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Positive periodic solutions and nonlinear eigenvalue problems for functional differential equations *†‡

Xuemei Zhang

Department of Mathematics and Physics, North China Electric Power University,

Beijing, 102206, PR China

Abstract: *This paper is devoted to investigate the existence of positive periodic solution for a functional differential equation in the form of* $\lambda \mathbb{L}x = -b(t)f(x(t-\tau(t)))$*, where* $\mathbb{L}x = x'(t) - a(t)g(x(t))x(t)$*. By using well-known fixed point index theory in a cone, values of* λ *are determined for which there exist positive periodic solutions for the above functional differential equation. The dependence of positive periodic solution* $x_\lambda(t)$ *on the parameter* λ *is also studied, i.e.,*

$$\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty \quad or \quad \lim_{\lambda\to+\infty}\|x_\lambda\| = 0.$$

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1 Introduction

Functional differential equations with periodic delays appear in some ecological models. For example, the model of the survival of red blood cells in an animal [1], and the model of dynamic disease [2], and so on. One of the important questions is whether these equations can support positive periodic solutions. In recent years, periodic population dynamics has become a very popular subject, and several different periodic models have been studied by many authors; see [3-25,34] and references therein.

In this article, we will study the existence of eigenvalues corresponding to positive periodic solutions of the first order functional differential equation with a parameter of the form

$$\lambda \mathbb{L}x = -b(t)f(x(t-\tau(t))), \tag{1.1}$$

*Corresponding author: Xuemei Zhang.

†E-mail address: zxm74@sina.com (X. Zhang); Tel: +86-010-61772874.

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where $\lambda > 0$ is a positive parameter, $\mathbb{L}x = x'(t) - a(t)g(x(t))x(t)$, $a, b \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic, $f, g \in C([0, +\infty), [0, +\infty))$ and $\tau(t)$ is a continuous ω-periodic function.

S.Chow [3], H.Freedman and J.Wu [4], K.Hadeler and J.Tomiuk [5], Y.Kuang [6,7], Y.Kuang and H.Smith [8], J.Mallet-Paret and R.Nussbaum [9] and many others studied the existence of periodic solutions of this type or its generalized forms. This type of equation has been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, the above references, and [2,10,11].

Recently, H.Wang [12] investigated the existence of positive solutions of $Eq.$(1.1) by a well-known result of the fixed point index under condition $f(x) > 0$ for $x > 0$. The author showed the relationship between the asymptotic behaviors of the quotient $\frac{f(x)}{x}$ (at zero and infinity) and the open intervals (eigenvalue intervals) of the parameter λ such that the problem has zero, one and two positive solution(s). However, to the best of our knowledge, little work has been done for the dependence of positive periodic solutions of $Eq.$(1.1) on the parameter λ without condition $f(x) > 0$ for $x > 0$. Thus, it is worthwhile to study $Eq.$(1.1) in this case.

On the other hand, some new results are obtained for the existence of positive periodic solution of $Eq.$(1.1) by using the fixed point index theory in a cone. Our results include and extend many results of X.Liu and W.Li [13], D.Jiang, J.Wei and B.Zhang [14], S.Cheng and G.Zhang [15] and D.Jiang and J.Wei [16] in the case of $g \equiv 1$.

At the same time, we notice that the dependence of positive solution $x_\lambda(t)$ on the parameter λ has received much attention, see [13,26-29] and the references cited therein. In [13], X.Liu and W.Li considered the existence and uniqueness of positive periodic solution for the periodic equation in the form of

$$x'(t) = -a(t)x(t) + \lambda f(x(t - \tau(t))).$$

They examined the uniqueness of the solutions and their dependence on the parameter λ under condition

(H) $f : [0, \infty) \to (0, \infty)$ is nondecreasing, and there exists $\nu \in (0, 1)$ such that

$$f(kx) \geq k^\nu f(x), \quad for\ k \in (0, 1)\ and\ x \in [0, +\infty).$$

Using a similar condition to that of (H) in [13], J.Graef, L.Kong, and H.Wang [26], L.Kong [27], T.He and Y.Su [28] and W.Li and X.Liu [29] also studied the dependence of positive solution $x_\lambda(t)$ on the parameter λ. But, to the best of our knowledge, there is no result for the dependence of positive periodic solution $x_\lambda(t)$ on the parameter λ of $Eq.$(1.1) without a similar condition to that of (H). The objective of the present paper is to fill this gap.

The main purpose of this paper is to establish some new sufficient conditions for the existence of positive periodic solution of $Eq.$ (1.1) by using well-known fixed point index theory in a cone. In particular, we examine the dependence of positive periodic solution $x_\lambda(t)$ on the parameter λ, i.e.,

$$\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty \quad or \quad \lim_{\lambda\to+\infty}\|x_\lambda\| = 0.$$

We remark that our methods are entirely different from those used in [12-16,26-29].

Let $\omega > 0$ and $\mathbb{R} = (-\infty, +\infty)$. We make the following hypotheses:

(H_1) $a,\ b \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic functions satisfying $\int_0^\omega a(t)dt > 0,\ \int_0^\omega b(t)dt > 0.$

(H_2) $\tau \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic functions.

(H_3) $f : [0, +\infty) \to [0, +\infty)$ is continuous and $f(0) = 0$;

(H_4) $g : [0, +\infty) \to [0, +\infty)$ is continuous and $0 < l \le g(x) < L < +\infty$, where $l,\ L$ are positive constants.

For ease of exposition, we set

$$f_0 = \lim_{x\to0^+}\frac{f(x)}{x},\quad f_\infty = \lim_{x\to+\infty}\frac{f(x)}{x}.$$

We will also need the function

$$m(r) = \min\left\{\frac{f(x)}{r} : x \in \left[\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}r, r\right]\right\},$$

where $\sigma = e^{-\int_0^\omega a(t)dt}$.

The main results of the present paper are as follows.

Theorem 1.1. Assume that $(H_1) - (H_4)$ hold. If $0 < f_\infty < +\infty$, then there exists $\beta_0 > 0$ such that, for every $R > \beta_0$, $Eq.$ (1.1) has a positive periodic solution $x_R(t)$ satisfying $\|x_R\| = R$ associated with

$$\lambda = \lambda_R \in [\lambda_0, \overline{\lambda}_0], \tag{1.2}$$

where λ_0 and $\overline{\lambda}_0$ are two positive finite numbers.

Remark 1.1. Some ideas of the proof of Theorem 1.1 are from Theorem 3.2.1 in [30] and Lemma 2.6 in [31].

Theorem 1.2. Assume that $(H_1) - (H_4)$ hold. If $f_\infty = +\infty$, then there exists $\overline{\beta}_0 > 0$ such that, for every $\overline{R} > \overline{\beta}_0$, $Eq.$ (1.1) has a positive periodic solution $x_{\overline{R}}(t)$ satisfying $\|x_{\overline{R}}\| = \overline{R}$ associated with

$$\lambda = \lambda_{\overline{R}} \ge \overline{\lambda}, \tag{1.3}$$

where $\overline{\lambda}$ is a positive finite number.

Theorem 1.3. Assume that $(H_1) - (H_4)$ hold. If $0 < f_0 < +\infty$, then there exists $\beta_0^* > 0$ such that, for every $0 < r < \beta_0^*$, $Eq.$ (1.1) has a positive periodic solution $x_r(t)$ satisfying $\|x_r\| = r$ associated with

$$\lambda = \lambda_r \in [\lambda_0^*, \overline{\lambda}_0^*], \tag{1.4}$$

where λ_0^* and $\overline{\lambda}_0^*$ are two positive finite numbers.

Theorem 1.4. Assume that $(H_1)-(H_4)$ hold. If $f_0 = +\infty$, then there exists $\beta_1 > 0$ such that, for any $0 < r^* < \beta_1$, $Eq.$ (1.1) has a positive periodic solution $x_{r^*}(t)$ satisfying $\|x_{r^*}\| = r^*$ associated with

$$\lambda = \lambda_{r^*} \geq \lambda^*, \tag{1.5}$$

where λ^* is a positive finite number.

Theorem 1.5. Assume that $(H_1)-(H_4)$ hold. If there exist $r^{**} > 0$ and $\beta_{r^{**}} > 0$ such that $m(r^{**}) \geq \beta_{r^{**}}$, then $Eq.$ (1.1) has a positive periodic solution $x_{r^{**}}(t)$ satisfying $\|x_{r^{**}}\| = r^{**}$ associated with

$$\lambda = \lambda_{r^{**}} \geq \lambda^{**}, \tag{1.6}$$

where λ^{**} is a positive finite number.

Finally we consider the dependence of positive periodic solution $x_\lambda(t)$ on the parameter λ.

Theorem 1.6. Assume that $(H_1)-(H_4)$ hold. Then the following two conclusions hold.

(H_5) If $f_0 = 0$ and $f_\infty = \infty$, then for every $\lambda > 0$ $Eq.$ (1.1) has a positive periodic solution $x_\lambda(t)$ satisfying $\lim\limits_{\lambda\to\infty} \|x_\lambda\| = \infty$;

(H_6) If $f_0 = \infty$ and $f_\infty = 0$, then for every $\lambda > 0$ $Eq.$ (1.1) has a positive periodic solution $x_\lambda(t)$ satisfying $\lim\limits_{\lambda\to\infty} \|x_\lambda\| = 0$.

Remark 1.2. Some ideas of the proof of Theorem 1.6 are from [30,32].

Remark 1.3. It is easy to point out some elementary functions, which satisfy conditions (H_3) and (H_5), or satisfy conditions (H_3) and (H_6); for example,

$$f(x) = k_1 x^2,$$

or

$$f(x) = k_2 x^{\frac{1}{2}},$$

where k_1 and k_2 are two positive real numbers.

2 Preliminaries

In order to establish the positive periodic solutions of $Eq.$ (1.1), we shall consider the following space:

$$X = \Big\{x : x(t) \in C(\mathbb{R}, \mathbb{R}),\ x(t+\omega) = x(t)\Big\}.$$

Then X is a real Banach space endowed with the usual linear structure as well as the norm

$$\|x\| = \sup_{t\in[0,\omega]} |x(t)|, \quad x \in X.$$

Define a cone $K \subset X$ by

$$K = \left\{ x \in X : x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\|x\|,\ t \in [0,\omega] \right\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \left\{ x \in X : \|x\| < r \right\}.$$

Note that $\partial\Omega_r = \left\{ x \in X : \|x\| = r \right\}$.

Definition 2.1. By a solution of $Eq.$ (1.1) we mean that a function $x \in X$ satisfying (1.1). x is a positive solution of $Eq.$ (1.1) if, in addition, $x(t) > 0$ for $t \in (0,\omega)$.

Let the map $A_\lambda : K \to X$ be defined by

$$A_\lambda x(t) = \frac{1}{\lambda}\int_t^{t+\omega} G_x(t,s)b(s)f(x(s-\tau(s)))ds, \tag{2.1}$$

where

$$G_x(t,s) = \frac{e^{-\int_t^s a(v)g(x(v))dv}}{1-e^{-\int_0^\omega a(v)g(x(v))dv}},\quad s \in [t, t+\omega]. \tag{2.2}$$

Further, it follows from (2.2) that

$$\frac{\sigma^L}{1-\sigma^L} \leq G_x(t,s) \leq \frac{1}{1-\sigma^l},\quad s \in [t,t+\omega]. \tag{2.3}$$

Lemma 2.1. (See[12]) Assume that $(H_1)-(H_4)$ hold. $Eq.$ (1.1) is equivalent to the fixed point problem of A_λ in K.

Lemma 2.2. (See[12]) Assume that $(H_1)-(H_4)$ hold. Then, $A_\lambda(K) \subset K$ and $A_\lambda : K \to K$ is completely continuous.

The following well-known results of the fixed point index and fixed point are crucial in our arguments.

Lemma 2.3. (See[30]) Let K be a cone in a real Banach space E, Ω be a bounded open set of E. Assume that operator $A : K \cap \overline{\Omega} \to K$ is completely continuous. If there exists a $x_0 > 0$ such that

$$x - Ax \neq tx_0,\quad \forall x \in K \cap \partial\Omega,\quad t \geq 0,$$

then $i(A, K \cap \Omega, K) = 0$.

Remark 2.1. It follows from the Corollary of Lemma 4.2 in [33] that $x_0 > 0$ implies that $x_0 \in K$ and $x_0 \neq 0$.

Lemma 2.4. (See[30]) Let P be a cone in a real Banach space E. Assume $\Omega_1,\ \Omega_2$ are bounded open sets in E with $0 \in \Omega_1,\ \overline{\Omega}_1 \subset \Omega_2$. If

$$A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

is completely continuous such that either

(i) $||Ax|| \leq ||x||, \ \ \forall x \in P \cap \partial\Omega_1$ and $||Ax|| \geq ||x||, \ \ \forall x \in P \cap \partial\Omega_2$, or

(ii) $||Ax|| \geq ||x||, \ \ \forall x \in P \cap \partial\Omega_1$ and $||Ax|| \leq ||x||, \ \ \forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Proof of main results

In this section, we prove the main results, and let us begin by introducing some notation:

$$K \cap \partial\Omega_r = \Big\{x : x \in K, \ \ ||x|| = r\Big\},$$

where $r > 0$ and

$$\gamma = \int_0^\omega b(t)dt.$$

It follows from (H_1) that $\gamma > 0$.

Proof of Theorem 1.1. It follows from $0 < f_\infty < +\infty$ that there exist $0 < l_1 < l_2, \ \ \mu > 0$ such that

$$l_1 x < f(x) < l_2 x \ \ (x \geq \mu). \tag{3.1}$$

Now, we prove that $\beta_0 = \mu\Big(\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\Big)^{-1}$ is required. Thus, when $x \in K \cap \partial\Omega_R$ we have

$$x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||x|| = \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}R, \ \ t \in [0, \omega].$$

Noticing $R > \beta_0$, we have

$$x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||x|| = \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}R > \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\beta_0 = \mu, \ \ t \in [0, \omega],$$

which also implies that $R > \mu$.

Let $\lambda_0 = l_1\gamma\frac{\sigma^L}{1-\sigma^L}$. Then we may assume that

$$x - A_{\lambda_0}x \neq 0 \ \ (\forall x \in K \cap \partial\Omega_R); \tag{3.2}$$

if not, then there exists $x_R \in K \cap \partial\Omega_R$ such that $A_{\lambda_0}x_R = x_R$ and therefore (1.1) already holds for $\lambda_R = \lambda_0$.

Define $\psi(t) \equiv 1$, for $t \in \mathbb{R}$. Then $\psi \in K$ with $||\psi|| \equiv 1$.

We now show that

$$x - A_{\lambda_0}x \neq \zeta\psi \ \ (\forall x \in K \cap \partial\Omega_R, \ \zeta \geq 0). \tag{3.3}$$

In fact, if there exist $x_1 \in K \cap \partial\Omega_R, \ \zeta_1 \geq 0$ such that $x_1 - A_{\lambda_0}x_1 = \zeta_1\psi$, then (3.2) implies that $\zeta_1 > 0$. On the other hand, $x_1 = \zeta_1\psi + A_{\lambda_0}x_1 \geq \zeta_1\psi$. So we can choose $\zeta^* = \sup\{\zeta | x_1 \geq \zeta\psi\}$, then $\zeta_1 \leq \zeta^* < +\infty, \ x_1 \geq \zeta^*\psi$. Therefore

$$\zeta^* = \zeta^*||\psi|| \leq ||x_1|| = R. \tag{3.4}$$

Consequently, for any $t \in [0, \omega]$, (2.3) and (3.1) imply

$$\begin{aligned} x_1(t) &= \lambda_0{}^{-1} \int_t^{t+\omega} G_{x_1}(t,s) b(s) f(x_1(s-\tau(s)))ds + \zeta_1 \psi(t) \\ &\geq \lambda_0{}^{-1} \tfrac{\sigma^L}{1-\sigma^L} \int_t^{t+\omega} b(s) f(x_1(s-\tau(s)))ds + \zeta_1 \psi(t) \\ &= \lambda_0{}^{-1} \tfrac{\sigma^L}{1-\sigma^L} \int_0^{\omega} b(s) f(x_1(s-\tau(s)))ds + \zeta_1 \psi(t) \\ &\geq \lambda_0{}^{-1} l_1 \tfrac{\sigma^L}{1-\sigma^L} \int_0^{\omega} b(s) x_1(s-\tau(s))ds + \zeta_1 \psi(t) \\ &\geq \lambda_0{}^{-1} l_1 \tfrac{\sigma^L}{1-\sigma^L} \int_0^{\omega} b(s) \zeta^* \psi(s-\tau(s))ds + \zeta_1 \psi(t) \\ &= \lambda_0{}^{-1} l_1 \tfrac{\sigma^L}{1-\sigma^L} \zeta^* \int_0^{\omega} b(s)ds + \zeta_1 \psi(t) \\ &= \lambda_0{}^{-1} l_1 \gamma \tfrac{\sigma^L}{1-\sigma^L} \zeta^* + \zeta_1 \psi(t) \\ &= \zeta^* + \zeta_1 \psi(t), \end{aligned}$$

which and (3.4) imply that $x_1(t) \geq (\zeta^* + \zeta_1)\psi(t), \quad t \in [0, \omega]$, which is a contradiction to the definition of ζ^*. Thus, (3.3) holds and, by Lemma 2.3, the fixed point index

$$i(A_{\lambda_0}, K \cap \Omega_R, K) = 0. \tag{3.5}$$

On the other hand, it is easy to see that

$$i(\theta, K \cap \Omega_R, K) = 1, \tag{3.6}$$

where θ is the zero operator.

It follows therefore from (3.5) and (3.6), and the homotopy invariance property that there exist $x_R \in K \cap \partial\Omega_R$ and $0 < \nu_R < 1$ such that $\nu_R A_{\lambda_0} x_R = x_R$, which implies that

$$\lambda_R = \lambda_0 \nu_R^{-1} > \lambda_0.$$

From the proof above, for any $R > \beta_0$, there exists a positive solution $x_R \in K \cap \partial\Omega_R$ associated with $\lambda = \lambda_R > 0$. Thus,

$$x_R(t) = \lambda_R^{-1} \int_t^{t+\omega} G_{x_R}(t,s) b(s) f(x_R(s-\tau(s)))ds,$$

with $\|x_R\| = R$.

On the other hand,

$$x_R(t) = \lambda_R^{-1} \int_0^{\omega} G_{x_R}(t,s) b(s) f(x_R(s-\tau(s)))ds \leq \frac{1}{1-\sigma^l} \lambda_R^{-1} l_2 R \int_0^{\omega} b(s)ds = \frac{1}{1-\sigma^l} \lambda_R^{-1} l_2 R \gamma,$$

which implies that

$$\|x_R\| = R \leq \frac{1}{1-\sigma^l} \lambda_R^{-1} l_2 R \gamma,$$

and hence,

$$\lambda_R \leq \frac{1}{1-\sigma^l} l_2 \gamma = \bar{\lambda}_0.$$

In conclusion, $\lambda_R \in [\lambda_0, \overline{\lambda_0}]$. The proof is complete. □

Remark 3.1. If we use the theory of Leray-Schauder degree, then we can replace (3.5) and (3.6) with

$$deg(I - A_{\lambda_0}, K \cap \Omega_R, K) = 0,$$

and

$$deg(I, K \cap \Omega_R, K) = 1,$$

respectively, where I is the identical operator.

Proof of Theorem 1.2. It follows from $f_\infty = +\infty$ that there exist $l^* > 0$, $\overline{\mu} > 0$ such that

$$f(x) > l^* x \ \ (x \geq \overline{\mu}). \tag{3.7}$$

Now, we prove that $\overline{\beta_0} = \overline{\mu}\left(\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\right)^{-1}$ is required. Thus, when $x \in K \cap \partial\Omega_{\overline{R}}$ we have

$$x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\|x\| = \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\overline{R}, \ \ t \in [0, \omega].$$

Noticing $\overline{R} > \overline{\beta_0}$, we have

$$x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\|x\| = \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\overline{R} > \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\overline{\beta_0} = \overline{\mu}, \ \ t \in [0, \omega].$$

Let $\overline{\lambda} = l^* \gamma \frac{\sigma^L}{1-\sigma^L}$, we proceed in the same way as in the proof of Theorem 1.1: replacing (3.2) we may assume that

$$x - A_{\overline{\lambda}} x \neq 0 \ \ (\forall x \in K \cap \partial\Omega_{\overline{R}}), \tag{3.8}$$

and replacing (3.3) we can prove

$$x - A_{\overline{\lambda}} x \neq \zeta\psi \ \ (\forall x \in K \cap \partial\Omega_{\overline{R}}, \ \zeta \geq 0). \tag{3.9}$$

Hence $i(A_{\overline{\lambda}}, K \cap \Omega_{\overline{R}}, K) = 0$. Observing $i(\theta, K \cap \Omega_{\overline{R}}, K) = 1$, we can show easily that there exist $x_{\overline{R}} \in K \cap \partial\Omega_{\overline{R}}$ and $0 < \nu_{\overline{R}} < 1$ such that $\nu_{\overline{R}} A_{\overline{\lambda}} x_{\overline{R}} = x_{\overline{R}}$. Hence (1.2) holds for $\lambda_{\overline{R}} = \overline{\lambda}\nu_{\overline{R}}^{-1} > \overline{\lambda}$, and the theorem is proved. □

Proof of Theorem 1.3. It follows from $0 < f_0 < +\infty$ that there exist $0 < d_1 < d_2$, $\mu_1 > 0$ such that

$$d_1 x < f(x) < d_2 x \ \ (0 < x \leq \mu_1). \tag{3.10}$$

Now, we prove that $\beta_0^* = \mu_1$ is required. Thus, when $x \in K \cap \partial\Omega_r$ we have

$$0 \leq x(t) \leq \|x\| = r.$$

Noticing $0 < r < \beta_0^*$, we have

$$0 \leq x(t) \leq \|x\| = r < \beta_0^* = \mu_1.$$

Let $\lambda_0^* = d_1\gamma\frac{\sigma^L}{1-\sigma^L}$. Then we may assume that

$$x - A_{\lambda_0^*}x \neq 0 \ \ (\forall x \in K \cap \partial\Omega_r); \tag{3.11}$$

if not, then there exists $x_r \in K \cap \partial\Omega_r$ such that $A_{\lambda_0^*}x_r = x_r$ and therefore (1.4) already holds for $\lambda_r = \lambda_0^*$.

We now show that

$$x - A_{\lambda_0^*}x \neq \zeta\psi \ \ (\forall x \in K \cap \partial\Omega_r,\ \zeta \geq 0), \tag{3.12}$$

where ψ is defined in the proof of Theorem 1.1.

In fact, if there exist $x_2 \in K \cap \partial\Omega_r,\ \zeta_2 \geq 0$ such that $x_2 - A_{\lambda_0^*}x_2 = \zeta_2\psi$, then (3.11) implies that $\zeta_2 > 0$. On the other hand, $x_2 = \zeta_2\psi + A_{\lambda_0^*}x_2 \geq \zeta_2\psi$. So we can choose $\zeta^* = \sup\{\zeta | x_2 \geq \zeta\psi\}$, then $\zeta_2 \leq \zeta^* < +\infty,\ x_2 \geq \zeta^*\psi$. Therefore

$$\zeta^* = \zeta^*\|\psi\| \leq \|x_2\| = r < \mu_1. \tag{3.13}$$

Consequently, for any $t \in [0, \omega]$, (2.3) and (3.10) imply

$$\begin{aligned}
x_2(t) &= \lambda_0^{*-1}\int_t^{t+\omega} G_{x_2}(t,s)b(s)f(x_2(s-\tau(s)))ds + \zeta_2\psi(t)\\
&\geq \lambda_0^{*-1}\frac{\sigma^L}{1-\sigma^L}\int_t^{t+\omega} b(s)f(x_2(s-\tau(s)))ds + \zeta_2\psi(t)\\
&= \lambda_0^{*-1}\frac{\sigma^L}{1-\sigma^L}\int_0^{\omega} b(s)f(x_2(s-\tau(s)))ds + \zeta_2\psi(t)\\
&\geq \lambda_0^{*-1}d_1\frac{\sigma^L}{1-\sigma^L}\int_0^{\omega} b(s)x_2(s-\tau(s))ds + \zeta_2\psi(t)\\
&\geq \lambda_0^{*-1}d_1\frac{\sigma^L}{1-\sigma^L}\int_0^{\omega} b(s)\zeta^*\psi(s-\tau(s))ds + \zeta_2\psi(t)\\
&= \lambda_0^{*-1}d_1\frac{\sigma^L}{1-\sigma^L}\zeta^*\int_0^{\omega} b(s)ds + \zeta_2\psi(t)\\
&= \lambda_0^{*-1}d_1\gamma\frac{\sigma^L}{1-\sigma^L}\zeta^* + \zeta_2\psi(t)\\
&= \zeta^* + \zeta_2\psi(t),
\end{aligned}$$

which and (3.13) imply that $x_2(t) \geq (\zeta^* + \zeta_2)\psi(t),\ \ t \in [0, \omega]$, which is a contradiction to the definition of ζ^*. Thus, (3.12) holds and, by Lemma 2.3, the fixed point index

$$i(A_{\lambda_0^*}, K \cap \Omega_r, K) = 0. \tag{3.14}$$

On the other hand, it is easy to see that

$$i(\theta, K \cap \Omega_r, K) = 1. \tag{3.15}$$

It follows therefore from (3.14) and (3.15), and the homotopy invariance property that there exist $x_r \in K \cap \partial\Omega_r$ and $0 < \nu_r < 1$ such that $\nu_r A_{\lambda_0^*}x_r = x_r$, which implies that

$$\lambda_r = \lambda_0^*\nu_r^{-1} > \lambda_0^*.$$

From the proof above, for any $0 < r < \beta_0^*$, there exists a positive solution $x_r \in K \cap \partial\Omega_r$ associated with $\lambda = \lambda_r > 0$. Thus,

$$x_r(t) = \lambda_r^{-1} \int_t^{t+\omega} G_{x_r}(t, s)b(s)f(x_r(s - \tau(s)))ds,$$

with $\|x_r\| = r$.

On the other hand,

$$x_r(t) \leq \frac{1}{1-\sigma^l}\lambda_r^{-1} \int_0^{\omega} b(s)f(x_r(s - \tau(s)))ds \leq \frac{1}{1-\sigma^l}\lambda_r^{-1}d_2 r \int_0^{\omega} b(s)ds = \frac{1}{1-\sigma^l}\lambda_r^{-1}d_2 r\gamma,$$

which implies that

$$\|x_r\| \leq \frac{1}{1-\sigma^l}\lambda_r^{-1}d_2 r\gamma,$$

and hence,

$$\lambda_r \leq \frac{1}{1-\sigma^l}d_2\gamma = \overline{\lambda}_0^*.$$

Hence $\lambda_r \in [\lambda_0^*, \overline{\lambda}_0^*]$. The proof is complete. □

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.3, we omit it here. □

Proof of Theorem 1.5. In fact, for any $x \in K \cap \partial\Omega_{r^{**}}$, we have $\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}r^{**} \leq x \leq r^{**}$, and hence it follows from the definition of $m(r^{**})$ and $m(r^{**}) \geq \beta_{r^{**}} > 0$ that

$$f(x) \geq r^{**}\beta_{r^{**}} \geq \beta_{r^{**}}x, \quad \forall x \in K \cap \partial\Omega_{r^{**}}.$$

Let $\lambda^{**} = \beta_{r^{**}}\gamma\frac{\sigma^L}{1-\sigma^L}$. Next by a similar manner as in Theorem 1.3 one can prove this theorem. So it is omitted. □

Proof of Theorem 1.6. We need only prove this theorem under condition (H_5) since the proof is similar when (H_6) holds. Let $\lambda > 0$. Considering $f_0 = 0$, there exists $r_1 > 0$ such that

$$f(x) \leq \varepsilon_1 x, \quad \forall 0 \leq x \leq r_1,$$

where $\varepsilon_1 > 0$ satisfies

$$\frac{1}{1-\sigma^l}\lambda^{-1}\varepsilon_1\gamma \leq 1.$$

Thus, for $x \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}(A_\lambda x)(t) &\leq \tfrac{1}{1-\sigma^l}\lambda^{-1} \textstyle\int_0^{\omega} b(s)f(x(s-\tau(s)))ds \\ &\leq \tfrac{1}{1-\sigma^l}\lambda^{-1}\varepsilon_1\|x\| \textstyle\int_0^{\omega} b(s)ds \\ &= \tfrac{1}{1-\sigma^l}\lambda^{-1}\varepsilon_1\|x\|\gamma \\ &\leq \|x\|,\end{aligned}$$

and therefore,

$$\|A_\lambda x\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_1}. \tag{3.16}$$

Next, turning to $f_\infty = \infty$, there exists $\hat{r}$ satisfying $0 < r_1 < \hat{r}$ such that

$$f(x) \geq \varepsilon_2 x, \quad \forall x \geq \hat{r},$$

where $\varepsilon_2 > 0$ satisfies

$$\lambda^{-1}\frac{\sigma^{2L}(1-\sigma^l)}{(1-\sigma^L)^2}\varepsilon_2\gamma \geq 1.$$

Let $r_2 = \frac{\hat{r}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}}$. Then, for $x \in K \cap \partial\Omega_{r_2}$, we have

$$x(t) \geq \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\|x\| = \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L} \cdot \frac{\hat{r}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}} = \hat{r},\ t \in [0,\omega].$$

Hence, for $x \in K \cap \partial\Omega_{r_2}$, it follows from (2.3) that

$$\begin{aligned}(A_\lambda x)(t) &= \lambda^{-1}\int_t^{t+\omega} G_x(t,s)b(s)f(x(s-\tau(s)))ds\\ &\geq \lambda^{-1}\tfrac{\sigma^L}{1-\sigma^L}\int_t^{t+\omega} b(s)f(x(s-\tau(s)))ds\\ &= \lambda^{-1}\tfrac{\sigma^L}{1-\sigma^L}\int_0^{\omega} b(s)f(x(s-\tau(s)))ds\\ &\geq \lambda^{-1}\tfrac{\sigma^L}{1-\sigma^L}\varepsilon_2\int_0^{\omega} b(s)x(s-\tau(s))ds\\ &\geq \lambda^{-1}\tfrac{\sigma^L}{1-\sigma^L}\varepsilon_2\tfrac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\|x\|\int_0^{\omega} b(s)ds\\ &= \lambda^{-1}\tfrac{\sigma^{2L}(1-\sigma^l)}{(1-\sigma^L)^2}\varepsilon_2\|x\|\gamma\\ &\geq \|x\|,\end{aligned}$$

and hence,

$$\|A_\lambda x\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_2}. \tag{3.17}$$

Applying (i) of Lemma 2.4 to (3.16) and (3.17) yields that operator A_λ has a fixed point $x_\lambda \in K \cap (\overline{\Omega}_{r_2} \backslash \Omega_{r_1})$. Thus it follows that for every $\lambda > 0$ *Eq.* (1.1) has a positive solution $x_\lambda(t)$.

It remains to prove $\|x_\lambda\| = +\infty$ as $\lambda \to +\infty$. In fact, if not, there exist a number $m > 0$ and a sequence $\lambda_n \to +\infty$ such that

$$\|x_{\lambda_n}\| \leq m\ (n = 1,2,3,\cdots).$$

Furthermore, the sequence $\{\|x_{\lambda_n}\|\}$ contains a subsequence that converges to a number $\eta (0 \leq \eta \leq m)$. For simplicity, suppose that $\{\|x_{\lambda_n}\|\}$ itself converges to η.

If $\eta > 0$, then $\|x_{\lambda_n}\| > \frac{\eta}{2}$ for sufficiently large n ($n > \mathbb{N}$), and therefore

$$\begin{aligned}\lambda_n &= \frac{\|\int_0^\omega G_{x_{\lambda_n}}(t,s)b(s)f(x_{\lambda_n}(s-\tau(s)))ds\|}{\|x_{\lambda_n}\|}\\ &\leq \frac{\frac{1}{1-\sigma^l}\int_0^\omega b(s)f(x_{\lambda_n}(s-\tau(s)))ds}{\|x_{\lambda_n}\|}\\ &\leq \frac{\frac{1}{1-\sigma^l}\int_0^\omega b(s)dsM}{\|x_{\lambda_n}\|}\\ &= \frac{\frac{1}{1-\sigma^l}\gamma M}{\|x_{\lambda_n}\|}\\ &\leq \frac{2\frac{1}{1-\sigma^l}\gamma M}{\eta}\ (n > \mathbb{N}),\end{aligned}$$

where, $M = \max_{\|x\|\le m} f(x)$, which contradicts $\lambda_n \to +\infty$.

If $\eta = 0$, then $\|x_{\lambda_n}\| \to 0$ for sufficiently large n ($n > \mathbb{N}$), and therefore it follows from (H_5) that for any $\varepsilon > 0$ there exists $r_3 > 0$ such that

$$f(x_{\lambda_n}) \le \varepsilon x_{\lambda_n},\ \forall 0 \le x_{\lambda_n} \le r_3,$$

and hence we obtain

$$\begin{aligned}\lambda_n &= \frac{\|\int_0^\omega G_{x_{\lambda_n}}(t,s)b(s)f(x_{\lambda_n}(s-\tau(s)))ds\|}{\|x_{\lambda_n}\|}\\ &\le \frac{\frac{1}{1-\sigma^l}\int_0^\omega b(s)f(x_{\lambda_n}(s-\tau(s)))ds}{\|x_{\lambda_n}\|}\\ &\le \frac{\frac{1}{1-\sigma^l}\gamma\varepsilon\|x_{\lambda_n}\|}{\|x_{\lambda_n}\|}\\ &= \frac{1}{1-\sigma^l}\gamma\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $\lambda_n \to 0$ ($n \to +\infty$) in contradiction with $\lambda_n \to +\infty$. Therefore, $\|x_\lambda\| \to +\infty$ as $\lambda \to +\infty$ and our proof is complete. □

Remark 3.2. Comparing with [12], the main features of this paper are as follows. First, the technique used in Theorem 1.1 (or Theorem 1.3) is different from that of Theorem 1.3 in [12]. Secondly, from the proof of Theorem 1.3 in [12], it is not difficult to see that professor Wang did not consider the existence of positive ω-periodic solution in the case $f_0 = 0$ and $f_\infty = \infty$ or $f_0 = \infty$ and $f_\infty = 0$. In Theorem 1.6, I not only consider the existence of positive periodic solution under the case $f_0 = 0$ and $f_\infty = \infty$ or $f_0 = \infty$ and $f_\infty = 0$, but also examine its dependence on the parameter λ. In the proof, it is easy to see that we allow that $f(0) = 0$, and it is a difficulty to overcome to prove that $\lambda_n \to 0$ ($n \to +\infty$) as $\eta = 0$.

4 Conclusion and discussion

In this paper, values of λ are determined for which there exist positive periodic solutions for a class of functional differential equations by using well-known fixed point index theory in a cone. The dependence of positive periodic solution $x_\lambda(t)$ on the parameter λ is also studied, i.e.,

$$\lim_{\lambda\to+\infty} \|x_\lambda\| = +\infty \quad or \quad \lim_{\lambda\to+\infty} \|x_\lambda\| = 0.$$

Needless to say, many more applications of theory of eigenvalue problems for fixed point index theory in a cone can be done, and some of them will be given in subsequent papers. Furthermore, many of the obtained results have direct generalizations to the study of positive periodic solutions for impulsive functional differential equations.

On the other hand, it is worth mentioning that there are still many problems that remain open in this vital field except for the results obtained in this paper: for example, whether or not we can obtain some new results for functional differential equations with p-Laplace operator by employing the same technique of this paper, and whether or not our concise criteria can guarantee the existence of positive periodic solutions for higher-order nonlinear functional differential equations. More efforts are still needed in the future.

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