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POSITIVE SOLUTIONS OF NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study the existence and multiplicity of positive solutions of a class of nonlinear fractional boundary value problems with Dirichlet boundary conditions. By applying the fixed point theory on cones we establish a series of criteria for the existence of one, two, any arbitrary finite number, and an infinite number of positive solutions. A criterion for the nonexistence of positive solutions is also derived. Several examples are given for demonstration.

1. INTRODUCTION

In this paper, we study the boundary value problem (BVP) consisting of the fractional differential equation

$$-D^{\alpha}_{0+}u = w(t)f(u),\ 0 < t < 1, \tag{1.1}$$

and the Dirichlet boundary condition (BC)

$$u(0) = 0,\ u(1) = 0, \tag{1.2}$$

where $1 < \alpha < 2$, $w \in L[0, 1]$ is bounded with $w(t) \geq 0$ on $[0, 1]$ and $w(t) > 0$ on $[1/4, 3/4]$, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$, and D^{α}_{0+} is the αth Riemann-Liouville fractional derivative of $h : [0, 1] \to \mathbb{R}$ defined as

$$D^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-s)^{1-\alpha}h(s)ds,$$

whenever the right-hand side is defined.

Fractional differential equations have attracted extensive attention as they can be applied in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., can be modeled as fractional differential equations. We refer the reader to [10, 19] and references therein for the detail.

Due to the needs in applications, people have special interests in the existence of positive solutions of BVPs. In the study on integer order nonlinear BVPs, the fixed point theory on cones is a powerful tool in dealing with the existence of positive solutions. The main idea is to construct a cone in a Banach space and a

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completely continuous operator defined on this cone based on the corresponding Green's function and then find fixed points of the operator. Many results have been obtained by this approach, see, for example, [5, 7, 8, 11, 14, 15, 16, 17, 18, 20, 21, 22]. This idea has also been used to the study of fractional BVPs, see [1, 2, 3, 6, 12, 13, 23, 25] and references therein for recent development.

In particular, Bai and Lü [3] proved that function $G : [0,1] \times [0,1] \to \mathbb{R}_+$ defined by

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{1.3}$$

is the Green's function for the BVP consisting of the fractional differential equation

$$D^{\alpha}_{0+} u = 0$$

and BC (1.2). Based on it, they studied the BVP consisting of

$$-D^{\alpha}_{0+} u = g(t,u), \ 0 < t < 1, \tag{1.4}$$

and BC (1.2) and obtained results on the existence of one or three positive solutions using the Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem. By further revealing the properties of the Green's function $G(t,s)$, Jiang and Yuan [12] obtained different conditions from those in [3] for BVP (1.4), (1.2) to have one positive solution and derived conditions for it to have two positive solutions.

To the best knowledge of the authors, there are no results yet on the existence of an arbitrary number of positive solutions of BVP (1.4), (1.2). This is due to the fact that although the Green's function $G(t,s)$ is positive in the interior of its domain $\{(t,s) : 0 \le t, s \le 1\}$, it becomes zero on the boundary, and does not satisfy the following condition

$$\gamma\Phi(s) \le G(t,s) \le \Phi(s), \ t \in [a,b] \subset [0,1], \ s \in [0,1], \tag{1.5}$$

with a positive Φ and $\gamma \in (0,1)$. The lack of (1.5) prevents us from constructing a needed cone for the natural application of the fixed point theory as with positive Green's functions.

In this paper, by carefully manipulating the Green's function $G(t,s)$, we obtain a weaker condition similar to (1.5) so that we are able to apply the fixed point theory on cones to BVP (1.1), (1.2), where the functions $w(t)$ and $f(x)$ satisfy certain conditions. A series of new criteria for BVP (1.1), (1.2) to have one, two, an arbitrary number, and even an infinite number of positive solutions are obtained. Our results reveal the fact that under our conditions, the existence of one or more positive solutions is determined by the behavior of f on certain intervals. Moreover, a theorem on the nonexistence of positive solutions is also derived.

This paper is organized as follows: After this introduction, our main results are stated in Section 2. Several examples are given in Section 3. All the proofs of the main results are given in the last section.

2. Main results

The following assumptions are needed in the presentations of our main results.

(H1) $f(x) > 0$ on $\mathbb{R}_+$ and there exists a function $B : (0, 1) \to (0, \infty)$ such that for any $r > 0$ and $k \in (0, 1)$

$$\frac{\max_{x\in[0,r]} f(x)}{\min_{x\in[kr,r]} f(x)} \le B(k).$$

(H2) there exists $k^* \in (0, 1)$ such that

$$\psi(k^*) := k^*(1 + B(k^*)M) \le \underline{\gamma}; \tag{2.1}$$

where $M = \max\{M_1, M_2\}$ with

$$M_1 := \max_{s\in[1/4,1/2]} \frac{G(s-1/4, s-1/4)w(s-1/4)}{G(s,s)w(s)} \quad \text{and}$$
$$M_2 := \max_{s\in[1/2,3/4]} \frac{G(s+1/4, s+1/4)w(s+1/4)}{G(s,s)w(s)},$$

here G is defined by (1.3); and $\underline{\gamma} = \min_{s\in[1/4,3/4]} \gamma(s)$ with

$$\gamma(s) = \begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}}, & 0 < s \le s_1, \\ \dfrac{1}{(4s)^{\alpha-1}}, & s_1 \le s < 1, \end{cases} \tag{2.2}$$

here s_1 is the unique solution of the equation

$$[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1} = \frac{1}{4^{\alpha-1}}(1-s)^{\alpha-1}.$$

From the proof of [3, Lemma 2.4] we know that $1/4 < s_1 < 3/4$.

Remark 2.1. The Assumptions (H1) and (H2) are satisfied by a variety of functions of f and w.

(i) We claim that Assumption (H1) is satisfied by the following functions

$$f_1(x) = bx^\theta,\ b > 0,\ \theta > 0,$$
$$f_2(x) = b_1x^{\theta_1} + b_2x^{\theta_2},\ b_1, b_2 > 0,\ 0 < \theta_1 < 1 < \theta_2, \quad \text{and}$$
$$f_3(x) = bx(\sin(x) + \mu),\ b > 0,\ \mu > 1.$$

In fact, for any $r > 0$ and $k \in (0,1)$, since

$$\frac{\max_{x\in[0,r]} f_1(x)}{\min_{x\in[kr,r]} f_1(x)} = \frac{br^\theta}{b(kr)^\theta} = k^{-\theta},$$

f_1 satisfies (H1) with $B(k) = k^{-\theta}$; since

$$\frac{\max_{x\in[0,r]} f_2(x)}{\min_{x\in[kr,r]} f_2(x)} = \frac{r^{\theta_2}(b_1 r^{\theta_1-\theta_2} + b_2)}{(kr)^{\theta_2}(b_1(kr)^{\theta_1-\theta_2} + b_2)} < k^{-\theta_2},$$

f_2 satisfies (H1) with $B(k) = k^{-\theta_2}$; and since

$$\frac{\max_{x\in[0,r]} f_3(x)}{\min_{x\in[kr,r]} f_3(x)} \le \frac{(\mu+1)br}{(\mu-1)bkr} = \frac{\mu+1}{(\mu-1)k},$$

f_3 satisfies (H1) with $B(k) = (\mu+1)(\mu-1)^{-1}k^{-1}$.

(ii) It is easy to see that for any function f satisfying Assumption (H1), Assumption (H2) holds when the function w is relatively small on $[0,1/4]\cup[3/4,1]$ compared with its values on $[1/4,3/4]$. In this case, the number M in (H2) can be sufficiently small and hence inequality (2.1) has a solution $k^* \in (0,1)$.

However, the choice of the cut-off points $1/4$ and $3/4$ in the definitions of M_1 and M_2 is only for convenience in computations. Actually, all results in this paper can be extended to the case when the function $w(t)$ is sufficiently small near the endpoints 0 and 1.

(iii) For some functions f satisfying (H1), (H2) holds for any $w \in C([0,1],\mathbb{R}_+)$. For example, consider $f_1(x) = bx^\theta$ with $\theta \in (0,1)$. Since $B(k) = k^{-\theta}$, $\psi(k) = k + k^{1-\theta}M \to 0$ as $k \to 0$. Hence there exists $k^* \in (0,1)$ such that (2.1) holds.

In the following, we assume that the Assumptions (H1) and (H2) hold. Let

$$L = \underline{\gamma}\int_{1/4}^{3/4} G(s,s)w(s)ds, \tag{2.3}$$

$$U = (1 + B(k^*)M)\int_{1/4}^{3/4} G(s,s)w(s)ds. \tag{2.4}$$

First, we state our basic result on the existence of a positive solution.

Theorem 2.1. *Assume there exist $r^*, r_* > 0$ with $[k^*r^*, r^*]\cap[k^*r_*, r_*] = \emptyset$, such that*

$$f(x) \le U^{-1}r^* \text{ on } [k^*r^*, r^*], \tag{2.5}$$

and

$$f(x) \ge L^{-1}r_* \text{ on } [k^*r_*, r_*]. \tag{2.6}$$

Then BVP (1.1), (1.2) *has at least one positive solution* u *with* $\min\{r^*, r_*\} \leq \|u\| \leq \max\{r^*, r_*\}$.

In the sequel, we will use the following notation:

$$f_0 = \liminf_{x\to 0} f(x)/x, \quad f_\infty = \liminf_{x\to\infty} f(x)/x;$$
$$f^0 = \limsup_{x\to 0} f(x)/x, \quad f^\infty = \limsup_{x\to\infty} f(x)/x.$$

The next three theorems are derived from Theorem 2.1 using f_0, f_∞, f^0, and f^∞.

Theorem 2.2. *BVP (1.1), (1.2) has at least one positive solution if either*

(a) $f^0 < U^{-1}$ *and* $f_\infty > (k^*L)^{-1}$; *or*
(b) $f_0 > (k^*L)^{-1}$ *and* $f^\infty < U^{-1}$.

Theorem 2.3. *Assume there exists* $r^* > 0$ *such that* (2.5) *holds.*

(a) *If* $f_0 > (k^*L)^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \leq r^*$;
(b) *if* $f_\infty > (k^*L)^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \geq r^*$.

Theorem 2.4. *Assume there exists* $r_* > 0$ *such that* (2.6) *holds.*

(a) *If* $f^0 < U^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \leq r_*$;
(b) *if* $f^\infty < U^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \geq r_*$.

Combining Theorems 2.3 and 2.4 we obtain results on the existence of at least two positive solutions.

Theorem 2.5. *Assume either*

(a) $f_0 > (k^*L)^{-1}$ *and* $f_\infty > (k^*L)^{-1}$, *and there exists* $r > 0$ *such that*

$$f(x) < U^{-1}r \text{ on } [k^*r, r]; \text{ or} \tag{2.7}$$

(b) $f^0 < U^{-1}$ *and* $f^\infty < U^{-1}$, *and there exists* $r > 0$ *such that*

$$f(x) > L^{-1}r \text{ on } [k^*r, r]. \tag{2.8}$$

Then BVP (1.1), (1.2) has at least two positive solutions u_1 *and* u_2 *with* $\|u_1\| < r < \|u_2\|$.

Note that in Theorem 2.5, the inequalities in (2.7) and (2.8) are strict and hence are different from those in (2.5) and (2.6) in Theorem 2.1. This is to guarantee that the two solutions u_1 and u_2 are different. By applying Theorem 2.1 repeatedly, we can generalize the conclusion to obtain criteria for the existence of multiple positive solutions.

Theorem 2.6. *Let $\{r_i\}_{i=1}^N \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots < r_N$ and $[k^* r_i, r_i]$, $i = 1, \ldots, N$, are mutually disjoint. Assume either*

(a) *f satisfies (2.7) with $r = r_i$ when i is odd, and satisfies (2.8) with $r = r_i$ when i is even; or*

(b) *f satisfies (2.7) with $r = r_i$ when i is even, and satisfies (2.8) with $r = r_i$ when i is odd.*

Then BVP (1.1), (1.2) has at least $N-1$ positive solutions u_i with $r_i < \|u_i\| < r_{i+1}$, $i = 1, 2, \ldots, N-1$.

Theorem 2.7. *Let $\{r_i\}_{i=1}^\infty \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots$ and $[k^* r_i, r_i]$, $i = 1, \ldots,$ are mutually disjoint. Assume either*

(a) *f satisfies (2.5) with $r_* = r_i$ when i is odd, and satisfies (2.6) with $r^* = r_i$ when i is even; or*

(b) *f satisfies (2.5) with $r_* = r_i$ when i is even, and satisfies (2.6) with $r^* = r_i$ when i is odd.*

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

The following is an immediate consequence of Theorem 2.7.

Corollary 2.1. *Let $\{r_i\}_{i=1}^\infty \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots$ and $[k^* r_i, r_i]$, $i = 1, \ldots,$ are mutually disjoint. Let $E_1 = \cup_{i=1}^\infty [k^* r_{2i-1}, r_{2i-1}]$ and $E_2 = \cup_{i=1}^\infty [k^* r_{2i}, r_{2i}]$. Assume*

$$\limsup_{E_1 \ni x \to \infty} \frac{f(x)}{x} < U^{-1} \quad \textit{and} \quad \liminf_{E_2 \ni x \to \infty} \frac{f(x)}{x} > (k^* L)^{-1}.$$

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

Our last theorem is about the nonexistence of positive solutions of BVP (1.1), (1.2).

Theorem 2.8. *BVP (1.1), (1.2) has no positive solutions if $f(x)/x < U^{-1}$ on $(0, \infty)$.*

3. Examples

Example 1. Let $f(x) = x^\theta$ with $\theta \in (0, 1)$. By Remark 2.1, Assumptions (H1) and (H2) are satisfied for any $w \in C([0, 1], \mathbb{R}_+)$. It is easy to see that $f_0 = \infty$ and $f^\infty = 0$. By Theorem 2.2 (b), BVP (1.1), (1.2) has at least one positive solution.

For easy computations, in Examples 2-4, we let $\alpha = 3/2$ and

$$w(t) = \begin{cases} c, & 0 \le t \le 1/4, \\ 1/G(t,t), & 1/4 < t < 3/4, \\ c, & 3/4 \le t \le 1, \end{cases}$$

where $G(t,s)$ is given in (1.3) and $c>0$ is a constant. By some computations, we know that $\underline{\gamma}=\sqrt{3}-2\sqrt{6}/3$ and $M=\sqrt{3}c/(4\Gamma(3/2))$.

Example 2. Let $f(x)=x^\theta$ with $\theta>1$. By a simple calculation we see that Assumptions (H1) and (H2) are satisfied when

$$c\le(4-8\sqrt{2}/3)\Gamma(3/2)((\theta-1)^{1/\theta}+(\theta-1)^{\frac{1-\theta}{\theta}})^{-\theta}.$$

In this case, since $f^0=0$ and $f_\infty=\infty$, by Theorem 2.2 (a), BVP (1.1), (1.2) has at least one positive solution.

Example 3. Let $f(x)=\lambda(x^{\theta_1}+x^{\theta_2})$, where $0<\theta_1<1<\theta_2<\infty$. By a simple calculation we see that Assumptions (H1) and (H2) are satisfied when

$$c\le(4-8\sqrt{2}/3)\Gamma(3/2)((\theta_2-1)^{1/\theta_2}+(\theta_2-1)^{\frac{1-\theta_2}{\theta_2}})^{-\theta_2}.$$

Furthermore, $L=\sqrt{3}/2-\sqrt{6}/3$ and $U=1/2+\sqrt{3}c/(8\Gamma(3/2)k^{\theta_2})$. In this case, we let $r_1=((1-\theta_1)/(\theta_2-1))^{1/(\theta_2-\theta_1)}$. We claim that BVP (1.1), (1.2) has

(a) at least one positive solution if $\lambda=\left(r_1^{\theta_1}+r_1^{\theta_2}\right)^{-1}U^{-1}$;

(b) at least two positive solutions if $\lambda\in(0,\left(r_1^{\theta_1}+r_1^{\theta_2}\right)^{-1}U^{-1})$.

We note that $f_0=f_\infty=\infty$, f is strictly increasing, and r_1 is the minimum point of $f(x)/x$ on $(0,\infty)$. When $\lambda=\left(r_1^{\theta_1}+r_1^{\theta_2}\right)^{-1}U^{-1}$, $f(x)\le U^{-1}r_1$ on $[k^*r_1,r_1]$. By Theorem 2.3 (a), BVP (1.1), (1.2) has a positive solution u_1 with $\|u_1\|\le r_1$. Similarly, by Theorem 2.3 (b), BVP (1.1), (1.2) has a positive solution u_2 with $\|u_2\|\ge r_1$. However, u_1 and u_2 may be the same when $\|u_1\|=\|u_2\|=r_1$.

Part (b) follows similarly from Theorem 2.5 (a).

Example 4. Let L and U be defined by (2.3) and (2.4), and

$$f(x)=\begin{cases}(2+\underline{\gamma})x(\sin(m\ln x)+\mu)/(2\underline{\gamma}L), & x>0,\\ 0, & x=0,\end{cases}$$

where

$$1<\mu<\min\{4/(2+\underline{\gamma}),1+2\underline{\gamma}L/((2+\underline{\gamma})U)\}$$

and

$$0<m<(\pi-2\sin^{-1}\delta)/\ln(2/\underline{\gamma})$$

with

$$\delta=\max\{|\delta_1|,|\delta_2|\},$$

here $\delta_1=(4-2\mu-\mu\underline{\gamma})/(2+\underline{\gamma})$ and $\delta_2=2\underline{\gamma}L/((2+\underline{\gamma})U)-\mu$. Note that $\delta_2<0$. By a simple calculation we see that Assumptions (H1) and (H2) are satisfied for $k^*=\underline{\gamma}/2$ when $c<\underline{\gamma}16^{\alpha-1}(\mu-1)\Gamma(\alpha)3^{1-\alpha}/(2(\mu+1))$. In this case, we claim that BVP (1.1), (1.2) has an infinite number of positive solutions.

In fact, it is easy to verify that $\delta \in (0,1)$. For $i \in \mathbb{N}$, let

$$\xi_i = \exp(m^{-1}(\sin^{-1}\delta + (i-1)\pi)),$$
$$\eta_i = \exp(m^{-1}(i\pi - \sin^{-1}\delta)).$$

Then

$$\frac{\eta_i}{\xi_i} = \exp(m^{-1}(\pi - 2\sin^{-1}\delta)) > \exp(\ln(2/\underline{\gamma})) = k^{*-1}.$$

Hence $\xi_i < k^*\eta_i$. Then for any $x \in [k^*\eta_i, \eta_i]$, $x \in [\xi_i, \eta_i]$.

When i is odd, for any $x \in [k^*\eta_i, \eta_i]$

$$\sin(m \ln x) \geq \sin(m \ln \xi_i) = \sin(\sin^{-1}\delta) = \delta > \delta_1.$$

So

$$f(x) \geq (2+\underline{\gamma})k^*\eta_i(\delta_1 + \mu)/(2\underline{\gamma}L) = L^{-1}\eta_i.$$

When i is even, for any $x \in [k^*\eta_i, \eta_i]$

$$\sin(m \ln x) \leq \sin(m \ln \eta_i) = \sin(-\sin^{-1}\delta) = -\delta \leq \delta_2.$$

So

$$f(x) \leq (2+\underline{\gamma})\eta_i(\delta_2 + \mu)/(2\underline{\gamma}L) = U^{-1}\eta_i.$$

Therefore, by Theorem 2.7 (b), BVP (1.1), (1.2) has an infinite number of positive solutions.

4. Proofs

Let X be a Banach space and $K \subset X$ a cone in X. For $r > 0$, define

$$K_r = \{u \in K \mid \|u\| < r\} \quad and \quad \partial K_r = \{u \in K \mid \|u\| = r\}.$$

For an operator $T : K_r \to K$, let $\mathfrak{i}(T, K_r, K)$ be the fixed point index of T on K_r with respect to K. We will use the following well-known lemmas on fixed point index to prove our main results. For the detail, see [4, 9] and [24, page 529, A2, A3].

Lemma 4.1. *Assume that for $r > 0$, $T : K_r \to K$ is a completely continuous operator such that $Tu \neq u$ for $u \in \partial K_r$.*

(a) *If $\|Tu\| \geq \|u\|$ for $u \in \partial K_r$, then $\mathfrak{i}(T, K_r, K) = 0$.*

(b) *If $\|Tu\| \leq \|u\|$ for $u \in \partial K_r$, then $\mathfrak{i}(T, K_r, K) = 1$.*

Lemma 4.2. *Let $0 < r_1 < r_2$ satisfy*

$$\mathfrak{i}(T, K_{r_1}, K) = 0 \quad and \quad \mathfrak{i}(T, K_{r_2}, K) = 1;$$

or

$$\mathfrak{i}(T, K_{r_1}, K) = 1 \quad and \quad \mathfrak{i}(T, K_{r_2}, K) = 0.$$

Then T has a fixed point in $K_{r_2} \setminus \overline{K}_{r_1}$.

We will also use the following property of the Green's function $G(t,s)$ for BVP (1.1), (1.2), see [3, Lemma 2.4] for the proof.

Lemma 4.3. *Let $G(t,s)$ be defined by* (1.3) *and $\underline{\gamma}$ defined in (H2). Then*

(a) $G(t,s) > 0$ *on* $(0,1) \times (0,1)$,

(b) $\max_{t\in[0,1]} G(t,s) = G(s,s)$ *for* $0 < s < 1$ *and* $\min_{1/4\le t\le 3/4} G(t,s) \ge \underline{\gamma}\, G(s,s)$ *for* $1/4 \le s \le 3/4$.

Let $X = C[0,1]$ and $\|u\| = \max_{t\in[0,1]} |u(t)|$ for any $u \in X$. Then $(X, \|\cdot\|)$ is a Banach space. Define

$$K = \{u \in X \mid u(t) \ge 0 \text{ on } [0,1] \text{ and } \min_{t\in[1/4,3/4]} u(t) \ge k^*\|u\|\}, \tag{4.1}$$

where k^* is defined in (H2), and $T : K \to X$ as

$$(Tu)(t) = \int_0^1 G(t,s)w(s)f(u(s))ds. \tag{4.2}$$

Clearly, $u(t)$ is a solution of BVP (1.1), (1.2) if and only if $u \in K$ is a fixed point of T, and $\|Tu\| \le \int_0^1 G(s,s)w(s)f(u(s))ds$.

Lemma 4.4. *Assume (H1), (H2) hold. Then $TK \subset K$ and T is a completely continuous operator.*

Proof. For any $u \in K$ with $\|u\| = r$, $k^*r \le u(t) \le r$ on $[1/4, 3/4]$. By (4.2) and Lemma 4.3, for $t \in [1/4, 3/4]$

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)w(s)f(u(s))ds \\ &\ge \underline{\gamma} \int_{1/4}^{3/4} G(s,s)w(s)f(u(s))ds. \end{aligned} \tag{4.3}$$

By (H1), $\max_{u\in[0,r]} f(u) \le B(k^*) \min_{u\in[k^*r,r]} f(u)$. Then by (H2)

$$\begin{aligned}&\int_0^{1/4} G(s,s)w(s)f(u(s))ds \le \max_{u\in[0,r]} f(u) \int_0^{1/4} G(s,s)w(s)ds \\ \le\ & B(k^*) \min_{u\in[k^*r,r]} f(u) \int_0^{1/4} G(s,s)w(s)ds \\ \le\ & B(k^*)M_1 \min_{u\in[k^*r,r]} f(u) \int_{1/4}^{1/2} G(s,s)w(s)ds \\ \le\ & B(k^*)M \int_{1/4}^{1/2} G(s,s)w(s)f(u(s))ds. \end{aligned} \tag{4.4}$$

Similarly,

$$\int_{3/4}^{1} G(s,s)w(s)f(u(s))ds \le B(k^*)M \int_{1/2}^{3/4} G(s,s)w(s)f(u(s))ds. \quad (4.5)$$

Therefore, by (4.4) and (4.5)

$$\begin{aligned} & \int_0^1 G(s,s)w(s)f(u(s))ds \\ = & \left(\int_0^{1/4} + \int_{1/4}^{1/2} + \int_{1/2}^{3/4} + \int_{3/4}^{1} \right) G(s,s)w(s)f(u(s))ds \\ & \le (1 + B(k^*)M) \int_{1/4}^{3/4} G(s,s)w(s)f(u(s))ds, \end{aligned} \quad (4.6)$$

which follows that

$$\int_{1/4}^{3/4} G(s,s)w(s)f(u(s))ds \ge \frac{1}{1 + B(k^*)M} \int_0^1 G(s,s)w(s)f(u(s))ds.$$

Hence by (H2) and (4.3)

$$\begin{aligned} (Tu)(t) & \ge \frac{\gamma}{1 + B(k^*)M} \int_0^1 G(s,s)w(s)f(u(s))ds \\ & \ge \frac{\gamma}{1 + B(k^*)M} \|Tu\| \ge k^* \|Tu\| \text{ on } [1/4, 3/4]. \end{aligned}$$

It is clear that $(Tu)(t) \ge 0$ on $[0,1]$. Therefore, $TK \subset K$. By Arzela-Ascoli Theorem, T is completely continuous. □

Proof of Theorem 2.1. Without loss of the generality, we assume $r_* < r^*$. For any $u \in \partial K_{r^*}$, $\|u\| = r^*$ and $k^* r^* \le u(t) \le r^*$ on $[1/4, 3/4]$. By (2.4), (2.5), (4.2), (4.6), and Lemma 4.3 (a)

$$\begin{aligned} \|Tu\| & = \max_{t \in [0,1]} \int_0^1 G(t,s)w(s)f(u(s))ds \\ & \le \int_0^1 G(s,s)w(s)f(u(s))ds \\ & \le U^{-1} r^* (1 + B(k^*)M) \int_{1/4}^{3/4} G(s,s)w(s)ds = r^*. \end{aligned}$$

By Lemma 4.1 (a), $\mathfrak{i}(T, K_{r^*}, K) = 1$.

For any $u \in \partial K_{r_*}$, $\|u\| = r_*$ and $k^* r_* \le u(t) \le r_*$ on $[1/4, 3/4]$. By (2.3), (2.6), (4.2), and Lemma 4.3 (b)

$$\begin{aligned} \|Tu\| \ge (Tu)(1/2) &= \int_0^1 G(1/2, s) w(s) f(u(s)) ds \\ \ge\ \underline{\gamma} \int_{1/4}^{3/4} G(s,s) w(s) f(u(s)) ds \\ \ge\ L^{-1} r_* \underline{\gamma} \int_{1/4}^{3/4} G(s,s) w(s) ds = r_*. \end{aligned}$$

By Lemma 4.1 (b), $\mathfrak{i}(T, K_{r_*}, K) = 0$.

By Lemma 4.2, T has a fixed point u in K with $r_* \le \|u\| \le r^*$. Hence BVP (1.1), (1.2) has at least one positive solution $u(t)$. □

Proof of Theorem 2.2. (a) If $f^0 < U^{-1}$, there exists a sufficiently small $r^* > 0$ such that

$$f(x) < U^{-1} x \le U^{-1} r^* \text{ on } [k^* r^*, r^*],$$

i.e., (2.5) holds.

If $f_\infty > (k^* L)^{-1}$, there exists $\hat{r} > r^*$ such that

$$f(x) > (k^* L)^{-1} x \text{ on } [\hat{r}, \infty).$$

Then for any r_* with $k^* r_* \ge \hat{r}$

$$f(x) > (k^* L)^{-1} x \ge L^{-1} r_* \text{ for all } x \in [k^* r_*, r_*],$$

i.e., (2.6) holds. Then the conclusion follows from Theorem 2.1.

(b) The proof is similar to Part (a) and hence is omitted. □

The proofs of Theorems 2.3 and 2.4 are in the same way and hence are omitted.

Proof of Theorem 2.5. (a) If there exists $r > 0$ such that (2.7) holds, then by the continuity of $f(x)/x$ on $(0, \infty)$, there exist $r_1, r_2 > 0$ such that $r_1 < r < r_2$ and $f(x) < U^{-1} r_i$ on $[k^* r_i, r_i]$, $i = 1, 2$. By Theorem 2.3 (a) and (b), BVP (1.1), (1.2) has two positive solutions u_1 and u_2 satisfying $\|u_1\| \le r_1$ and $\|u_2\| \ge r_2$.

Similarly, Case (b) follows from Theorem 2.4. □

The proofs of Theorems 2.6 and 2.7 are in the same way and are hence omitted.

Proof of Corollary 2.1. From the assumption we see that for sufficiently large i

$$\frac{f(x)}{x} < U^{-1} \text{ on } [k^* r_{2i-1}, r_{2i-1}]$$

and

$$\frac{f(x)}{x} > (k^*L)^{-1} \text{ on } [k^*r_{2i}, r_{2i}].$$

This shows that for sufficiently large i

$$f(x) < U^{-1}x \leq U^{-1}r_{2i-1} \text{ on } [k^*r_{2i-1}, r_{2i-1}]$$

and

$$f(x) > (k^*L)^{-1}x \geq L^{-1}r_{2i} \text{ on } [k^*r_{2i}, r_{2i}].$$

Therefore, the conclusion follows from Theorem 2.7. □

Proof of Theorem 2.8. Assume BVP (1.1), (1.2) has a positive solution u with $\|u\| = r$ for some $r > 0$. Then u is a fixed point of the operator T defined by (4.2). For any $t \in [0, 1]$, by (4.6)

$$\begin{aligned} u(t) &= (Tu)(t) = \int_0^1 G(t,s)w(s)f(u(s))ds \\ &\leq (1 + B(k^*)M) \int_{1/4}^{3/4} G(s,s)w(s)f(u(s))ds \\ &< U^{-1}r\,(1 + B(k^*)M) \int_{1/4}^{3/4} G(s,s)w(s)ds = r, \end{aligned}$$

which contradicts $\|u\| = r$. Therefore, BVP (1.1), (1.2) has no positive solutions. □

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