Almost periodic solutions of N-th order neutral differential difference equations with piecewise constant arguments^{*}

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Abstract: In this paper, we study the existence of almost periodic solutions of neutral differential difference equations with piecewise constant arguments via difference equation methods.

Keywords: almost periodic functions; almost periodic sequences; piecewise constant arguments; neutral differential difference equationsMSC2000: 34K14.

1 Introduction

In this paper, consider neutral differential difference equations with piecewise constant argument of the forms

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + f(t),$$
(1)

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + g(t, x(t), x([t-1])).$$
(2)

Here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Throughout this paper, we use the following notations: \mathbb{R} is the set of reals; \mathbb{Z} the set of integers; i.e., $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$; \mathbb{Z}^+ the set of positive integers; \mathbb{C} denotes the set of complex numbers. A function $x : \mathbb{R} \to \mathbb{R}$ is called a solution of Eq. (1) (or (2)) if the following conditions are satisfied:

(i) x is continuous on \mathbb{R} ;

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(ii) the N-th order derivative of x(t) + p(t)x(t-1) exists on \mathbb{R} except possibly at the points t = n, $n \in \mathbb{Z}$, where one-sided N-th order derivatives of x(t) + p(t)x(t-1) exist;

(iii) x satisfies Eq. (1) (or (2)) on each interval (n, n+1) with integer $n \in \mathbb{Z}$.

Differential equations with piecewise constant arguments describe hybrid dynamical systems (a combination of continuous and discrete systems) and, therefore, combine the properties of both differential equations and difference equations. For a survey of work on differential equations with piecewise constant arguments we refer to [1], and for some excellent works in this field we refer to [2-5] and references therein.

In paper [2], Yuan studied the existence of almost periodic solutions for second-order equations involving the argument 2[(t+1)/2] in the unknown function. In paper [3], Piao studied the existence of almost periodic solutions for second-order equations involving the argument [t-1] in the unknown function. In paper [4], Seifert intensively studied the existence of almost periodic solutions for secondorder equations involving the argument [t] in the unknown function by using different methods. However, to the best of our knowledge, there are no results regarding the existence of almost periodic solutions for N-th order neutral differential equations with piecewise constant argument as Eq. (1) (or (2)) up to now. Motivated by the ideas of Yuan [2], Piao[3] and Seifert [4], in this paper we will investigate the existence of almost periodic solutions to Eq. (1) and (2).

Our present paper is organized as follows: in Section 2, we state some definitions and lemmas; in Section 3, we state our main results and prove them.

2 Definitions and Lemmas

Now we start with some definitions.

Definition 2.1 ([6]). A set $K \subset \mathbb{R}$ is said to be relatively dense if there exists L > 0 such that $[a, a + L] \cap K \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 2.2 ([6]). A bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ (resp. \mathbb{C}) is said to be almost periodic if the ε -translation set of f

$$T(f,\varepsilon) = \{ \tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{R} \},\$$

is relatively dense for each $\varepsilon > 0$. We denote the set of all such function f by $\mathcal{AP}(\mathbb{R},\mathbb{R})$ (resp. $\mathcal{AP}(\mathbb{R},\mathbb{C})$).

Definition 2.3 ([6]). A bounded continuous function $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ (resp. \mathbb{C}) is said to be almost periodic function for t uniformly on $\mathbb{R} \times \Omega$, where Ω is an open subset of \mathbb{R}^2 , if, for any compact subset $W \subset \Omega$, the ε -translation set of g,

$$T(g,\varepsilon,W)=\{\tau\in\mathbb{R}:|g(t+\tau,x)-g(t,x)|<\varepsilon,\quad\forall(t,x)\in\mathbb{R}\times W\},$$

is relatively dense in \mathbb{R} . τ is called the ε -period for g. Denote by $\mathcal{AP}(\mathbb{R} \times W, \mathbb{R})$ (resp. $\mathcal{AP}(\mathbb{R} \times W, \mathbb{C})$) the set of all such functions.

Definition 2.4 ([6]). A sequence $x : \mathbb{Z} \to \mathbb{R}^k$ (resp. \mathbb{C}^k), $k \in \mathbb{Z}, k > 0$, denoted by $\{x_n\}$, is called an almost periodic sequence if the ε -translation set of $\{x_n\}$

$$T(\{x_n\},\varepsilon) = \{\tau \in \mathbb{Z} : |x_{n+\tau} - x_n| < \varepsilon, \quad \forall n \in \mathbb{Z}\},\$$

is relatively dense for each $\varepsilon > 0$, here $|\cdot|$ is any convenient norm in \mathbb{R}^k (resp. \mathbb{C}^k). We denote the set of all such sequences $\{x_n\}$ by $\mathcal{APS}(\mathbb{Z}, \mathbb{R}^k)$ (resp. $\mathcal{APS}(\mathbb{Z}, \mathbb{C}^k)$).

Proposition 2.5. $\{x_n\} = \{(x_{n1}, x_{n2}, \cdots, x_{nk})\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^k)$ (resp. $\mathcal{APS}(\mathbb{Z}, \mathbb{C}^k)$) if and only if $\{x_{ni}\} \in \mathcal{APS}(\mathbb{Z},\mathbb{R}) \text{ (resp. } \mathcal{APS}(\mathbb{Z},\mathbb{C})), i = 1, 2, \cdots, k.$

Proposition 2.6. Suppose that $\{x_n\} \in \mathcal{APS}(\mathbb{Z},\mathbb{R}), f \in \mathcal{AP}(\mathbb{R},\mathbb{R})$. Then the sets $T(f,\varepsilon) \cap \mathbb{Z}$ and $T(\{x_n\}, \varepsilon) \cap T(f, \varepsilon)$ are relatively dense.

Lemma 2.7 ([2, 3]). Let $x : \mathbb{R} \to \mathbb{R}$ is a continuous function, and w(t) = x(t) + px(t-1). Then

$$|x(t)| \le e^{-a(t-t_0)} \sup_{-1 \le \theta \le 0} |x(t_0 + \theta)| + b \sup_{t_0 \le u \le t} |w(u)|, \quad t \ge t_0,$$

where $|p| < 1, a = \log 1/|p|, b = 1/(1-|p|)$, or

$$|x(t)| \le e^{\log|p|(t-t_0)} \sup_{0 \le \theta \le 1} |x(t_0 + \theta)| + b \sup_{t \le u \le t_0} |w(u+1)|, \quad t \le t_0,$$

where |p| > 1, b = 1/(|p| - 1).

3 Main Results

Now we rewrite Eq. (1) as the following equivalent system

 (3_1)

$$\begin{cases} \vdots & (3) \\ I & (0) & (0) \end{cases}$$

$$y_{N-2}'(t) = y_{N-1}(t), (3_{N-1})$$

$$(y'_{N-1}(t) = qx([t-1]) + f(t).$$
(3_N)

Lemma 3.1. If $(x(t), y_1(t), \dots, y_{N-1}(t))$ be a solutions of system (3) on \mathbb{R} , (defined similarly as above for (1)), then for $n \in \mathbb{Z}$,

$$\begin{aligned} x(n+1) &= (1-p)x(n) + y_1(n) + \frac{1}{2!}y_2(n) + \dots + \frac{1}{(N-1)!}y_{N-1}(n) + (p + \frac{q}{N!})x(n-1) \\ &+ f_n^{(1)}, & (4_1) \end{aligned}$$

$$y_1(n+1) &= y_1(n) + y_2(n) + \frac{1}{2!}y_3(n) + \dots + \frac{1}{(N-2)!}y_{N-1}(n) + \frac{q}{(N-1)!}x(n-1) + f_n^{(2)}, (4_2) \\ \vdots & \vdots \end{aligned}$$

$$y_{N-2}(n+1) &= y_{N-2}(n) + y_{N-1}(n) + \frac{q}{2}x(n-1) + f_n^{(N-1)}, & (4_{N-1}) \\ y_{N-1}(n+1) &= y_{N-1}(n) + qx(n-1) + f_n^{(N)}, & (4_N) \end{aligned}$$

$$y_{N-2}(n+1) = y_{N-2}(n) + y_{N-1}(n) + \frac{q}{2}x(n-1) + f_n^{(N-1)},$$

$$y_{N-1}(n+1) = y_{N-1}(n) + qx(n-1) + f_n^{(N)},$$

$$(4_N)$$

where

$$f_n^{(1)} = \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N, \quad \cdots, \quad f_n^{(N-1)} = \int_n^{n+1} \int_n^{t_2} f(t_1) dt_1 dt_2,$$

$$f_n^{(N)} = \int_n^{n+1} f(t_1) dt_1.$$
 (5)

Proof. For $n \leq t < n+1, n \in \mathbb{Z}$, using (3_N) we obtain

$$y_{N-1}(t) = y_{N-1}(n) + qx(n-1)(t-n) + \int_{n}^{t} f(t_1)dt_1,$$
(6)

and using this with (3_{N-1}) we obtain

$$y_{N-2}(t) = y_{N-2}(n) + y_{N-1}(n)(t-n) + \frac{1}{2}qx(n-1)(t-n)^2 + \int_n^t \int_n^{t_2} f(t_1)dt_1dt_2.$$
 (7)

Continuing this way, and at last we get

$$x(t) + px(t-1) = x(n) + px(n-1) + y_1(n)(t-n) + \frac{1}{2}y_2(n)(t-n)^2 + \cdots + \frac{1}{(N-1)!}y_{N-1}(n)(t-n)^{N-1} + \frac{1}{N!}qx(n-1)(t-n)^N + \int_n^t \int_n^{t} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_2 \cdots dt_N.$$
(8)

Since x(t) must be continuous at n + 1, these equations yield system (4).

Lemma 3.2. If $f \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$, then sequences $\{f_n^{(i)}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}), i = 1, 2, \dots, N$. **Proof.** We typically consider $\{f_n^{(1)}\}$. $\forall \varepsilon > 0$ and $\tau \in T(f, \varepsilon) \cap \mathbb{Z}$, we have

$$\begin{aligned} \left| f_{n+\tau}^{(1)} - f_n^{(1)} \right| &= \left| \int_{n+\tau}^{n+\tau+1} \int_{n+\tau}^{t_N} \cdots \int_{n+\tau}^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N - \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1) dt_1 dt_2 \cdots dt_N \right| \\ &\leq \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} |f(t_1 + \tau) - f(t_1)| dt_1 dt_2 \cdots dt_N \\ &\leq \frac{\varepsilon}{N!}. \end{aligned}$$

From Definition 2.4, it follows that $\{f_n^{(1)}\}\$ is an almost periodic sequence. In a manner similar to the proof just completed, we know that $\{f_n^{(2)}\}, \{f_n^{(3)}\}, \dots, \{f_n^{(N)}\}\$ are also almost periodic sequences. This completes the proof of lemma.

Next we express system (4) in terms of an equivalent system in \mathbb{R}^{N+1} given by

$$v_{n+1} = Av_n + h_n,\tag{9}$$

where

$$A = \begin{pmatrix} 1-p & 1 & \frac{1}{2!} & \cdots & \frac{1}{(N-1)!} & p + \frac{q}{N!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(N-2)!} & \frac{q}{(N-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{q}{2!} \\ 0 & 0 & 0 & \cdots & 1 & q \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and $v_n = (x(n), y_1(n), y_2(n), \dots, y_{N-1}, x(n-1))^T, h_n = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(N)}, 0)^T.$

Lemma 3.3. Suppose that all eigenvalues of A are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$) and $|\lambda_i| \neq 1, 1 \leq i \leq N+1$. Then system (9) has a unique almost periodic solution.

Proof. From our hypotheses, there exists a $(N + 1) \times (N + 1)$ nonsingular matrix P such that $PAP^{-1} = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$ are the distinct eigenvalues of A. Define $\bar{v}_n = Pv_n$, then (9) becomes

$$\bar{v}_{n+1} = \Lambda \bar{v}_n + \bar{h}_n,\tag{10}$$

where $\bar{h}_n = Ph_n$.

For the sake of simplicity, we consider first the case $|\lambda_1| < 1$. Define

$$\bar{v}_{n1} = \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m-1)1}$$

where $\bar{h}_n = (\bar{h}_{n1}, \bar{h}_{n2}, \cdots, \bar{h}_{n(N+1)})^T$, $n \in \mathbb{Z}$. Clearly $\{\bar{h}_{n1}\}$ is almost periodic, since $\bar{h}_n = Ph_n$, and

 $\{h_n\}$ is. For $\tau \in T(\{\bar{h}_{n1}\}, \varepsilon)$, we have

$$\begin{aligned} \left| \bar{v}_{(n+\tau)1} - \bar{v}_{n1} \right| \\ &= \left| \sum_{m \le n+\tau} \lambda_1^{n+\tau-m} \bar{h}_{(m-1)1} - \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m-1)1} \right| \\ &\text{(letting } m = m' + \tau, \text{then replacing } m' \text{ by } m\text{)} \\ &= \left| \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m+\tau-1)1} - \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m-1)1} \right| \\ &= \left| \sum_{m \le n} \lambda_1^{n-m} \left(\bar{h}_{(m+\tau-1)1} - \bar{h}_{(m-1)1} \right) \right| \\ &\le \frac{\varepsilon}{1 - |\lambda_1|} \end{aligned}$$

this shows that $\{\bar{v}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C}).$

If $|\lambda_i| < 1, 2 \le i \le N + 1$, in a manner similar to the proof just completed for λ_1 , we know that $\{\bar{v}_{ni}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C}), 2 \le i \le N + 1$, and so $\{\bar{v}_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C}^{N+1})$. It follows easily that then $\{P^{-1}\bar{v}_n\} = \{v_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$ and our lemma follows.

Assume now $|\lambda_1| > 1$. Now define

$$\bar{v}_{n1} = \sum_{m \le n} \lambda_1^{m-n} \bar{h}_{(m-1)1}, n \in \mathbb{Z}.$$

As before, the fact that $\{\bar{v}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C})$ follows easily from the fact that $\{\bar{h}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C})$. So in every possible case, we see that each component $v_{ni}, i = 1, 2, \dots, N+1$, of v_n is almost periodic and so $\{v_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$.

The uniqueness of this almost periodic solution $\{v_n\}$ of (9) follows from the uniqueness of the solution \bar{v}_n of (10) since $P^{-1}\bar{v}_n = v_n$, and the uniqueness of \bar{v}_n of (10) follows, since if \tilde{v}_n were a solution of (10) distinct from $\bar{v}_n, u_n = \bar{v}_n - \tilde{v}_n$ would also be almost periodic and solve $u_{n+1} = \Lambda u_n, n \in \mathbb{Z}$. But by our condition on Λ , it follows that each component of u_n must become unbounded either as $n \to \infty$ or as $n \to -\infty$, and that is impossible, since it must be almost periodic. This proves the lemma.

Lemma 3.4. Let $(c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of (9) given by Lemma 3.3, then there exists a solution $(x(t), y_1(t), y_2(t), \dots, y_{N-1}(t)), t \in \mathbb{R}$, of (3) such that $x(n) = c_n, y_1(n) = d_n^{(1)}, \dots, y_{N-1}(n) = d_n^{(N-1)}, n \in \mathbb{Z}$.

Proof. Define

$$w(t) = c_n + pc_{n-1} + d_n^{(1)}(t-n) + \frac{1}{2!}d_n^{(2)}(t-n)^2 + \dots + \frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1} + \frac{1}{N!}qc_{n-1}(t-n)^N + \int_n^t \int_n^{t_N} \dots \int_n^{t_2} f(t_1)dt_1dt_1 \dots dt_N,$$
(11)

for $n \le t < n + 1, n \in \mathbb{Z}$. It can easily be verified that w(t) is continuous on \mathbb{R} . we omit the details. Define $x(t) = \phi(t), -1 \le t \le 0$, where $\phi(t)$ is continuous, and $\phi(0) = c_0, \phi(-1) = c_{-1}$;

$$x(t) = (w(t+1) - \phi(t+1))/p, \quad -2 \le t < -1,$$
$$x(t) = (w(t+1) - x(t+1))/p, \quad -3 \le t \le -2.$$

Continuing this way, we can define x(t) for t < 0. Similarly, define

$$\begin{aligned} x(t) &= -p\phi(t-1) + w(t), \quad 0 \leq t < 1, \\ x(t) &= -px(t-1) + w(t), \quad 1 \leq t < 2; \end{aligned}$$

continuing in this way x(t) is defined for $t \ge 0$, and so x(t) is defined for all $t \in \mathbb{R}$.

Next, define $y_1(t) = w'(t), y_2(t) = w''(t), \dots, y_{N-1}(t) = w^{(N-1)}(t), t \neq n \in \mathbb{Z}$, and by the appropriate one sided derivative of $w'(t), w''(t), \dots, w^{(N-1)}(t)$ at $n \in \mathbb{Z}$. It is easy to see that $y_1(t), y_2(t), \dots, y_{N-1}(t)$ are continuous on \mathbb{R} , and $(x(n), y_1(n), y_2(n), \dots, y_{N-1}(n)) = (c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ for $n \in \mathbb{Z}$; we omit the details.

It is easy to see that x(t) is continuous on $\mathbb{R}, x(n) = c_n$ and satisfies Eq. (1). We don't know if x(t) is an almost periodic one, but we can show that w(t) := x(t) + px(t-1) is an almost periodic function.

Lemma 3.5. Suppose that conditions of Lemma 3.3 hold, x(t) is as defined as above with $(c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of (9) given by Lemma 3.3, then w(t) := x(t) + px(t-1) is almost periodic.

Proof. Indeed, for $\tau \in T(\{c_n\}, \varepsilon) \cap T(\{d_n^{(1)}\}, \varepsilon) \cap T(\{d_n^{(2)}\}, \varepsilon) \cap \cdots \cap T(\{d_n^{(N-1)}\}, \varepsilon) \cap T(f, \varepsilon),$

$$|w(t+\tau) - w(t)| = |(c_{n+\tau} - c_n) + p(c_{n+\tau-1} - c_{n-1}) + (d_{n+\tau}^{(1)} - d_n^{(1)})(t-n) + \frac{1}{2!}(d_{n+\tau}^{(2)} - d_n^{(2)})(t-n)^2 + \cdots + \frac{1}{(N-1)!}(d_{n+\tau}^{(N-1)} - d_n^{(N-1)})(t-n)^{N-1} + \frac{q}{N!}(c_{n+\tau-1} - c_{n-1})(t-n)^N + \int_{n+\tau}^{t+\tau} \int_{n+\tau}^{t_N} \cdots \int_{n+\tau}^{t_2} f(t_1)dt_1dt_2 \cdots dt_N - \int_n^t \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_2 \cdots dt_N| \\ \leq \left(|p| + \frac{|q|}{N!} + \sum_{i=0}^N \frac{1}{i!}\right)\varepsilon.$$
(12)

It follows from definition that w(t) is almost periodic.

Theorem 3.6. Suppose that $|p| \neq 1$ and all eigenvalues of A in (9) are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$) and satisfy $|\lambda_i| \neq 1, 1 \leq i \leq N+1$. Then Eq. (1) has a unique almost periodic solution $\bar{x}(t)$.

Proof. Case I: |p| < 1. For each $m \in \mathbb{Z}^+$ define $x_m(t)$ as follows:

$$x_m(t) = w(t) - px_m(t-1), \quad t > -m,$$
(13)

$$x_m(t) = \phi(t), \quad t \le -m, \tag{14}$$

here w(t) is as defined in the proof of Lemma 3.4, and

$$\phi(t) = c_n + (c_{n+1} - c_n)(t - n), \quad n \le t < n + 1, n \in \mathbb{Z},$$

where c_n is the first component of the solution v_n of (9) given by Lemma 3.3. Let $l \in \mathbb{Z}^+$, then from (13) we get

$$(-p)^{l}x_{m}(t-l) = (-p)^{l}w(t-l) + (-p)^{l+1}x_{m}(t-l-1), \quad t > -m.$$
(15)

It follows

$$x_m(t) = \sum_{j=0}^{l-1} (-p)^j w(t-j) + (-p)^l x_m(t-l), \qquad t > -m.$$

If l > t + m, $x_m(t - l) = \phi(t - l)$, and so for such l,

$$\left| x_m(t) - \sum_{j=0}^{l-1} (-p)^j w(t-j) \right| \le |p|^l |\phi(t-l)|.$$

Let $l \to \infty$, we get

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$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty} (-p)^j w(t-j), & t > -m, \\ \phi(t), & t \le -m. \end{cases}$$
(16)

Since w(t) and $\phi(t)$ are uniformly continuous on \mathbb{R} , it follows that $\{x_m(t) : m \in \mathbb{Z}^+\}$ is equicontinuous on each interval $[-L, L], L \in \mathbb{Z}^+$, and by the Ascoli-Arzelá Theorem, there exists a subsequence, which we again denote by $x_m(t)$, and a function $\bar{x}(t)$ such that $x_m(t) \to \bar{x}(t)$ uniformly on [-L, L], and by a familiar diagonalization procedure, can find a subsequence, again denoted by $x_m(t)$ which is such that $x_m(t) \to \bar{x}(t)$ for each $t \in \mathbb{R}$. From (16) it follows that

$$\bar{x}(t) = \sum_{j=0}^{\infty} (-p)^j w(t-j),$$
(17)

and so $\bar{x}(t)$ is almost periodic since w(t-j) is almost periodic in t for each $j \ge 0$, and |p| < 1. From (13), letting $m \to \infty$, we get $\bar{x}(t) + p\bar{x}(t-1) = w(t), t \in \mathbb{R}$, and since w(t) solves Eq. (1), $\bar{x}(t)$ does also.

The uniqueness of $\bar{x}(t)$ as an almost periodic solution of (1) follows from the uniqueness of the almost periodic solution $v_n : \mathbb{Z} \to \mathbb{R}^{N+1}$ of (9) given by Lemma 3.3, which determines the uniqueness of w(t), and therefore from (17) the uniqueness of $\bar{x}(t)$.

Case II: |p| > 1. Rewriting (15) as

$$\left(\frac{-1}{p}\right)^{l} x_{m}(t-l) = \left(\frac{-1}{p}\right)^{l} w(t-l) + \left(\frac{-1}{p}\right)^{l+1} x_{m}(t-l-1), \quad t > -m,$$
(18)

we deduce in a similar manner that

$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty} \left(\frac{-1}{p}\right)^j w(t-j), & t > -m, \\ \phi(t), & t \le -m. \end{cases}$$
(19)

The remainder of the proof is similar to that of Case I, we omit the details.

Theorem 3.7. Suppose that the conditions of Theorem 3.6 are satisfied, and $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is almost periodic for t uniformly on $\mathbb{R} \times \mathbb{R}$, then there exists $\eta^* > 0$ such that if g satisfies

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le \eta [|x_1 - x_2| + |y_1 - y_2|], 0 \le \eta < \eta^*,$$

for $(t, x_i, y_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, i = 1, 2. Eq. (2) has a unique almost periodic solution.

Proof. It is easy to see that the space $\mathcal{AP}(\mathbb{R},\mathbb{R})$ is a Banach space with supremum norm $||\phi|| = \sup_{t \in \mathbb{R}} |\phi(t)|$ (see [6, 7]). For any $\phi \in \mathcal{AP}(\mathbb{R},\mathbb{R})$, $g(t,\phi(t),\phi([t-1]))$ is an almost periodic function (see [6]). We consider the following equation:

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + g(t,\phi(t),\phi([t-1])).$$
(20)

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From Theorem 3.6, it follows that Eq. (20) has a unique almost periodic solution, denote by $T\phi$. Thus, we obtain a mapping $T : \mathcal{AP}(\mathbb{R}, \mathbb{R}) \to \mathcal{AP}(\mathbb{R}, \mathbb{R})$. For any $\varphi, \psi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}), T\varphi - T\psi$ satisfies the following equation:

$$(z(t) + pz(t-1))^{(N)} = qz([t-1]) + g(t,\varphi(t),\varphi([t-1])) - g(t,\psi(t),\psi([t-1])).$$
(21)

Since $T\varphi$ and $T\psi$ are almost periodic, there exists a constant B > 0 such that $|T\varphi|, |T\psi| \leq B$ and there exists a sequence $\{\alpha_k\}, \alpha \to +\infty$ as $k \to +\infty$, such that $(T\varphi)(t + \alpha_k) \to (T\varphi)(t)$ and $(T\psi)(t + \alpha_k) \to (T\psi)(t)$ as $k \to \infty$. Setting $c_n = (T\varphi)(n) - (T\psi)(n)$, and repeating the preceding steps of Lemma 3.1, we have

$$\begin{cases} c_{n+1} = (1-p)c_n + d_n^{(1)} + \frac{1}{2!}d_n^{(2)} + \dots + \frac{1}{(N-1)!}d_n^{(N-1)} + (p + \frac{q}{N!})c_{n-1} + \widetilde{f}_n^{(1)}, \\ d_{n+1}^{(1)} = d_n^{(1)} + d_n^{(2)} + \frac{1}{2!}d_n^{(3)} + \dots + \frac{1}{(N-2)!}d_n^{(N-1)} + \frac{q}{(N-1)!}c_{n-1} + \widetilde{f}_n^{(2)}, \\ \vdots \\ d_{n+1}^{(N-2)} = d_n^{(N-2)} + d_n^{(N-1)} + \frac{q}{2!}c_{n-1} + \widetilde{f}_n^{(N-1)}, \\ d_{n+1}^{(N-1)} = d_n^{(N-1)} + qc_{n-1} + \widetilde{f}_n^{(N)}, \end{cases}$$

$$(22)$$

where

$$\widetilde{f}_{n}^{(1)} = \int_{n}^{n+1} \int_{n}^{t_{N}} \cdots \int_{n}^{t_{2}} [g(t_{1}, \varphi(t_{1}), \varphi([t_{1}-1])) - g(t_{1}, \psi(t_{1}), \psi([t_{1}-1]))] dt_{1} dt_{2} \cdots dt_{N},$$

$$\cdots$$

$$\widetilde{f}_{n}^{(N-1)} = \int_{n}^{n+1} \int_{n}^{t_{2}} [g(t_{1}, \varphi(t_{1}), \varphi([t_{1}-1])) - g(t_{1}, \psi(t_{1}), \psi([t_{1}-1]))] dt_{1} dt_{2},$$

$$\widetilde{f}_{n}^{(N)} = \int_{n}^{n+1} [g(t_{1}, \varphi(t_{1}), \varphi([t_{1}-1])) - g(t_{1}, \psi(t_{1}), \psi([t_{1}-1]))] dt_{1}.$$

Systems (22) equivalent to

$$v_{n+1} = Av_n + \widetilde{h}_n,\tag{23}$$

where $v_n = (c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)}, c_{n-1})^T$, $\tilde{h}_n = (\tilde{f}_n^{(1)}, \tilde{f}_n^{(2)}, \dots, \tilde{f}_n^{(N)}, 0)^T$. Let P be the nonsingular matrix such that $PAP^{-1} = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$ are the distinct eigenvalues of A. Define $\bar{v}_n = Pv_n$, then (23) becomes

$$\bar{v}_{n+1} = \Lambda \bar{v}_n + \bar{h}_n,\tag{24}$$

where $\bar{h}_n = P\tilde{h}_n$. Repeating the steps of lemma 3.3, we can prove that Eq. (23) has a unique almost periodic solution v_n . Let the elements of the first column of matrix P^{-1} be k_1, k_2, \dots, k_{N+1} , then we have

$$c_n = v_{n1} = k_1 \overline{v}_{n1} + k_2 \overline{v}_{n2} + \dots + k_{N+1} \overline{v}_{N+1}.$$

Let $L = \{l | |\lambda_l| < 1, 1 \le l \le N+1\}, L' = \{l | |\lambda_l| > 1, 1 \le l \le N+1\}$. Then $L \cap L' = \emptyset, L \cup L' = \{1, 2, \dots, N+1\}$, and

$$c_n = \sum_{l \in L} k_l \sum_{m \le n} \lambda_l^{n-m} \bar{h}_{(m-1)l} + \sum_{l \in L'} k_l \sum_{m \le n} \lambda_l^{m-n} \bar{h}_{(m-1)l}.$$

So, there exists $K_0 > 0, K_1 > 0$ such that

$$|(T\varphi)(n) - (T\psi)(n)| \le K_0 \sup_{n \in \mathbb{Z}} |\bar{h}_n| = K_0 \sup_{n \in \mathbb{Z}} |P\tilde{h}_n| \le K_1 \eta |\varphi - \psi|, n \in \mathbb{Z}.$$

Denote

$$W(t) = c_n + pc_{n-1} + d_n^{(1)}(t-n) + \frac{1}{2!}d_n^{(2)}(t-n)^2 + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1} + \frac{1}{N!}qc_{n-1}(t-n)^N + \int_n^t \int_n^{t_N} \cdots \int_n^{t_2} [g(t_1,\varphi(t_1),\varphi([t_1-1])) - g(t_1,\psi(t_1),\psi([t_1-1]))]dt_1dt_1 \cdots dt_N,$$

Then there exists a sufficiently large $K_2 > 0$ such that $|W(t)| \le K_2 \eta |\varphi - \psi|$. We easily conclude that

$$(T\varphi)(t) - (T\psi)(t) + p[(T\varphi)(t-1) - (T\psi)(t-1)] = W(t).$$

We typically consider the case when |p| < 1. Using the first inequality in Lemma 2.7, we have

$$\begin{aligned} |(T\varphi)(t) - (T\psi)(t)| &\leq e^{-a(t-t_0)} \sup_{-1 \leq \theta \leq 0} |(T\varphi)(t_0 + \theta) - (T\psi)(t_0 + \theta)| + b \sup_{t_0 \leq u \leq t} |W(u)| \\ &\leq 2e^{-a(t-t_0)}B + bK_2\eta |\varphi - \psi|, t \geq t_0. \end{aligned}$$

Furthermore, we have

$$|(T\varphi)(t+\alpha_k) - (T\psi)(t+\alpha_k)| \le 2e^{-a(t+\alpha_k-t_0)}B + bK_2\eta|\varphi - \psi|, t+\alpha_k \ge t_0.$$

Note $(T\varphi)(t+\alpha_k) \to (T\varphi)(t)$ and $(T\psi)(t+\alpha_k) \to (T\psi)(t)$ as $k \to \infty$. we have

$$|(T\varphi)(t) - (T\psi)(t)| \le bK_2\eta|\varphi - \psi|.$$

Let $\eta^* = (bK_2)^{-1}$. Then $0 < \eta < \eta^*$ implies that $T : \mathcal{AP}(\mathbb{R}, \mathbb{R}) \to \mathcal{AP}(\mathbb{R}, \mathbb{R})$ is a contraction mapping. It follows that there exists a unique $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$ such that $T\phi = \phi$, that is, Eq. (2) has a unique almost periodic solution.

For the case when |p| > 1, we use the second inequality in Lemma 2.7, the rest of the proof is virtually the same as the above.

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