1-.)\right) \\ & +e^{(1-x)Q}H\int_0^1 e^{sQ}\left(f(s) - f(0)\right)ds \\ & -\frac{1}{2}e^{(1-x)Q}Q^{-1}\int_0^1 e^{sQ}\left(f(1-s) - f(1)\right)ds, \end{aligned} \tag{15}$$

(note that since $u_0 \in D(A)$ then $HQu_0 = -HQ^{-1}Au_0$ is well defined).

2.3 Regularity results

To study the regularity of the solution we need some technical lemmas. First recall that if $g \in C^{\theta}([0,1], X), \varphi \in X, \varkappa \in D_Q(\theta, +\infty), \psi \in D(Q), \widetilde{\psi} \in D(Q^2)$ then

$$\begin{cases} S(\cdot, \varphi + \varkappa, g) \simeq_{\theta} S(\cdot, \varphi, g) \\ QS(\cdot, \psi + Q^{-1}\varkappa, g) \simeq_{\theta} QS(\cdot, \psi, g) \\ Q^2S\left(\cdot, \widetilde{\psi} + Q^{-2}\varkappa, g\right) \simeq_{\theta} Q^2S\left(\cdot, \widetilde{\psi}, g\right) \end{cases} \tag{16}$$

$$QS(\cdot,\psi,g)\simeq_\theta e^{\cdot Q}(Q\psi+g(0)), \tag{17}$$

$$\mathfrak{I}_g := Q\int_0^1 e^{sQ}(g(s)-g(0))\,ds \in D_Q(\theta,+\infty) \tag{18}$$

and

$$e^{\cdot Q}Q\int_0^1 e^{sQ}(g(s)-g(0))\,ds \in C^\theta([0,1];X), \tag{19}$$

(see Propositions 3 and 5).

Lemma 8 *Assume (2)∼(5). Let* $f\in C^\theta([0,1],X)$ *and* $u_0\in D(A)$*. Then*

1. $u_R, Au_R \in C^\infty([0,1],X)$.
2. $v\in C^2(]0,1],X)\cap C(]0,1],D(A))$.
3. $Av\simeq_\theta e^{\cdot Q}[Au_0-f(0)]$ *and thus*

$$\begin{cases} Av\in C([0,1];X)\Leftrightarrow Au_0-f(0)\in\overline{D(A)} \\ Av\in C^\theta([0,1];X)\Leftrightarrow Au_0-f(0)\in D_A(\theta/2,+\infty). \end{cases}$$

Proof.

1. For any $\varphi\in X$ we have $e^Q\varphi, W\varphi\subset\bigcap_{k=1}^{+\infty}D(Q^k)$ from which we deduce that
$$\begin{cases} e^{\cdot Q}e^Q\varphi, e^{\cdot Q}W\varphi\in C^\infty([0,1],X) \text{ and} \\ -Q^2e^{\cdot Q}e^Q\varphi, -Q^2e^{\cdot Q}W\varphi\in C^\infty([0,1],X), \end{cases}$$
thus $u_R\in C^\infty([0,1],X)$ and $Au_R=-Q^2u_R\in C^\infty([0,1],X)$.

2. Obvious since for $\varphi\in X$ and $x>0$ we have $e^{xQ}\varphi\subset\bigcap_{k=1}^{+\infty}D(Q^k)$.

3. Since $A=-Q^2$, we have
$$Av=QS\left(\cdot,-Qu_0-\frac{1}{2}Q^{-1}f(0),-\frac{1}{2}f\right)+\frac{1}{2}e^{\cdot Q}Q\int_0^1 e^{sQ}(f(s)-f(0))\,ds,$$

then, by (17) and (19) we get

$$\begin{aligned} Av &\simeq_\theta QS\left(\cdot, -Qu_0 - \frac{1}{2}Q^{-1}f(0), \frac{1}{2}f\right) \\ &\simeq_\theta e^{\cdot Q}\left(Q\left(-Qu_0 - \frac{1}{2}Q^{-1}f(0)\right) - \frac{1}{2}f(0)\right) \\ &\simeq_\theta e^{\cdot Q}\left[Au_0 - f(0)\right]. \end{aligned}$$

■

Lemma 9 *Assume (2)∼(5). Let* $f \in C^\theta([0,1], X)$ *and* $u_0, u_{1,0} \in D(A)$. *Then*

1. $w \in C^2([0,1[, X) \cap C([0,1[, D(A))$.
2. $w \simeq_\theta e^{(1-\cdot)Q} HQ^{-1}(Au_0 - f(0))$ *and thus*
$$\begin{cases} w \in C([0,1], X) \Leftrightarrow HQ^{-1}[Au_0 - f(0)] \in \overline{D(A)} \\ w \in C^\theta([0,1], X) \Leftrightarrow HQ^{-1}[Au_0 - f(0)] \in D_A(\theta/2, +\infty). \end{cases}$$
3. $w([0,1]) \subset D(Q) \Longleftrightarrow HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q)$.
4. *Assuming* $HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q)$ *we get*
$$Qw \simeq_\theta e^{(1-\cdot)Q} QHQ^{-1}[Au_0 - f(0) + \Im_f],$$
and thus
$$\begin{cases} w', Qw \in C([0,1], X) \Leftrightarrow QHQ^{-1}[Au_0 - f(0) + \Im_f] \in \overline{D(A)} \\ w', Qw \in C^\theta([0,1], X) \Leftrightarrow QHQ^{-1}[Au_0 - f(0) + \Im_f] \in D_A(\theta/2, +\infty). \end{cases}$$
5. $w([0,1]) \subset D(Q^2) \Longleftrightarrow HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q^2)$.
6. *Assuming* $HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q^2)$ *we get*
$$Q^2 w \simeq_\theta e^{(1-\cdot)Q}\left(Q^2 HQ^{-1}[Au_0 - f(0) + \Im_f] - [Au_{1,0} - f(1)]\right),$$
and thus
$$\begin{cases} Aw \in C([0,1], X) \text{ if and only if} \\ Q^2 HQ^{-1}[Au_0 - f(0) + \Im_f] - [Au_{1,0} - f(1)] \in \overline{D(A)} \\ \\ Aw \in C^\theta([0,1], X) \text{ if and only if} \\ Q^2 HQ^{-1}[Au_0 - f(0) + \Im_f] - [Au_{1,0} - f(1)] \in D_A(\theta/2, +\infty). \end{cases}$$

Proof. Setting $\widetilde{f} = f(1-\cdot)$ and noting that $HQ^{-1} \in \mathcal{L}(X)$ we have

$$w(x) = S\left(1-x, \psi_0, \frac{1}{2}Q^{-1}\widetilde{f}\right),$$

where

$$\begin{aligned} \psi_0 &= HQ^{-1}Au_0 - HQ^{-1}f(0) + u_{1,0} + \frac{1}{2}Q^{-2}f(1) + HQ^{-1}\mathfrak{I}_f - \frac{1}{2}Q^{-2}\mathfrak{I}_{\widetilde{f}}. \\ &= HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] + u_{1,0} + \frac{1}{2}Q^{-2}f(1) - \frac{1}{2}Q^{-2}\mathfrak{I}_{\widetilde{f}}. \end{aligned}$$

1. Obvious, since for $\varphi \in X$ and $x \in [0,1[$ we have $e^{(1-x)Q}\varphi \subset \bigcap_{k=1}^{+\infty} D\left(Q^k\right)$.

2. Due to (18), $\psi_0 = HQ^{-1}\left[Au_0 - f(0)\right] + \varkappa_0$ with $\varkappa_0 \in D_Q(\theta, +\infty)$. So, From (16), we get

$$\begin{aligned} w &\simeq_\theta S\left(1-\cdot, HQ^{-1}\left[Au_0 - f(0)\right], \frac{1}{2}Q^{-1}\widetilde{f}\right) \\ &\simeq_\theta QS\left(1-\cdot, Q^{-1}HQ^{-1}\left[Au_0 - f(0)\right], \frac{1}{2}Q^{-2}\widetilde{f}\right), \end{aligned}$$

   which gives, in virtue of (17)

$$\begin{aligned} w &\simeq_\theta e^{(1-\cdot)Q}\left(HQ^{-1}\left[Au_0 - f(0)\right] + \frac{1}{2}Q^{-2}\widetilde{f}(0)\right). \\ &\simeq_\theta e^{(1-\cdot)Q}HQ^{-1}\left[Au_0 - f(0)\right]. \end{aligned}$$

3. $w([0,1[) \subset D(Q)$. Moreover $w(1) \in D(Q)$ if and only if

$$\psi_0 = HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] + u_{1,0} + \frac{1}{2}Q^{-2}f(1) - \frac{1}{2}Q^{-2}\mathfrak{I}_{\widetilde{f}} \in D(Q),$$

   so

$$w(1) \in D(Q) \quad \Leftrightarrow \quad HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] \in D(Q).$$

4. From (16), (17) and (19), we get

$$\begin{aligned} Qw &\simeq_\theta QS\left(1-\cdot, HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right], \frac{1}{2}Q^{-1}\widetilde{f}\right) \\ &\simeq_\theta e^{(1-\cdot)Q}\left(QHQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] + \frac{1}{2}Q^{-1}\widetilde{f}(0)\right) \\ &\simeq_\theta e^{(1-\cdot)Q}QHQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right], \end{aligned} \tag{20}$$

Morerover for $g \in C^{\theta}([0,1], X), \psi \in D(Q)$ we have

$$S'(\cdot, \psi, g) = QS(\cdot, \psi, g) + g,$$

so here

$$\begin{cases} Qw \in C([0,1], X) \Leftrightarrow w' \in C([0,1], X) \\ Qw \in C^{\theta}([0,1], X) \Leftrightarrow w' \in C^{\theta}([0,1], X). \end{cases}$$

Then (20) furnishes the desired equivalences.

5. See statement 3.

6. From (16), (17) and (19), we get

$$\begin{aligned} Q^2 w &\simeq_{\theta} Q^2 S\left(1-\cdot, \psi_0, \frac{1}{2} Q^{-1} \widetilde{f}\right) \\ &\simeq_{\theta} QS\left(1-\cdot, Q\psi_0, \frac{1}{2} \widetilde{f}\right) \\ &\simeq_{\theta} e^{(1-\cdot)Q}\left(Q^2 H Q^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] - \left[Au_{1,0} - f(1)\right]\right), \end{aligned}$$

since

$$\begin{aligned} & Q(Q\psi_0) + \frac{1}{2}\widetilde{f}(0) \\ = \;& Q^2 H Q^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] \\ & + Q^2 u_{1,0} + \frac{1}{2} f(1) - \frac{1}{2}\mathfrak{I}_{\widetilde{f}} + \frac{1}{2}\widetilde{f}(0) \\ = \;& Q^2 H Q^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] - \left[Au_{1,0} - f(1)\right] - \frac{1}{2}\mathfrak{I}_{\widetilde{f}}. \end{aligned}$$

■

Lemma 10 *Assume (2)∼(5). Let* $f \in C^{\theta}([0,1], X)$ *and* $u_0, u_{1,0} \in D(A)$.

1. *If* $H \in \mathcal{L}(X)$ *then* $Qw \simeq_{\theta} e^{(1-\cdot)Q} H\left[Au_0 - f(0)\right]$.

2. *If* $H \in \mathcal{L}(X)$ *with* $H(X) \subset D(Q)$ *then* $QH \in \mathcal{L}(X)$ *and*

$$Q^2 w \simeq_{\theta} e^{(1-\cdot)Q}\left(QH\left[Au_0 - f(0)\right] - \left[Au_{1,0} - f(1)\right]\right).$$

Proof.

1. Here $HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] = Q^{-1}H\left[Au_0 - f(0) + \mathfrak{I}_f\right] \in D(Q)$ then, from Lemma 9, statement 4, we deduce

$$\begin{array}{rcl} Qw & \simeq_\theta & e^{(1-\cdot)Q}QHQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] \\ & \simeq_\theta & e^{(1-\cdot)Q}H\left[Au_0 - f(0) + \mathfrak{I}_f\right], \end{array}$$

but, from (18), $e^{(1-\cdot)Q}H\mathfrak{I}_f = He^{(1-\cdot)Q}\mathfrak{I}_f \in C^\theta([0,1];X)$ which gives the result.

2. Since if $QH \in \mathcal{L}(X)$ then

$$HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] = Q^{-2}QH\left[Au_0 - f(0) + \mathfrak{I}_f\right] \in D(Q^2),$$

and from Lemma 9, statement 6, we deduce

$$\begin{array}{rcl} Q^2w & \simeq_\theta & e^{(1-\cdot)Q}\left(Q^2HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] - \left[Au_{1,0} - f(1)\right]\right) \\ & \simeq_\theta & e^{(1-\cdot)Q}\left(QH\left[Au_0 - f(0) + \mathfrak{I}_f\right] - \left[Au_{1,0} - f(1)\right]\right) \end{array}$$

but, from (18), $e^{(1-\cdot)Q}QH\mathfrak{I}_f = QHe^{(1-\cdot)Q}\mathfrak{I}_f \in C^\theta([0,1];X)$ which gives the result.

■

These two last cases correspond, for example, to operators $H = \alpha I$ and $H = -\alpha Q^{-1}$ $(\alpha \in \mathbb{C}\setminus\{0\}, \operatorname{Re}\alpha \geq 0)$ which are studied in subsection 2.5.

Lemma 11 *Assume (2)∼(5) and let $u_0, u_{1,0} \in D(A)$.*

1. *If $f \in C^\theta([0,1], D(Q))$ then*

$$w([0,1]) \subset D(Q) \Longleftrightarrow HQ^{-1}Au_0 \in D(Q),$$

and when $HQ^{-1}Au_0 \in D(Q)$ we have

$$Qw \simeq_\theta e^{(1-\cdot)Q}QHQ^{-1}\left[Au_0 - f(0)\right].$$

2. *If $f \in C^\theta([0,1], D(Q^2))$ then*

$$w([0,1]) \subset D(Q^2) \Longleftrightarrow HQ^{-1}Au_0 \in D(Q^2),$$

and when $HQ^{-1}Au_0 \in D(Q^2)$ we have

$$Q^2w \simeq_\theta e^{(1-\cdot)Q}\left(Q^2HQ^{-1}\left[Au_0 - f(0)\right] - \left[Au_{1,0} - f(1)\right]\right).$$

Proof.

1. Here $f(0) \in D(Q)$ and
$$Q\mathfrak{I}_f = Q\int_0^1 e^{sQ}\left(Qf(s) - Qf(0)\right)ds \in D_Q(\theta,+\infty),$$
then, from Lemma 9, statement 3, we get
$$\begin{aligned} w([0,1]) \subset D(Q) &\Leftrightarrow HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] \in D(Q) \\ &\Leftrightarrow HQ^{-1}Au_0 - Q^{-1}HQ^{-1}\left[Qf(0) + Q\mathfrak{I}_f\right] \in D(Q) \\ &\Leftrightarrow HQ^{-1}Au_0 \in D(Q). \end{aligned}$$
Now when $HQ^{-1}Au_0 \in D(Q)$, Lemma 9, statement 4, furnish
$$Qw \simeq_\theta e^{(1-\cdot)Q}QHQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right],$$
and we conclude noting that
$$e^{(1-\cdot)Q}QHQ^{-1}\mathfrak{I}_f = HQ^{-1}e^{(1-\cdot)Q}Q\mathfrak{I}_f \in C^\theta([0,1];X).$$

2. Here $f(0) \in D(Q^2)$ and
$$Q^2\mathfrak{I}_f = Q\int_0^1 e^{sQ}\left(Q^2f(s) - Q^2f(0)\right)ds \in D_Q(\theta,+\infty),$$
then, from Lemma 9, statement 5, we get
$$\begin{aligned} w([0,1]) \subset D(Q^2) &\Leftrightarrow HQ^{-1}\left[Au_0 - f(0) + \mathfrak{I}_f\right] \in D(Q^2) \\ &\Leftrightarrow HQ^{-1}Au_0 \in D(Q^2). \end{aligned}$$
Now, when $HQ^{-1}Au_0 \in D(Q^2)$, Lemma 9, statement 6, furnish
$$Q^2w \simeq_\theta e^{(1-\cdot)Q}\left(Q^2HQ^{-1}\left[Au_0 - f(0)\right] + Q^2HQ^{-1}\mathfrak{I}_f - \left[Au_{1,0} - f(1)\right]\right),$$
and we conclude noting that
$$e^{(1-\cdot)Q}Q^2HQ^{-1}\mathfrak{I}_f = HQ^{-1}e^{(1-\cdot)Q}Q^2\mathfrak{I}_f \in C^\theta([0,1];X).$$

■

By similar arguments, we can also prove the following Lemma.

Lemma 12 *Assume (2)∼(5) and* $u_0, u_{10} \in D(A)$. *If* $H \in \mathcal{L}(X)$ *and* $f \in C^\theta([0,1], D(Q))$ *then*
$$w([0,1]) \subset D(Q^2) \Longleftrightarrow HAu_0 \in D(Q),$$
and when $HAu_0 \in D(Q)$ *we have*
$$Q^2w \simeq_\theta e^{(1-\cdot)Q}\left(QH\left[Au_0 - f(0)\right] - \left[Au_{1,0} - f(1)\right]\right).$$

2.4 Main results

Theorem 13 *Assume (2)∼(5), suppose that* $u_0, u_{1,0} \in D(A)$ *and*

$$f \in C^{\theta}([0,1], X) \ \text{ with } \theta \in]0,1[.$$

Then:

1. *there exists a semiclassical solution* u *of problem (1) if and only if*

$$Au_0 - f(0) \in \overline{D(A)},$$

2. *there exists a semiclassical solution* u *of problem (1) having the maximal regularity property (7) if and only if*

$$Au_0 - f(0) \in D_A(\theta/2, +\infty),$$

3. *there exists a semistrict solution* u *of problem (1) if and only if*

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)} \\ HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q) \ \text{and} \\ QHQ^{-1}[Au_0 - f(0) + \Im_f] \in \overline{D(A)}, \end{cases}$$

4. *there exists a semistrict solution* u *of problem (1) having the maximal regularity property (7)-(8) if and only if*

$$\begin{cases} Au_0 - f(0) \in D_A(\theta/2, +\infty) \\ HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(Q) \ \text{and} \\ QHQ^{-1}[Au_0 - f(0) + \Im_f] \in D_A(\theta/2, +\infty), \end{cases}$$

5. *there exists a strict solution* u *of problem (1) if and only if*

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)} \\ HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(A) \ \text{and} \\ Q^2HQ^{-1}[Au_0 - f(0) + \Im_f] - [Au_{1,0} - f(1)] \in \overline{D(A)}. \end{cases}$$

6. *there exists a strict solution* u *of problem (1) having the maximal regularity property (6) if and only if*

$$\begin{cases} Au_0 - f(0) \in D_A(\theta/2, +\infty) \\ HQ^{-1}[Au_0 - f(0) + \Im_f] \in D(A) \ \text{and} \\ Q^2HQ^{-1}[Au_0 - f(0) + \Im_f] - [Au_{1,0} - f(1)] \in D_A(\theta/2, +\infty). \end{cases}$$

Moreover, in the 6 cases u is unique and given by $u = u_R + v + w$ where u_R, v, w are defined in (13),(14) and (15).

Proof. For statements 1 and 2, we first remark that, from subsection 2.2, if there is a semiclassical solution u of problem (1) then u is uniquely determined by $u = u_R + v + w$. We conclude by applying Lemmas 8 and 9 and noting that, since $u'' + Au = f$, then

$$\begin{cases} Au \in C([0,1], X) \Leftrightarrow u'' \in C([0,1], X) \\ Au \in C^{\theta}([0,1], X) \Leftrightarrow u'' \in C^{\theta}([0,1], X). \end{cases}$$

Statements 3∼6 are similarly proved. ■

We now study some situations where more regularity is given on H or f which allow us to drop the conditions on $\mathfrak{I}_f$.

Corollary 14 *Assume (2)∼(5). Let $f \in C^{\theta}([0,1], X)$ and $u_0, u_{1,0} \in D(A)$.*

1. *Suppose that $H \in \mathcal{L}(X)$ then: there exists a semistrict solution u of problem (1) if and only if*

$$Au_0 - f(0) \in \overline{D(A)}.$$

2. *Suppose that $H \in \mathcal{L}(X)$ with $H(X) \subset D(Q)$ then: there exists a strict solution u of problem (1) if and only if*

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)} \text{ and} \\ QH[Au_0 - f(0)] - [Au_{1,0} - f(1)] \in \overline{D(A)}. \end{cases}$$

3. *Suppose that $f \in C^{\theta}([0,1], D(Q))$ then: there exists a semistrict solution u of problem (1) if and only if*

$$Au_0 \in D(QHQ^{-1}) \cap \overline{D(A)} \text{ and } QHQ^{-1}[Au_0 - f(0)] \in \overline{D(A)}.$$

4. *Suppose that $f \in C^{\theta}([0,1], D(Q^2))$ then: there exists a strict solution u of problem (1) if and only if*

$$\begin{cases} Au_0 \in D(Q^2HQ^{-1}) \cap \overline{D(A)} \text{ and} \\ Q^2HQ^{-1}[Au_0 - f(0)] - [Au_{1,0} - f(1)] \in \overline{D(A)}. \end{cases}$$

5. *Suppose that* $H \in \mathcal{L}(X)$ *and* $f \in C^{\theta}([0,1], D(Q))$ *then: there exists a unique strict solution* u *of problem (1) if and only if*

$$Au_0 \in \overline{D(A)} \cap D(QH) \text{ and } [Au_{1,0} - f(1)] - QH[Au_0 - f(0)] \in \overline{D(A)}.$$

Proof. For statement 1 and 2, we apply Lemmas 8 and 10, noting that

$$\left[Au_0 - f(0) \in \overline{D(A)} \text{ and } H[Au_0 - f(0)] \in \overline{D(A)}\right] \Longleftrightarrow \left[Au_0 - f(0) \in \overline{D(A)}\right].$$

For statement 3, we use Lemmas 8, 11 and also the fact that

$$f(0) \in D(Q) \subset \overline{D(A)},$$

which gives

$$\left[Au_0 - f(0) \in \overline{D(A)} \text{ and } HQ^{-1}Au_0 \in D(Q^2)\right] \Longleftrightarrow Au_0 \in D(QHQ^{-1}) \cap \overline{D(A)}.$$

Statement 4 and 5 are similarly treated. ■

In the previous corollary, we will obtain, in each case, maximal regularity for the solution u if we replace $\overline{D(A)}$ by $D_A(\theta/2, +\infty)$.

2.5 Particular case for Problem (1)

We first study the particular case $H = \alpha I$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re} \alpha \geq 0$ that is

$$\begin{cases} u''(x) + Au(x) = f(x), & x \in]0,1[\\ u(0) = u_0 \\ \alpha u'(0) + u(1) = u_{1,0}. \end{cases} \tag{21}$$

The main difficulty is assumption (5) and we need some results of functional calculus.

Here, our main assumption on A is

$$\begin{cases} A \text{ is a closed linear operator in } X, \ \sigma(A) \subset]-\infty, 0[\text{ and} \\ \text{for any } \theta \in]0,\pi[, \ \sup\limits_{\lambda \in S_\theta} \left\| \lambda (A - \lambda I)^{-1} \right\|_{\mathcal{L}(X)} < +\infty, \end{cases} \tag{22}$$

where $S_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. Since $H = \alpha I$ then

$$\Lambda = I - 2\alpha Q e^{Q} - e^{2Q},$$

and we have to study the functions F, G defined by

$$\begin{cases} F(z) = 1 + G(z) \\ G(z) = 2\alpha z e^{-z} - e^{-2z}, \quad z \in \mathbb{C}. \end{cases}$$

First we fix $\varepsilon_0 > 0$ such that $B(0, 4\varepsilon_0^2) \subset \rho(A)$.

Lemma 15 *Setting $S = S_{\pi/4}$, we get:*

1. *F, G are holomorphic on a neighborhood of $\overline{S}$.*
2. *$x > 0$ implies $|F(x)| > 0$.*
3. $\lim_{\operatorname{Re} z \to +\infty,\ z \in \overline{S}} 2\alpha z e^{-z} + e^{-2z} = 0$ *and then*
   (a) *there exists $x_0 > 0$ such that $z \in \overline{S}$ and $\operatorname{Re} z \geqslant x_0$ imply*
   $$2 \geq |F(z)| \geq 1/2.$$
   (b) *F is bounded on $\overline{S}$.*
4. *There exists $\theta_0 \in]0, \pi/4[$ such that $F(z)$ does not vanish on*
   $$\Sigma_0 = \{z \in \mathbb{C} : \operatorname{Re} z \geq \varepsilon_0 \text{ and } |\arg(z)| \leq \theta_0\},$$
   and $\min_{z \in \Sigma_0} |F(z)| = r > 0$.

Proof.

1. It is obvious
2. We have, for $x > 0$
   $$\operatorname{Re} F(x) = \left(1 - e^{-2x}\right) + 2(\operatorname{Re}\alpha)\, x e^{-x} > 0.$$
3. We just write for $z \in \overline{S}$
   $$\begin{aligned} \left|2\alpha z e^{-z} + e^{-2z}\right| &\leq 2|\alpha|\,|z|\, e^{-\operatorname{Re} z} + e^{-2\operatorname{Re} z} \\ &\leq 2\sqrt{2}\,|\alpha|\,(\operatorname{Re} z)\, e^{-\operatorname{Re} z} + e^{-2\operatorname{Re} z}. \end{aligned}$$

4. We have $|F(z)| \geq 1/2$ for any

$$z \in \Sigma_1 = \{z \in \mathbb{C} : \operatorname{Re} z \geq x_0 \text{ and } |\arg(z)| < \pi/4\}.$$

Moreover F is holomorphic on a neighborhood of

$$\Sigma_2 = \{z \in \mathbb{C} : \varepsilon_0 \geq \operatorname{Re} z \geq x_0 \text{ and } |\arg(z)| \leq \pi/4\},$$

so, on Σ_2, F has at most a finite number of zeros (which are not on the real axis, see statement 2). Thus, we can find $\theta_0 \in]0, \pi/4]$, small enough such that F does not vanishes on

$$\Sigma_2 = \{z \in \mathbb{C} : \varepsilon_0 \geq \operatorname{Re} z \geq x_0 \text{ and } |\arg(z)| \leq \theta_0\}.$$

Moreover

$$\min_{z \in \Sigma_0} |F(z)| = \min\left(\min_{z \in \Sigma_2} |F(z)|, 1/2\right) > 0.$$

■

Now we set for $z \in \Sigma_0$

$$\Psi(z) = \frac{G(z)}{1 + G(z)}.$$

Lemma 16 *Under assumption (22), the operator $\Lambda = I - 2\alpha Q e^Q - e^{2Q}$ is boundedly invertible and $\Lambda^{-1} = I - \Psi(-Q)$.*

Proof. Choose $\theta \in]0, \theta_0[$ such that $\sigma(-Q) \subset S_\theta \backslash B(0, 2\varepsilon_0)$. Note that G is holomorphic and bounded in a neighborhood of $S_\theta \backslash B(0, 2\varepsilon_0)$. Moreover there exists $\sigma > 0$ such that

$$|\Psi(z)| = O\left(|z|^{-\sigma}\right) \text{ when } z \to +\infty, z \in S_\theta \backslash B(0, 2\varepsilon_0).$$

So we can define $\Psi(-Q)$ and also $G(-Q)$ (see for instance [12], subsection 2.5.1, p. 45, together with Remark 2.5.1 and fig. 6, p. 46).

We have also $\Lambda = I + G(-Q)$ and

$$\begin{aligned}
(I - \Psi(-Q))\,\Lambda &= (1 - \Psi)(-Q) \circ (1 + G)(-Q) \\
&= [(1 - \Psi)(1 + G)](-Q) \\
&= \left(1 - \frac{G}{1 + G}\right)(1 + G)(-Q) \\
&= 1(-Q) \\
&= I.
\end{aligned}$$

Similarly $\Lambda\left(I-\Psi(-Q)\right)=I$. ■

If we assume (22), $f\in C^{\theta}\left([0,1],X\right)$, $u_0,u_{1,0}\in D\left(A\right)$ and consider $H=\alpha I$ $(\alpha\in\mathbb{C}\backslash\left\{0\right\},\operatorname{Re}\alpha\geq 0)$ then, due to the previous Lemma assumptions (2)∼(5) are satisfied and we can apply Propositions 2, 3 and Corollary 14, statement 1, to obtain:

Theorem 17 *Under (22), we suppose that* $u_0,u_{1,0}\in D\left(A\right)$ *and*

$$f\in C^{\theta}\left([0,1],X\right)\ \text{ with } \theta\in\left]0,1\right[.$$

Then:

1. *there exists a unique semistrict solution* u *of problem (21) if and only if* $Au_0-f\left(0\right)\in\overline{D\left(A\right)}$,
2. *there exists a unique semistrict solution* u *of problem (21) having (7)-(8) if and only if* $Au_0-f\left(0\right)\in D_A\left(\theta/2,+\infty\right)$.
3. *there exists a unique strict solution* u *of problem (21) if and only if*

$$\begin{cases} Au_0-f\left(0\right)\in\overline{D\left(A\right)}, \\ Au_0-f\left(0\right)+\Im_f\in D(Q)\ \text{ and} \\ \alpha Q\left[Au_0-f\left(0\right)+\Im_f\right]-\left[Au_{1,0}-f\left(1\right)\right]\in\overline{D\left(A\right)}. \end{cases}$$

Remark 18 *Let* $\alpha\in\mathbb{C}\backslash\left\{0\right\},\operatorname{Re}\alpha\geq 0$.

1. *By the same techniques we can consider* $H=-\alpha Q$ *under hypothesis (22), study functions* $\widetilde{F},\widetilde{G}$ *defined by*

$$\widetilde{F}(z)=1+\widetilde{G}(z),\quad \widetilde{G}(z)=-2\alpha z^2e^{-z}-e^{-2z}$$

and thus prove that $\Lambda=I+2\alpha Q^2e^{Q}-e^{2Q}$ *is boundedly invertible with*

$$\Lambda^{-1}=I-\frac{\widetilde{G}}{1+\widetilde{G}}(-Q),$$

then (2)∼(5) will be satisfied and we can apply Theorem 13.

2. Notice that we can also solve the Problem

$$\left\{\begin{array}{l} u''(x) + Au(x) = f(x), \quad x \in [0,1[\\ u(0) = u_0 \\ -\alpha u'(0) + Qu(1) = u_{1,0}, \end{array}\right. \tag{23}$$

since second boundary condition can be written

$$-\alpha Q^{-1} u'(0) + u(1) = Q^{-1} u_{1,0},$$

here $H = -\alpha Q^{-1}$, $\Lambda = I + 2\alpha e^{Q} - e^{2Q} \in \mathcal{L}(X)$ *and, assuming (22), we can apply Corollary 10 statement 2.*

3 Problem with a spectral parameter

In order to provide results for general H satisfying (5), we will consider some large positive number ω and the problem

$$\left\{\begin{array}{l} u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in [0,1] \\ u(0) = u_0 \\ u(1) + Hu'(0) = u_{1,0}. \end{array}\right. \tag{24}$$

3.1 Study of Problem (24)

We consider some fixed $\omega_0 \geq 0$ and we set, for $\omega \geq \omega_0$

$$A_\omega = A - \omega I,$$

then Problem (24) is Problem (1) with A replaced by A_ω.

Our main assumptions on the operators are

$$\left\{\begin{array}{l} A_{\omega_0} \text{ is a closed linear operator in } X,\ [0,+\infty[\subset \rho(A_{\omega_0}) \text{ and} \\ \sup\limits_{\lambda \geqslant 0} \left\| \lambda (A_{\omega_0} - \lambda I)^{-1} \right\|_{L(X)} < +\infty, \end{array}\right. \tag{25}$$

this assumption implies that $Q_{\omega_0} = -(-A_{\omega_0})^{\frac{1}{2}}$, is the infnitesimal generator of a generalized analytic semigroup on X.

$$\forall \zeta \in D(H) : A_{\omega_0}^{-1} H\zeta = H A_{\omega_0}^{-1}\zeta, \tag{26}$$

$$D(Q_{\omega_0}) \subset D(H). \tag{27}$$

Remark 19

1. *Assumption (25) implies that for any* $\omega \geq \omega_0$

$$\begin{cases} A_\omega \text{ is a closed linear operator in } X, \\ [\omega_0 - \omega, +\infty[\subset \rho(A_\omega) \text{ and} \\ \sup_{\lambda \geqslant \omega_0 - \omega} \left\| (\lambda + \omega - \omega_0)(A_\omega - \lambda I)^{-1} \right\|_{\mathcal{L}(X)} < +\infty. \end{cases} \tag{28}$$

   but

$$\sup_{\lambda \geqslant 0} \left\| \lambda (A_\omega - \lambda I)^{-1} \right\|_{\mathcal{L}(X)} \leq \sup_{\lambda \geqslant \omega_0 - \omega} \left\| (\lambda + \omega - \omega_0)(A_\omega - \lambda I)^{-1} \right\|_{\mathcal{L}(X)},$$

   so, for any $\omega \geq \omega_0$, $Q_\omega = -(-A_\omega)^{\frac{1}{2}}$, *is the infnitesimal generator of a generalized analytic semigroup on* X. *Note that*

$$\begin{aligned} c_0 &= \sup_{\lambda \geqslant \omega_0 - \omega} \left\| (\lambda + \omega - \omega_0)(A_\omega - \lambda I)^{-1} \right\|_{\mathcal{L}(X)} \\ &= \sup_{\lambda \geqslant 0} \left\| \lambda (A_{\omega_0} - \lambda I)^{-1} \right\|_{\mathcal{L}(X)}, \end{aligned}$$

   and then c_0 *does not depend of* ω.

2. *Assumption (26) implies that* $\omega \geq \omega_0$

$$\forall \lambda \geqslant \omega_0 - \omega, \forall \zeta \in D(H), \quad (\lambda I - A_\omega)^{-1} H\zeta = H(\lambda I - A_\omega)^{-1} \zeta,$$

Lemma 20 *Assume (25)* $\sim$ *(27), then there exists* $\omega^* \geq \omega_0$ *such that, for* $\omega \geqslant \omega^*$, *the operator* $\Lambda_\omega = -2HQ_\omega e^{Q_\omega} + I - e^{2Q_\omega}$ *has a bounded inverse.*

Proof. We can write $\Lambda_\omega = I - T_\omega$ with $T_\omega = 2HQ_\omega e^{Q_\omega} + e^{2Q_\omega}$. Thus, to show that the operator Λ_ω has a bounded inverse, it is enough to have $\|T_\omega\|_{\mathcal{L}(X)} < 1$.

By using Lemma p. 103 in G. Dore and S. Yakubov [8], we have

$$\begin{cases} \exists c > 0 \text{ et } k > 0 : \\ \left\| Q_\omega^3 e^{Q_\omega} \right\|_{L(X)} \leq ce^{-k\sqrt{\omega}} \text{ and } \left\| e^{2Q_\omega} \right\|_{L(X)} \leq ce^{-k\sqrt{\omega}}. \end{cases} \tag{29}$$

Moreover

$$\begin{aligned}
\left\|Q_{\omega_0}^2 Q_\omega^{-2}\right\|_{\mathcal{L}(X)} &= \left\|A_{\omega_0} A_\omega^{-1}\right\|_{\mathcal{L}(X)} \\
&= \left\|(A-\omega_0 I)(A-\omega I)^{-1}\right\|_{\mathcal{L}(X)} \\
&= \left\|(A-\omega I-(\omega_0-\omega) I)(A-\omega I)^{-1}\right\|_{\mathcal{L}(X)} \\
&= \left\|I-(\omega_0-\omega)(A-\omega I)^{-1}\right\|_{\mathcal{L}(X)} \\
&\leq 1+c_0,
\end{aligned}$$

and, since $HQ_{\omega_0}^{-2}$ is bounded, then

$$\begin{aligned}
\|T_\omega\|_{\mathcal{L}(X)} &= \left\|2HQ_{\omega_0}^{-2}Q_{\omega_0}^2 Q_\omega^{-2} Q_\omega^3 e^{Q_\omega}+e^{2Q_\omega}\right\|_{\mathcal{L}(X)} \\
&\leq 2\left\|HQ_{\omega_0}^{-2}\right\|_{\mathcal{L}(X)}\left\|Q_{\omega_0}^2 Q_\omega^{-2}\right\|_{\mathcal{L}(X)}\left\|Q_\omega^3 e^{2Q_\omega}\right\|_{\mathcal{L}(X)}+\left\|e^{2Q_\omega}\right\|_{\mathcal{L}(X)} \\
&\leq 2\left\|HQ_{\omega_0}^{-2}\right\|_{\mathcal{L}(X)}(1+c_0)\left\|Q_\omega^3 e^{2Q_\omega}\right\|_{\mathcal{L}(X)}+\left\|e^{2Q_\omega}\right\|_{\mathcal{L}(X)},
\end{aligned}$$

and due to (29) there exists $\omega^* \geq \omega_0$ such that for $\omega \geq \omega^*$

$$\|T_\omega\|_{\mathcal{L}(X)} < 1.$$

■

We can now solve Problem (24)

Theorem 21 *Assume (25)∼(27), suppose that $u_0, u_{1,0} \in D(A)$ and*

$$f \in C^\theta([0,1], X) \text{ with } \theta \in \,]0,1[\,.$$

For any $\omega \geqslant \omega^*$

1. *there exists a semiclassical solution u_ω of problem (1) if and only if*

$$Au_0 - f(0) \in \overline{D(A)},$$

2. *there exists a semistrict solution u_ω of problem (1) if and only if*

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)} \\ HQ_\omega^{-1}[Au_0 - f(0) + \mathfrak{I}_f] \in D(Q) \text{ and} \\ Q_\omega HQ_\omega^{-1}[Au_0 - f(0) + \mathfrak{I}_f] \in \overline{D(A)}, \end{cases}$$

3. a strict solution u_ω of problem (1) if and only if

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)} \\ HQ_\omega^{-1}\left[A_\omega u_0 - f(0) + \mathfrak{I}_f\right] \in D(A) \text{ and} \\ Q_\omega^2 HQ_\omega^{-1}\left[A_\omega u_0 - f(0) + \mathfrak{I}_f\right] - \left[A_\omega u_{1,0} - f(1)\right] \in \overline{D(A)}. \end{cases}$$

Moreover, in the 3 cases u is unique and given by $u_\omega = u_{\omega,R} + v_\omega + w_\omega$ where $u_{\omega,R}, v_\omega, w_\omega$ are defined as in (13),(14) and (15) with A, Q, Λ replaced respectively by $A_\omega, Q_\omega, \Lambda_\omega$.

Proof. Let $\omega \geqslant \omega^*$. Notice that if we replace A by.A_ω then Problem (24) corresponds to Problem (1), assumptions (25) $\sim$ (27) correspond to (2) $\sim$ (5), indeed due to Lemma 20, hypotheses (25) $\sim$ (27) implie (5). Then, it is enough to apply Theorem 13 with A replaced by A_ω noting that $\overline{D(A_\omega)} = \overline{D(A)}, D(Q_\omega) = D(Q)$ and

$$A_\omega u_0 - f(0) \in \overline{D(A_\omega)} \Longleftrightarrow Au_0 - f(0) \in \overline{D(A)}.$$

■

Remark 22 *In Theorem 21, we will obtain, in each case, maximal regularity for the solution u_ω if we replace $\overline{D(A)}$ by $D_A(\theta/2, +\infty)$.*

3.2 Particular case for Problem (24)

We consider here $H = \alpha (Q_{\omega_0})^\beta$ with $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in]-\infty, 1]$. So Problem (24) becomes

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in]0,1[\\ u(0) = u_0 \\ u(1) + \alpha (Q_{\omega_0})^\beta u'(0) = u_{1,0}. \end{cases} \tag{30}$$

If we assume (25) then (26), (27) are satisfied and we can apply Theorem 21 (moreover $H \in \mathcal{L}(X)$ if $\beta \in]-1,0]$ and $H \in \mathcal{L}(X)$ with $H(X) \subset D(Q)$ for $\beta \in]-\infty,-1]$. In these cases we can apply Corollary 10). For example if $\beta = 0$ we get the following abstract problem

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in]0,1[\\ u(0) = u_0 \\ u(1) + \alpha u'(0) = u_{1,0}, \end{cases} \tag{31}$$

and the result:

Theorem 23 *Under (25), we suppose that* $u_0, u_{1,0} \in D(A)$ *and*

$$f \in C^{\theta}([0,1], X) \text{ with } \theta \in]0,1[.$$

We have that for any $\omega \geqslant \omega^*$

1. *there exists a semistrict solution* u_ω *of problem (31) if and only if*

$$Au_0 - f(0) \in \overline{D(A)}$$

2. *there exists a strict solution* u_ω *of problem (31) if and only if*

$$\begin{cases} Au_0 - f(0) \in \overline{D(A)}, \\ Au_0 - f(0) + \mathfrak{I}_f \in D(Q) \text{ and} \\ \alpha Q_\omega [A_\omega u_0 - f(0) + \mathfrak{I}_f] - [A_\omega u_{1,0} - f(1)] \in \overline{D(A)}. \end{cases}$$

4 Example

Let M be the linear operator in $X = C([0,1])$ defined by

$$\begin{cases} D(M) = \{v \in C^2([0,1]) : v(0) = v(1) = 0\} \\ Mv = v'', \quad v \in D(M), \end{cases}$$

and for a fixed $c > 0$ set

$$A = -M^2 - cI \text{ and } H = M. \tag{32}$$

A satisfies (22) and thus (2). Moreover, from (32) we deduce (3) $\sim$ (4). Here $Q = -\sqrt{M^2 + cI}$ and setting

$$G(z) = 2\alpha\sqrt{z^2 - c}ze^{-z} - e^{-2z},$$

we prove, as in subsection 2.5, that

$$\begin{aligned} \Lambda &= I - 2MQe^{Q} - e^{2Q} \\ &= I + 2M\sqrt{M^2 + cI}e^{-\sqrt{M^2+cI}} - e^{-2\sqrt{M^2+cI}} \\ &= I + G(-Q), \end{aligned}$$

is boundedly invertible and so (5) is verified. We can then apply Theorem 13 to Problem

$$\begin{cases} u''(x) + Au(x) = f(x), \quad x \in [0,1[\\ u(0) = u_0 \\ u(1) + Mu'(0) = u_{1,0}, \end{cases} \tag{33}$$

Since

$$\begin{cases} D(A) = \{v \in C^4([0,1]) : v(0) = v(1) = v''(0) = v''(1) = 0\} \\ Av = -v^{(4)} - cv, v \in D(A), \end{cases}$$

we can thus deal with the following PDE's nonlocal Problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - \dfrac{\partial^4 u}{\partial y^4}(x,y) - cu(x,y) = f(x,y), (x,y) \in (0,1) \times (0,1) \\ u(x,0) = u(x,1) = \dfrac{\partial^2 u}{\partial y^2}(x,0) = \dfrac{\partial^2 u}{\partial y^2}(x,1) = 0, \ x \in (0,1) \\ u(0,y) = u_0(y), \ y \in (0,1) \\ u(1,y) + \dfrac{\partial^3 u}{\partial y^2 \partial x}(0,y) = u_{1,0}(y), \ y \in (0,1). \end{cases} \tag{34}$$

$D(A)$ is not dense in $X = C([0,1])$ since

$$\overline{D(A)} = \overline{D(M)} = \{v \in C([0,1]) : v(0) = v(1) = 0\}.$$

and for $\theta \in]0,1[$

$$D_A(\theta/2, +\infty) = \left\{v \in C^\theta([0,1]) : v(0) = v(1) = 0\right\}.$$

By applying Theorem 13 we obtain:

Theorem 24 *For any* $f \in C^\theta([0,1], X), \theta \in]0,1[$ *and* $u_0, u_{1,0} \in C^4([0,1])$, *such that*

$$\begin{cases} u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0 \\ u_{0,1}(0) = u_{1,0}(1) = u_{1,0}''(0) = u_{1,0}''(1) = 0 \end{cases},$$

we have

1. *If* $u_0(.) + f(0,.) \in C([0,1])$ *and*

$$u_0^{(4)}(0) + f(0,0) = u_0^{(4)}(1) + f(0,1) = 0,$$

then there exists a unique semiclassical solution u *of problem (34).*

2. *If* $u_0^{(4)}(.) + f(0,.) \in C^\theta([0,1])$ *and*

$$u_0^{(4)}(0) + f(0,0) = u_0^{(4)}(1) + f(0,1) = 0,$$

then the unique semiclassical solution u *of problem (34) has the maximal regularity property (7).*

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