Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **15**, 1–20; https://doi.org/10.14232/ejqtde.2024.1.15 www.math.u-szeged.hu/ejqtde/

Slow divergence integral in regularized piecewise smooth systems

Renato Huzak✉[1], **Kristian Uldall Kristiansen**[2] and **Goran Radunović**[3]

[1]Hasselt University, Campus Diepenbeek, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium
[2]Department of Applied Mathematics and Computer Science, Technical University of Denmark, 2800 Kgs. Lyngby, Denmark
[3]University of Zagreb, Faculty of Science, Horvatovac 102a, 10000 Zagreb, Croatia

Received 9 October 2023, appeared 26 March 2024
Communicated by Armengol Gasull

Abstract. In this paper we define the notion of slow divergence integral along sliding segments in regularized planar piecewise smooth systems. The boundary of such segments may contain diverse tangency points. We show that the slow divergence integral is invariant under smooth equivalences. This is a natural generalization of the notion of slow divergence integral along normally hyperbolic portions of curve of singularities in smooth planar slow–fast systems. We give an interesting application of the integral in a model with visible-invisible two-fold of type VI_3. It is related to a connection between so-called Minkowski dimension of bounded and monotone "entry-exit" sequences and the number of sliding limit cycles produced by so-called canard cycles.

Keywords: sliding limit cycles, piecewise smooth systems, regularization function, slow divergence integral, Minkowski dimension.

2020 Mathematics Subject Classification: 34E15, 34E17, 34C40.

1 Introduction

The notion of slow divergence integral [6, Chapter 5] has proved to be an important tool to study lower and upper bounds of limit cycles in smooth planar slow–fast systems (see e.g. [5–7, 10, 11] and references therein). In this paper by "smooth", we mean differentiable of class C^∞. One of the main goals of this paper is to define the slow divergence integral in regularized planar piecewise smooth (PWS) systems with sliding and to prove its invariance under smooth equivalences (by smooth equivalence we mean smooth coordinate change and a multiplication by a smooth positive function). This is a natural generalization of [25] where the slow divergence integral is defined only for a PWS two-fold bifurcation of type visible-invisible called VI_3 and the switching boundary is a straight line (for more details about two-fold singularity VI_3 see [28] and Sections 2 and 3). In this paper we define the slow

✉Corresponding author. Email: renato.huzak@uhasselt.be

divergence integral along sliding segments with a regular sliding vector field [16] (Section 2.1), and extend it to tangency points when only one vector field is tangent to switching boundary (Section 2.3), two-fold singularities of sliding type VV_1, II_1, VI_2 and VI_3 [28], and to a visible-invisible two-fold singularity when the sliding vector field vanishes at the two-fold point (Section 2.2).

Consider a smooth planar slow–fast system

$$X_{\epsilon,\lambda} = f_\lambda Y_\lambda + \epsilon Q_\lambda + O(\epsilon^2)$$

defined on open set $V \subset \mathbb{R}^2$, where $0 < \epsilon \ll 1$ is the singular perturbation parameter, $\lambda \sim \lambda_0 \in \mathbb{R}^l$, f_λ is a smooth family of functions and Y_λ and Q_λ are smooth families of vector fields. We suppose that $X_{0,\lambda}$ has a curve of singularities C_λ for all $\lambda \sim \lambda_0$ (Fig. 1.1). We further assume that $\nabla f_\lambda(p) \neq (0,0)$ for all $p \in \{(x,y) \in V \mid f_\lambda(x,y) = 0\}$ and that Y_λ has no singularities for each $\lambda \sim \lambda_0$. Then we have $C_\lambda = \{f_\lambda = 0\}$ and C_λ is a smooth family of 1-dimensional manifolds.

The orbits of the flow of $X_{0,\lambda}$ are located inside the leaves of a smooth 1-dimensional foliation $\mathcal{F}_\lambda$ on V tangent to Y_λ ($\mathcal{F}_\lambda$ is called the fast foliation of $X_{0,\lambda}$). If $p \in C_\lambda$, then $DX_{0,\lambda}(p)$ has one eigenvalue equal to zero, with eigenspace T_pC_λ, and the other one equal to $\operatorname{div} X_{0,\lambda}(p)$ (i.e., the trace of $DX_{0,\lambda}(p)$) with eigenspace $T_p l_{\lambda,p}$ ($l_{\lambda,p}$ is the leaf through p). We say that $p \in C_\lambda$ is normally hyperbolic if $\operatorname{div} X_{0,\lambda}(p) \neq 0$ (attracting when $\operatorname{div} X_{0,\lambda}(p) < 0$ and repelling when $\operatorname{div} X_{0,\lambda}(p) > 0$). We can define the notion of slow vector field on normally hyperbolic segments of C_λ. Let $p \in C_\lambda$ be a normally hyperbolic singularity and let $\bar{Q}_\lambda(p) \in T_pC_\lambda$ be the projection of $Q_\lambda(p)$ on T_pC_λ in the direction of $T_p l_{\lambda,p}$. $\bar{Q}_\lambda$ is called the slow vector field and its flow the slow dynamics. The time variable of the slow dynamics is the slow time $\bar{t} = \epsilon t$ where t is the time variable of the flow of $X_{\epsilon,\lambda}$. We point out that the classical definition of the slow vector field using center manifolds is equivalent to this definition. For more details see [6, Chapter 3].

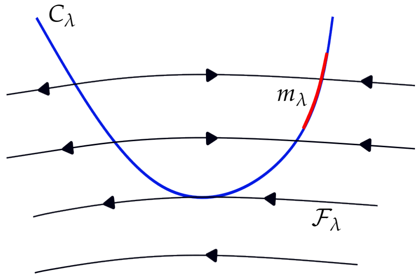

Figure 1.1: The dynamics of $X_{0,\lambda}$ with the curve of singularities C_λ (blue) and the fast foliation $\mathcal{F}_\lambda$. A normally hyperbolic segment $m_\lambda \subset C_\lambda$ (red) along which the slow divergence integral can be defined.

We define now the notion of slow divergence integral (see [6, Chapter 5]). If $m_\lambda \subset C_\lambda$ is a normally hyperbolic segment not containing singularities of $\bar{Q}_\lambda$, then the slow divergence integral along m_λ is defined by

$$I(m_\lambda) = \int_{\bar{t}_1}^{\bar{t}_2} \operatorname{div} X_{0,\lambda}(z_\lambda(\bar{t}))d\bar{t} \tag{1.1}$$

where $z_\lambda : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^2$, $z'_\lambda(\bar{t}) = \bar{Q}_\lambda(z_\lambda(\bar{t}))$ and $z_\lambda(\bar{t}_1)$ and $z_\lambda(\bar{t}_2)$ are the end points of m_λ (we parameterize m_λ by $\bar{t}$). This definition is independent of the choice of parameterization z_λ of

m_λ and the slow divergence integral is invariant under smooth equivalences (see [6, Section 5.3]).

If both eigenvalues of the linear part of $X_{0,\lambda}$ at $p \in C_\lambda$ are zero, then we say that p is a (nilpotent) contact point between C_λ and $\mathcal{F}_\lambda$. The slow divergence integral can also be defined along parts of C_λ that contain contact points, using its invariance under smooth equivalences and normal forms near contact points (see [6, Section 5.5]).

We come now to a natural question: can we define the notion of slow divergence integral if we replace the slow–fast system $X_{\epsilon,\lambda}$ with a regularized planar PWS system? In Section 2 we give a positive answer to the question. Instead of $X_{0,\lambda}$ we consider a λ-family of planar PWS systems (2.1) defined in Section 2. The switching boundary Σ_λ defined after (2.1) plays the role of the curve of singularities C_λ, and the Filippov sliding vector field Z_λ^{sl} on sliding subsets of Σ_λ (see (2.2)) plays the role of the slow vector field $\bar{Q}_\lambda$ on normally hyperbolic portions of C_λ (see Proposition 2.1). The function that will be integrated (Definition 2.2 in Section 2.1) is the divergence of a smooth slow–fast vector field visible in the scaling chart of a cylindrical blow-up applied to regularized PWS system (2.4) (for more details see [25] and the proof of Proposition 2.1). The notion of slow divergence integral in the PWS setting is well-defined when the sliding vector field Z_λ^{sl} has no singularities (see Definition 2.2).

We show that the slow divergence integral from Definition 2.2 is invariant under smooth equivalences (see Theorem 2.4 in Section 2.1).

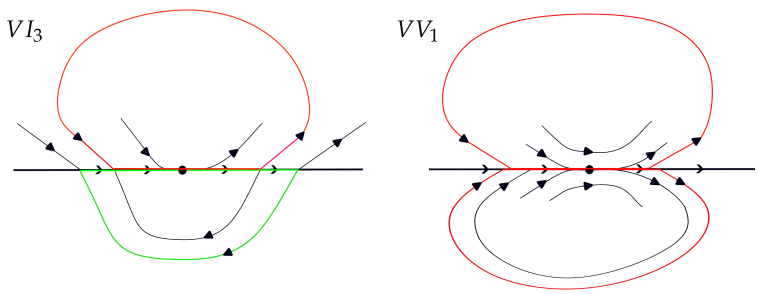

Figure 1.2: Limit periodic sets in planar PWS systems through two-fold points with sliding (the VI_3 case and the VV_1 case). They can be located in a region with invisible fold point (green) or in a region with visible fold point (red).

In Sections 2.2 and 2.3 we define the slow divergence integral near tangency points, as already mentioned above (tangency points in Σ_λ play the role of contact points between C_λ and $\mathcal{F}_\lambda$). We use the invariance of the slow divergence integral under smooth equivalence. The extension of the slow divergence integral to tangency points has proved to be crucial when we study the number of sliding limit cycles (i.e., isolated closed orbits with sliding segments) of a regularized planar PWS system produced by so-called canard limit periodic sets or canard cycles (for more details see [25]). In [25] only the VI_3 case has been studied, with canard cycles located in the region with invisible fold point (see the green closed curve in Fig. 1.2). Canard cycles contain both stable and unstable sliding portions of the discontinuity manifold (often called switching boundary). For example, it has been proved in [25] that the number of sliding limit cycles (produced by the canard cycles) of regularized quadratic PWS systems is unbounded.

A canard cycle can also be located in a region with visible fold point (for example, red closed curves in Fig. 1.2), and again the slow divergence integral associated to the segment of the switching boundary contained in the canard cycle plays an important role when studying sliding limit cycles (see [24]).

Besides sliding cycles, crossing limit cycles can exist for PWS systems and for example J. Llibre and co-workers have obtained upper bounds for a number of classes [13, 29, 32]. See also [2, 4, 17, 18, 20, 21, 30, 31] and references therein.

Section 3 is devoted to applications of the slow divergence integral from Section 2. In Section 3 we focus on the model used in [25] (visible-invisible two-fold VI_3) and read upper bounds of the number of sliding limit cycles and type of bifurcations near so-called generalized canard cycles (Fig. 3.1) from fractal properties of a bounded and monotone sequence in $\mathbb{R}$ defined using the slow divergence integral and the notion of slow relation function (see Section 3.1). The main advantage of this fractal approach is that, instead of computing the multiplicity of zeros of the slow divergence integral (like in [25]), it suffices to find the Minkowski dimension [14] of the sequence. There is a bijective correspondence between the multiplicity and the Minkowski dimension (see Section 4). A similar fractal approach has been used in [8, 22, 23, 26] where one deals with smooth slow–fast systems. See also [12, 35] and the references therein. We point out that there exist simple formulas for numerical computation of the Minkowski dimension of the sequence (see e.g. [23]). In Section 3.2 we state the main fractal results (Theorems 3.4–3.6), and in Section 4 we prove them.

For the sake of readability, in this paper we work in $\mathbb{R}^2$ using the Euclidean metric. We believe that the notion of slow divergence integral in regularized PWS systems on a smooth surface could also be defined. We point out that the slow divergence integral [6] is defined for slow–fast systems on a smooth surface.

2 The slow divergence integral in PWS systems with sliding

First we recall the basic definitions in PWS theory [9, 19] (switching boundary, sliding set, crossing set, sliding vector field, tangency point, two-fold singularity, etc.). Then we define the notion of slow divergence integral of a regularized PWS system along a sliding segment (not containing singularities of the sliding vector field) and prove that the integral is invariant under smooth equivalences (see Section 2.1). In Section 2.2 we extend the definition of the slow divergence integral to segments consisting of a stable sliding region, an unstable sliding region and a two-fold singularity between them. If the two-fold singularity is visible-invisible, then we assume that the sliding vector field is regular or has a hyperbolic singularity in the two-fold point. In Section 2.3 we define the slow divergence integral near a tangency point where the tangency (quadratic or more degenerate) appears only in one vector field.

Consider a λ-family of PWS systems in the plane

$$\dot{z} = \begin{cases} Z_\lambda^+(z) & \text{for } z \in \Sigma_\lambda^+, \\ Z_\lambda^-(z) & \text{for } z \in \Sigma_\lambda^-, \end{cases} \tag{2.1}$$

where $z = (x, y)$, $\lambda \sim \lambda_0 \in \mathbb{R}^l$ and the switching boundary is a smooth λ-family of 1-dimensional manifolds Σ_λ given by

$$\Sigma_\lambda = \{z \in \mathbb{R}^2 \mid h_\lambda(z) = 0\} \cap V,$$

with an open set V and a smooth family of functions h_λ such that $\nabla h_\lambda(z) \neq (0,0)$, $\forall z \in \Sigma_\lambda$. The switching boundary Σ_λ separates the open set $\Sigma_\lambda^+ = \{z \in V \mid h_\lambda(z) > 0\}$ from the open set $\Sigma_\lambda^- = \{z \in V \mid h_\lambda(z) < 0\}$. We assume that the λ-family of vector fields $Z_\lambda^+ = (X_\lambda^+, Y_\lambda^+)$ (resp. $Z_\lambda^- = (X_\lambda^-, Y_\lambda^-)$) is smooth in the closure of the λ-family Σ_λ^+ (resp. Σ_λ^-). In this paper "smooth" means "C^∞-smooth".

The subset $\Sigma_\lambda^{sl} \subset \Sigma_\lambda$ consisting of all points $z \in \Sigma_\lambda$ such that

$$Z_\lambda^+(h_\lambda)(z) Z_\lambda^-(h_\lambda)(z) < 0$$

is called the sliding set, where $Z_\lambda^\pm(h_\lambda)(z) := \nabla h_\lambda(z) \cdot Z_\lambda^\pm(z)$ is the Lie-derivative of h_λ with respect to the vector field $Z_\lambda^\pm$ at z. A sliding point $z \in \Sigma_\lambda^{sl}$ is stable (resp. unstable) if $Z_\lambda^+(h_\lambda)(z) < 0$ and $Z_\lambda^-(h_\lambda)(z) > 0$ (resp. $Z_\lambda^+(h_\lambda)(z) > 0$ and $Z_\lambda^-(h_\lambda)(z) < 0$). We write $\Sigma_\lambda^{sl} = \Sigma_\lambda^s \cup \Sigma_\lambda^u$ where Σ_λ^s (resp. Σ_λ^u) is the set of all stable (resp. unstable) sliding points. In Σ_λ^s (resp. Σ_λ^u) the vector fields $Z_\lambda^\pm$ point toward (resp. away from) the switching boundary. We call the set $\Sigma_\lambda^{cr} \subset \Sigma_\lambda$ of all points $z \in \Sigma_\lambda$ such that

$$Z_\lambda^+(h_\lambda)(z) Z_\lambda^-(h_\lambda)(z) > 0$$

the crossing set.

At each point $z \in \Sigma_\lambda^{cr}$ the orbit of (2.1) crosses the switching boundary Σ_λ (it is the concatenation of the orbit of Z_λ^+ and the orbit of Z_λ^- through z). Along the sliding set Σ_λ^{sl}, the flow is given by the Filippov sliding vector field [16]

$$Z_\lambda^{sl}(z) = \frac{1}{(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)} \left(Z_\lambda^+(h_\lambda) Z_\lambda^- - Z_\lambda^-(h_\lambda) Z_\lambda^+ \right)(z), \ z \in \Sigma_\lambda^{sl}. \tag{2.2}$$

The sliding vector field Z_λ^{sl} defined in (2.2) is tangent to Σ_λ^{sl}, i.e., $Z_\lambda^{sl}(z)$ is equal to the convex combination $\tau Z_\lambda^+(z) + (1-\tau) Z_\lambda^-(z)$ with

$$\tau = \tau_\lambda(z) = \frac{-Z_\lambda^-(h_\lambda)}{(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)}(z) \in]0,1[. \tag{2.3}$$

We say that $z \in \Sigma_\lambda^{sl}$ is a pseudo-equilibrium of (2.1) if $Z_\lambda^{sl}(z) = 0$.

A point $z \in \Sigma_\lambda$ where $Z_\lambda^+(h_\lambda)(z) = 0$ or $Z_\lambda^-(h_\lambda)(z) = 0$ is a PWS singularity called tangency. We say that $z \in \Sigma_\lambda$ is a fold singularity (or a fold point) of Z_λ^+ (resp. Z_λ^-) if $Z_\lambda^+(h_\lambda)(z) = 0$ and $(Z_\lambda^+)^2(h_\lambda)(z) \neq 0$ (resp. $Z_\lambda^-(h_\lambda)(z) = 0$ and $(Z_\lambda^-)^2(h_\lambda)(z) \neq 0$). The fold point is visible if $(Z_\lambda^+)^2(h_\lambda)(z) > 0$ (resp. $(Z_\lambda^-)^2(h_\lambda)(z) < 0$) and invisible if $(Z_\lambda^+)^2(h_\lambda)(z) < 0$ (resp. $(Z_\lambda^-)^2(h_\lambda)(z) > 0$).

We say that $z \in \Sigma_\lambda$ is a two-fold singularity if z is a fold point of both $Z_\lambda^\pm$. A two-fold singularity $z \in \Sigma_\lambda$ is said to be visible-visible (VV) if z is visible in both $Z_\lambda^\pm$, invisible-invisible (II) if z is invisible in both $Z_\lambda^\pm$, and visible-invisible (VI) if z is visible in Z_λ^+ and invisible in Z_λ^- or visible in Z_λ^- and invisible in Z_λ^+. Following [28], there exist 7 (generic) cases for two-fold singularities taking into account the direction of the flow of $Z_\lambda^\pm$ and Z_λ^{sl}: 2 visible-visible cases (denoted by VV_1 and VV_2 in [28]), 2 invisible-invisible cases (II_1 and II_2) and 3 visible-invisible cases (VI_1, VI_2 and VI_3). For more details we refer to [1,19,27,28]. In Section 2.2 we define the notion of slow divergence integral near two-fold singularities of sliding type (VV_1, II_1, VI_2 and VI_3). The four sliding cases are illustrated in Fig. 2.2. We also treat a visible-invisible two-fold singularity where the sliding vector field points toward (or away from) the two-fold singularity on both sides (Fig. 2.3).

We consider a regularized PWS system [25]

$$\dot{z} = \phi(h_\lambda(z)\epsilon^{-1}) Z_\lambda^+(z) + (1 - \phi(h_\lambda(z)\epsilon^{-1})) Z_\lambda^-(z) \tag{2.4}$$

where $0 < \epsilon \ll 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth regularization function. We assume that ϕ is strictly monotone, i.e.,

$$\phi'(u) > 0 \quad \text{for all } u \in \mathbb{R}, \tag{2.5}$$

and ϕ has the following asymptotics for $u \to \pm\infty$:

$$\phi(u) \to \begin{cases} 1 & \text{for } u \to \infty, \\ 0 & \text{for } u \to -\infty. \end{cases} \tag{2.6}$$

Moreover, we assume that ϕ is smooth at $\pm\infty$ in the following sense: The functions

$$\phi_+(u) := \begin{cases} 1 & \text{for } u = 0, \\ \phi(u^{-1}) & \text{for } u > 0, \end{cases} \quad \phi_-(u) := \begin{cases} \phi(-u^{-1}) & \text{for } u > 0, \\ 0 & \text{for } u = 0, \end{cases}$$

are smooth at $u = 0$.

Using the asymptotics of ϕ given in (2.6), the system (2.4) becomes the PWS system (2.1) in the limit $\epsilon \to 0$. Combining this with the fact that ϕ is smooth at $\pm\infty$, we have that, for z kept in any fixed compact set in $V \setminus \Sigma_\lambda$, the right hand side of (2.4) is an $o(1)$-perturbation of the right hand side of (2.1) where $o(1)$ is a smooth function in (z, ϵ, λ) and tends to 0 as $\epsilon \to 0$, uniformly in (z, λ). Thus, the PWS system (2.1) describes the dynamics of (2.4), for $\epsilon > 0$ small, as long as z is kept uniformly away from Σ_λ.

Near Σ_λ^{sl}, the dynamics of (2.4), with $0 < \epsilon \ll 1$, is given by Proposition 2.1 (see also [33]).

2.1 Definition and invariance of the slow divergence integral

Proposition 2.1. *Suppose that the PWS system (2.1) has a stable (resp. unstable) sliding point $p \in \Sigma_{\lambda_0}^{sl}$. Then, for each $0 < \epsilon \ll 1$ and $\lambda \sim \lambda_0$, (2.4) has a locally invariant manifold near p with foliation by stable (resp. unstable) fibers, and the reduced dynamics on this manifold (when $\epsilon \to 0$) is given by sliding vector field Z_λ^{sl} defined in (2.2).*

Proof. Without loss of generality, we can assume that $\frac{\partial h_{\lambda_0}}{\partial y}(p) \neq 0$. Then the switching boundary Σ_λ (locally near p) is the graph of a smooth function $y = f_\lambda(x)$. Using $h_\lambda(x, y) = \epsilon \tilde{y}$, the system (2.4) multiplied by $\epsilon > 0$ becomes a slow–fast system

$$\begin{aligned} \dot{x} &= \epsilon \left(\phi(\tilde{y}) X_\lambda^+(x, f_\lambda(x)) + (1 - \phi(\tilde{y})) X_\lambda^-(x, f_\lambda(x)) + O(\epsilon \tilde{y}) \right), \\ \dot{\tilde{y}} &= \phi(\tilde{y}) Z_\lambda^+(h_\lambda)(x, f_\lambda(x)) + (1 - \phi(\tilde{y})) Z_\lambda^-(h_\lambda)(x, f_\lambda(x)) + O(\epsilon \tilde{y}). \end{aligned} \tag{2.7}$$

When $\epsilon = 0$, the curve of singularities of (2.7) is given by $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$ where τ_λ is defined in (2.3). Each singularity $(x, \phi^{-1}(\tau_\lambda(x, f_\lambda(x))))$ is semi-hyperbolic with the nonzero eigenvalue equal to the divergence of the vector field (2.7), with $\epsilon = 0$, computed in that singularity:

$$(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)(x, f_\lambda(x)) \phi' \left(\phi^{-1}(\tau_\lambda(x, f_\lambda(x))) \right). \tag{2.8}$$

The reason why the eigenvalue in (2.8) is nonzero is because $Z_\lambda^+(h_\lambda) Z_\lambda^-(h_\lambda) < 0$ and $\phi' > 0$ (see (2.5)). The curve of singularities is attracting (resp. repelling) if p is a stable (resp. unstable) sliding point. The result follows now from Fenichel's theory [15]. Notice that the reduced dynamics of (2.7) along the curve of singularities is given by the vector field

$$\left(\tau_\lambda X_\lambda^+ + (1 - \tau_\lambda) X_\lambda^- \right)(x, f_\lambda(x)). \tag{2.9}$$

We divided the x-component in (2.7) by ϵ and let $\epsilon \to 0$ with $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$. We get the same expression (2.9) if we use the definition of the slow vector field introduced in Section 1. This completes the proof. □

Following [6, Chapter 5] or Section 1 in the smooth slow–fast system (2.7) one can define the notion of slow divergence integral along normally hyperbolic curve of singularities $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$ when the sliding vector field in (2.9) has no singularities: it is the integral of the divergence in (2.8) where the variable of integration is the time variable of the flow of the sliding vector field. This is our motivation for the definition of the notion of slow divergence integral of regularized PWS system (2.4) (see also [25]).

Definition 2.2 (Slow divergence integral). Let $m_\lambda \subset \Sigma_\lambda^{sl}$ be a bounded segment (Fig. 2.1) not containing pseudo-equilibria of the PWS system (2.1). Let $z_\lambda : [t_1, t_2] \to \mathbb{R}^2$ be a solution of $z'(t) = Z_\lambda^{sl}(z(t))$ where $z_\lambda(t_1)$ and $z_\lambda(t_2)$ are the end points of m_λ (z_λ is a parameterization of m_λ). Then we define the slow divergence integral of regularized PWS system (2.4) associated to m_λ as

$$I(m_\lambda) = \int_{t_1}^{t_2} E_\lambda(z_\lambda(t))dt \tag{2.10}$$

where

$$E_\lambda(z) = (Z_\lambda^+ - Z_\lambda^-)(h_\lambda)(z)\phi'\left(\phi^{-1}(\tau_\lambda(z))\right), \qquad z \in \Sigma_\lambda^{sl}.$$

Remark 2.3. Note that the definition of the slow divergence integral given by (2.10) is independent of the choice of z_λ. Indeed, if $\hat{z}_\lambda$ is another solution to $z'(t) = Z_\lambda^{sl}(z(t))$ and $p \in m_\lambda$, then there exist $\tilde{t} \in [t_1, t_2]$ and $\bar{t}$ such that $z_\lambda(\tilde{t}) = \hat{z}_\lambda(\bar{t}) = p$. Then we have $z_\lambda(t) = \hat{z}_\lambda(t + \bar{t} - \tilde{t})$ due to uniqueness of solutions. Now, we get

$$\int_{t_1+\bar{t}-\tilde{t}}^{t_2+\bar{t}-\tilde{t}} E_\lambda(\hat{z}_\lambda(s))ds = \int_{t_1}^{t_2} E_\lambda(z_\lambda(t))dt,$$

where we use the change of variable $s = t + \bar{t} - \tilde{t}$.

If m_λ is stable (resp. unstable), then $I(m_\lambda)$ is negative (resp. positive).

The slow divergence integral from Definition 2.2 is invariant under smooth equivalences (Theorem 2.4.1 and Theorem 2.4.2). Theorem 2.4.3 tells us how to compute $I(m_\lambda)$ for an arbitrary parameterization of m_λ (see also [6, Proposition 5.3]).

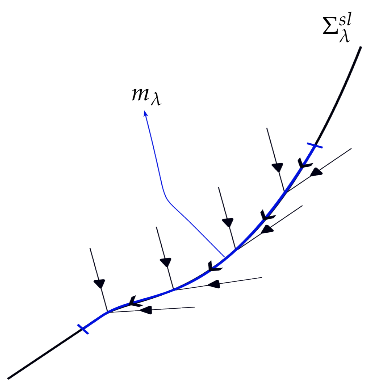

Figure 2.1: A segment $m_\lambda \subset \Sigma_\lambda^{sl}$ (blue).

Theorem 2.4 (Invariance of the slow divergence integral). *Let us denote the family of vector field in* (2.4) *by* $Z_{\epsilon,\lambda}$ *and let* $m_\lambda \subset \Sigma_\lambda^{sl}$ *be as in Definition* 2.2*. The following statements are true.*

1. *Let* $T : V_w \to V_z \subset V$ $(w \mapsto z = T(w))$ *be a smooth coordinate transformation, with open sets* $V_w, V_z \subset \mathbb{R}^2$*. Let* $I(m_\lambda)$ *be the slow divergence integral of* $Z_{\epsilon,\lambda}$ *along* $m_\lambda \subset V_z$*. Then the slow divergence integral* $I(T^{-1}(m_\lambda))$ *of the pullback of the vector field* $Z_{\epsilon,\lambda}|_{V_z}$ *along* $T^{-1}(m_\lambda) \subset V_w$ *is equal to* $I(m_\lambda)$*.*

2. *Let* g *be a smooth strictly positive function defined in a neighborhood of* m_λ*. Then the slow divergence integral of* $Z_{\epsilon,\lambda}$ *along* m_λ *is equal to the slow divergence integral of the equivalent vector field* $g.Z_{\epsilon,\lambda}$ *along* m_λ*.*

3. *Let* $p_\lambda : [v_1, v_2] \to \mathbb{R}^2$ *be a parameterization of* m_λ*. Then we have*

$$I(m_\lambda) = \int_{v_1}^{v_2} \frac{E_\lambda(p_\lambda(v))dv}{|\tilde{p}_\lambda(v)|},$$

where $\tilde{p}_\lambda$ *is a smooth* λ*-family of nowhere zero functions satisfying*

$$Z_\lambda^{sl}(p_\lambda(v)) = \tilde{p}_\lambda(v)p'_\lambda(v).$$

Proof. *Statement 1*. The pullback of the vector field $Z_{\epsilon,\lambda}|_{V_z}$ can be written as

$$T^*(Z_{\epsilon,\lambda}|_{V_z})(w) = \phi(h_\lambda \circ T(w)\epsilon^{-1})W_\lambda^+(w) + (1 - \phi(h_\lambda \circ T(w)\epsilon^{-1}))W_\lambda^-(w)$$

where $W_\lambda^\pm(w) = DT(w)^{-1}(Z_\lambda^\pm \circ T)(w)$. It is not difficult to see that the Lie-derivative of $h_\lambda \circ T$ with respect to the vector field $W_\lambda^\pm$ is given by

$$W_\lambda^\pm(h_\lambda \circ T)(w) = Z_\lambda^\pm(h_\lambda)(T(w)). \tag{2.11}$$

Using (2.11) and Definition 2.2 we find that the slow divergence integral of $T^*(Z_{\epsilon,\lambda}|_{V_z})$ along $T^{-1}(m_\lambda)$ is given by

$$I(T^{-1}(m_\lambda)) = \int_{t_1}^{t_2} E_\lambda(T(w_\lambda(t)))dt$$

where $w_\lambda : [t_1, t_2] \to T^{-1}(m_\lambda)$ is a solution of $w'(t) = W_\lambda^{sl}(w(t))$ (the Filippov sliding vector field along $T^{-1}(m_\lambda)$ is given by $W_\lambda^{sl}(w) = DT(w)^{-1}Z_\lambda^{sl}(T(w))$). Since $T \circ w_\lambda : [t_1, t_2] \to m_\lambda$ is a solution to $z'(t) = Z_\lambda^{sl}(z(t))$, the result follows.

Statement 2. From Definition 2.2 it follows that the slow divergence integral of $g.Z_{\epsilon,\lambda}$ along m_λ is equal to

$$I(m_\lambda) = \int_{\hat{t}_1}^{\hat{t}_2} E_\lambda(\hat{z}_\lambda(\hat{t}))g(\hat{z}_\lambda(\hat{t}))d\hat{t} \tag{2.12}$$

where $\hat{z}_\lambda : [\hat{t}_1, \hat{t}_2] \to m_\lambda$ and $\hat{z}'_\lambda(\hat{t}) = g(\hat{z}_\lambda(\hat{t}))Z_\lambda^{sl}(\hat{z}_\lambda(\hat{t}))$. We make in the integral in (2.12) the change of variable $t = \rho(\hat{t}) = \int_{\hat{t}_1}^{\hat{t}} g(\hat{z}_\lambda(v))dv$ with $\hat{t} \in [\hat{t}_1, \hat{t}_2]$. Then we have

$$\int_{\hat{t}_1}^{\hat{t}_2} E_\lambda(\hat{z}_\lambda(\hat{t}))g(\hat{z}_\lambda(\hat{t}))d\hat{t} = \int_0^{\rho(\hat{t}_2)} E_\lambda(\hat{z}_\lambda \circ \rho^{-1}(t))dt.$$

Since $(\hat{z}_\lambda \circ \rho^{-1})'(t) = Z_\lambda^{sl}((\hat{z}_\lambda \circ \rho^{-1})(t))$, $t \in [0, \rho(\hat{t}_2)]$, this integral is the slow divergence integral of $Z_{\epsilon,\lambda}$ associated to m_λ. This completes the proof of Statement 2.

Statement 3. The proof of Statement 3 is similar to the proof of Statement 2. □

Remark 2.5. It follows directly from Definition 2.2 that the slow divergence integral of $-Z_{\epsilon,\lambda}$ along m_λ is equal to the slow divergence integral of $Z_{\epsilon,\lambda}$ along m_λ multiplied by -1.

We will use the invariance of the slow divergence integral under smooth equivalences in Section 2.2 and Section 2.3.

If $m_\lambda \subset \Sigma_\lambda^{sl}$ contains pseudo-equilibria, then the slow divergence integral associated to m_λ is not well-defined.

2.2 The slow divergence integral near two-fold singularities

In this section we suppose that the sliding vector field Z_λ^{sl}, given by (2.2), is defined around a two-fold singularity. Our goal is to define the notion of slow divergence integral near such a two-fold singularity. Since the slow divergence integral (2.10) is invariant under smooth equivalences (Theorem 2.4), we use a normal form of (2.1), locally near the two-fold singularity, in which $h_\lambda(x,y) = y$ and the two-fold point corresponds to the origin $p = (0,0)$. Notice that such normal form coordinates exist because $\nabla h_\lambda(z) \neq (0,0)$, $\forall z \in \Sigma_\lambda$, in (2.1).

Using $h_\lambda(x,y) = y$ the two-fold p satisfies

$$Z_\lambda^\pm(h_\lambda)(0) = Y_\lambda^\pm(0) = 0, \quad (Z_\lambda^\pm)^2(h_\lambda)(0) = X_\lambda^\pm(0)\partial_x Y_\lambda^\pm(0) \neq 0, \tag{2.13}$$

and the sliding vector field Z_λ^{sl} near p can be written as

$$X_\lambda^{sl}(x) = \frac{\det Z_\lambda(x)}{(Y_\lambda^- - Y_\lambda^+)(x,0)} \tag{2.14}$$

where

$$\det Z_\lambda(x) := (X_\lambda^+ Y_\lambda^- - X_\lambda^- Y_\lambda^+)(x,0).$$

Remark 2.6. The notation $\det Z_\lambda$ comes from [1]. In [25] a similar notation has been used for $-(X_\lambda^+ Y_\lambda^- - X_\lambda^- Y_\lambda^+)$.

Since we assumed that the sliding vector field X_λ^{sl} is defined around the two-fold p, we find that $X_\lambda^+(0)X_\lambda^-(0) > 0$ if the folds have the same visibility (visible-visible or invisible-invisible) and $X_\lambda^+(0)X_\lambda^-(0) < 0$ if the folds have opposite visibility. We have $p \in \partial\Sigma_\lambda^s \cap \partial\Sigma_\lambda^u$. These properties follow directly from (2.13) and the definition of visible and invisible folds (see [1, Lemma 2.8]), and imply that $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ and $\partial_x Y_\lambda^+(0)\partial_x Y_\lambda^-(0) < 0$.

Using $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ it is clear that the sliding vector field in (2.14) has a removable singularity in $x = 0$ and

$$X_\lambda^{sl}(x) = \nu + O(x), \quad \nu = \frac{(\det Z_\lambda)'(0)}{\partial_x(Y_\lambda^- - Y_\lambda^+)(0)}. \tag{2.15}$$

From [1, Lemma 2.9] and [1, Corollary 2.10] it follows that $\nu \neq 0$ and $\operatorname{sgn}(\nu) = \operatorname{sgn}(X_\lambda^+(0))$ if the folds have the same visibility (VV_1 and II_1 in Fig. 2.2), and that $\nu \neq 0$ and $\operatorname{sgn}(\nu) = -\operatorname{sgn}\left(X_\lambda^+(0)(\det Z_\lambda)'(0)\right)$ if the folds have opposite visibility and $(\det Z_\lambda)'(0) \neq 0$ (VI_2 and VI_3 in Fig. 2.2). *If the folds have opposite visibility, we assume that* $\nu \neq 0$ *in* (2.15) *or* $x = 0$ *is a hyperbolic singularity of the sliding vector field* X_λ^{sl} *(or, equivalently,* $x = 0$ *is a zero of multiplicity* 1 *or* 2 *of the function* $\det Z_\lambda$*).* We refer to Fig. 2.3 (the multiplicity of the zero $x = 0$ of $\det Z_\lambda$ is 2).

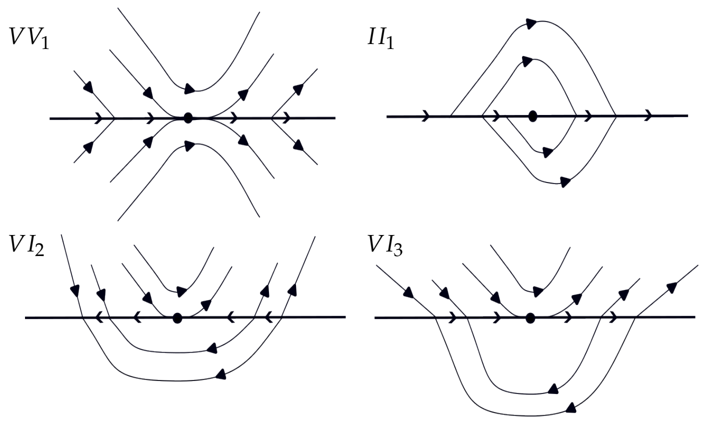

Figure 2.2: The different types of two-fold singularities with sliding: the folds in VV_1 and II_1 have the same visibility, while the folds in VI_2 and VI_3 have opposite visibility.

From $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ and $\partial_x Y_\lambda^+(0)\partial_x Y_\lambda^-(0) < 0$ it follows that the function τ_λ defined in (2.3) has the following property when $x \to 0$:

$$\lim_{x\to 0} \tau_\lambda(x,0) = \lim_{x\to 0} \frac{-Y_\lambda^-}{Y_\lambda^+ - Y_\lambda^-}(x,0) = \frac{\partial_x Y_\lambda^-(0)}{\partial_x(Y_\lambda^- - Y_\lambda^+)(0)} \in]0,1[. \tag{2.16}$$

Let us now compute the slow divergence integral along $[x_0, x_1]$, with $0 < x_0 < x_1$. We assume that x_1 is small enough such that $[x_0, x_1]$ does not contain any singularities of the sliding vector field X_λ^{sl}. We use Theorem 2.4.3. We take $p_\lambda(x) = (x,0)$, $x \in [x_0, x_1]$, in Theorem 2.4.3. Then we have

$$\tilde{p}_\lambda(x) = \frac{\det Z_\lambda(x)}{(Y_\lambda^- - Y_\lambda^+)(x,0)},\ E_\lambda(p_\lambda(v)) = (Y_\lambda^+ - Y_\lambda^-)(x,0)\phi'\left(\phi^{-1}\left(\tau_\lambda(x,0)\right)\right).$$

This implies

$$I([x_0,x_1]) = \int_{x_0}^{x_1} \frac{|Y_\lambda^- - Y_\lambda^+|(Y_\lambda^+ - Y_\lambda^-)(x,0)}{|\det Z_\lambda|(x)}\phi'\left(\phi^{-1}\left(\frac{-Y_\lambda^-}{Y_\lambda^+ - Y_\lambda^-}(x,0)\right)\right)dx. \tag{2.17}$$

Finally, we define the slow divergence integral along $[0, x_1]$ (the left end point of $[0, x_1]$ is the two-fold point).

Definition 2.7. Let $m_\lambda = [0, x_1]$. Then the slow divergence integral along m_λ is defined as

$$I(m_\lambda) = \lim_{x_0\to 0+} I([x_0, x_1])$$

where $I([x_0, x_1])$ is given in (2.17).

Remark 2.8. Notice that the function $x \mapsto \phi'\left(\phi^{-1}\left(\tau_\lambda(x,0)\right)\right)$ in (2.17) can be defined at $x = 0$ such that this function is smooth and positive on the segment m_λ (see (2.5), (2.6) and (2.16)). If the folds have the same visibility, then $I(m_\lambda)$ is well-defined (finite) because $\nu \neq 0$ in (2.15). Since we assume that the multiplicity of the zero $x = 0$ of $\det Z_\lambda$ does not exceed 2 when the folds have opposite visibility, $I(m_\lambda)$ is finite.

Remark 2.9. The slow divergence integral along $m_\lambda = [x_0, 0]$, with $x_0 < 0$, can be defined in a similar way: $I(m_\lambda) = \lim_{x_1\to 0-} I([x_0, x_1])$ where $I([x_0, x_1])$ has the same form (2.17).

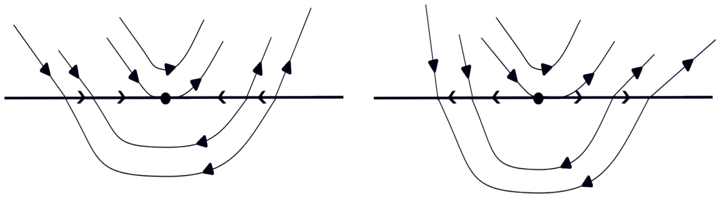

Figure 2.3: Non-generic visible-invisible two-fold singularities. The (extended) sliding vector field has a hyperbolic singularity at the two-fold points. (a) The sliding vector field points toward the two-fold singularity. (b) The sliding vector field points away from the two-fold singularity.

2.3 The slow divergence integral near one-sided tangency points

In this section we define the slow divergence integral near a tangency point $p \in \partial\Sigma_\lambda^s \cup \partial\Sigma_\lambda^u$ where both vectors $Z_\lambda^\pm(p)$ are nonzero and precisely one of them is tangent to Σ_λ at p (see e.g. Fig. 2.4). Like in Section 2.2, the switching boundary Σ_λ is locally given by $h_\lambda(x,y) = y$ and $p = (0,0)$. Since we suppose that $p \in \partial\Sigma_\lambda^s \cup \partial\Sigma_\lambda^u$, there is a side of p (without loss of generality we take $x > 0$) where the sliding vector field is defined, and given by (2.14). If the (nonzero) vector $Z_\lambda^+(0)$ (resp. $Z_\lambda^-(0)$) is tangent to Σ_λ, then $X_\lambda^+(0) \neq 0$, $Y_\lambda^+(0) = 0$ and $Y_\lambda^-(0) \neq 0$ (resp. $X_\lambda^-(0) \neq 0$, $Y_\lambda^-(0) = 0$ and $Y_\lambda^+(0) \neq 0$) and

$$X_\lambda^{sl}(x) = X_\lambda^+(0) + O(x) \quad \left(\text{resp. } X_\lambda^{sl}(x) = X_\lambda^-(0) + O(x)\right). \tag{2.18}$$

Since $X_\lambda^+(0) \neq 0$ (resp. $X_\lambda^-(0) \neq 0$), the sliding vector field X_λ^{sl} in (2.18) is regular near $x = 0$. Thus, the segment $[x_0, x_1]$, with $0 < x_0 < x_1$, does not contain any singularities of X_λ^{sl} if x_1 is small enough and we can define the slow divergence integral along $[x_0, x_1]$ exactly in the same way as in Section 2.2. The slow divergence integral is given by (2.17) and we use the same notation $I([x_0, x_1])$.

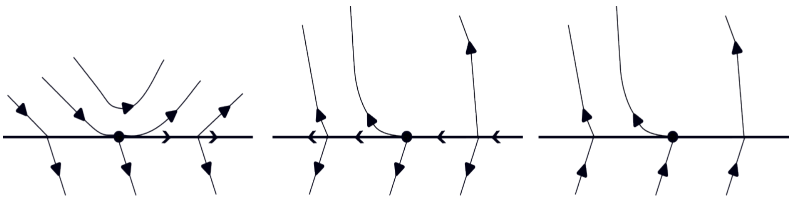

Figure 2.4: (a) The sliding vector field is defined on one side of the tangency point. (b) The sliding vector field is defined on both sides of the tangency point. (c) A crossing region around the tangency point (in this case the slow divergence integral near the tangency point is not defined).

We can now define the slow divergence integral near the tangency point p.

Definition 2.10. Let $m_\lambda = [0, x_1]$. Then the slow divergence integral along m_λ is defined as

$$I(m_\lambda) = \lim_{x_0 \to 0+} I([x_0, x_1]).$$

Remark 2.11. The slow divergence integral $I(m_\lambda)$ from Definition 2.10 is well-defined. Indeed, $\lim_{u\to\pm\infty}\phi'(u)=0$ (due to the smoothness of ϕ at $\pm\infty$ given after (2.6)). Moreover, we have (a) $\lim_{u\to 1-}\phi^{-1}(u)=+\infty$, (b) $\lim_{u\to 0+}\phi^{-1}(u)=-\infty$ (see (2.6)) and finally (c) $\frac{-Y_\lambda^-}{Y_\lambda^+-Y_\lambda^-}(x,0)$ tends to 1 (resp. 0) as $x\to 0+$ when $Z_\lambda^+(0)$ (resp. $Z_\lambda^-(0)$) is tangent to Σ_λ. It follows from (a), (b) and (c) that the integrand in (2.17) can be defined at $x=0$ (0 for $x=0$) and that the integrand is continuous on the segment m_λ. This implies that $I(m_\lambda)$ is well-defined.

3 Limit cycles and fractal analysis through visible-invisible two-fold VI_3

3.1 Model and assumptions

We consider a PWS system (2.1) where we assume that $\lambda\sim\lambda_0\in\mathbb{R}$, and $h_\lambda(x,y)=y$ (the switching boundary is the line $y=0$).

Assumption A. Suppose that there are $\eta_-<0$ and $\eta_+>0$ such that the PWS system (2.1) for $\lambda=\lambda_0$ has stable sliding for all $x\in[\eta_-,0[$ (i.e., $Y_{\lambda_0}^+(x,0)<0$ and $Y_{\lambda_0}^-(x,0)>0$ for $x\in[\eta_-,0[$) and unstable sliding for all $x\in]0,\eta_+]$ (i.e., $Y_{\lambda_0}^+(x,0)>0$ and $Y_{\lambda_0}^-(x,0)<0$ for $x\in]0,\eta_+]$). Moreover, we assume that the Filippov sliding vector field X_λ^{sl} given by (2.14) is positive for $x\in[\eta_-,\eta_+]\setminus\{0\}$ and $\lambda=\lambda_0$.

Assumption A implies that $Y_{\lambda_0}^\pm(0)=0$ and the origin $z=0$ is therefore a tangency point (see Section 2). We assume that $z=0$ for $\lambda=\lambda_0$ is a two-fold singularity. Moreover, we suppose that the two-fold singularity is visible from "above" and invisible from "below", i.e., the orbit of $Z_{\lambda_0}^+$ through $z=0$ is contained within $y>0$ near $z=0$, and the orbit of $Z_{\lambda_0}^-$ through $z=0$ is not contained within $y<0$ (Section 2).

Assumption B. We assume that the origin $z=0$ in the PWS system (2.1) is a visible-invisible two-fold for $\lambda=\lambda_0$: $Y_{\lambda_0}^\pm(0)=0$ and

$$\begin{cases} X_{\lambda_0}^+(0) & >0,\\ \partial_x Y_{\lambda_0}^+(0) & >0,\end{cases}\quad \begin{cases} X_{\lambda_0}^-(0) & <0,\\ \partial_x Y_{\lambda_0}^-(0) & <0.\end{cases} \tag{3.1}$$

Additionally, we assume that $(\det Z_{\lambda_0})'(0)<0$ where $\det Z_\lambda$ is defined in (2.14).

Remark 3.1. From (3.1) it follows that $\partial_x(Y_{\lambda_0}^- - Y_{\lambda_0}^+)(0)<0$. This, together with $(\det Z_{\lambda_0})'(0)<0$ and (2.15), implies that $X_{\lambda_0}^{sl}(0)>0$. Thus, $X_{\lambda_0}^{sl}(x)>0$ for all $x\in[\eta_-,\eta_+]$ (see Assumption A).

Assumption B and the Implicit Function Theorem imply the existence of smooth λ-families of fold singularities $z_+=(x_+(\lambda),0)$ from above and fold singularities $z_-=(x_-(\lambda),0)$ from below, for $\lambda\sim\lambda_0$, with $x_\pm(\lambda_0)=0$. The following assumption deals with non-zero velocity of the collision between z_+ and z_- for $\lambda=\lambda_0$ at the origin $z=0$:

$$x_+'(\lambda_0)-x_-'(\lambda_0)=\left(-\frac{\partial_\lambda Y_{\lambda_0}^+}{\partial_x Y_{\lambda_0}^+}+\frac{\partial_\lambda Y_{\lambda_0}^-}{\partial_x Y_{\lambda_0}^-}\right)(0)\neq 0$$

where $\partial_\lambda Y_{\lambda_0}^\pm$ means the partial derivative of $Y_\lambda^\pm$ w.r.t. λ, computed in $\lambda=\lambda_0$.

Assumption C. We assume that

$$\partial_\lambda Y_\lambda^- \partial_x Y_\lambda^+ \neq \partial_\lambda Y_\lambda^+ \partial_x Y_\lambda^- \tag{3.2}$$

at $(z, \lambda) = (0, \lambda_0)$.

We consider a regularized PWS system (2.4) with $h_\lambda(x, y) = y$:

$$\dot{z} = \phi(y\epsilon^{-2}) Z_\lambda^+(z) + (1 - \phi(y\epsilon^{-2})) Z_\lambda^-(z) \tag{3.3}$$

where $0 < \epsilon \ll 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth regularization function that satisfies the assumptions given after (2.4). More precisely, we have

Assumption D. We suppose that ϕ satisfies (2.5) and (2.6) and that ϕ is smooth at $\pm\infty$.

It is more convenient to write ϵ^{-2} in (3.3) instead of ϵ^{-1} so that we can directly use results from [25] (see Section 4).

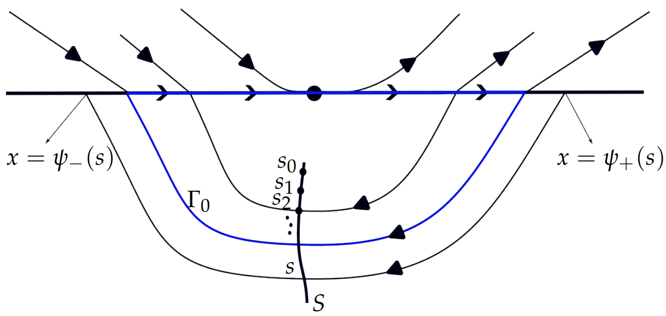

Figure 3.1: A fractal sequence $(s_n)_{n\in\mathbb{N}}$ near the canard cycle Γ_0.

Let S be an open section transversally cutting orbits of Z_λ^-, parametrized by a regular parameter $s \sim 0$ (Fig. 3.1). We assume that s increases as we approach the origin $z = 0$. For $\lambda = \lambda_0$, let Γ_s be the limit periodic set consisting of the orbit of $Z_{\lambda_0}^-$ connecting $(\psi_+(s), 0)$ and $(\psi_-(s), 0)$, and the segment $[\psi_-(s), \psi_+(s)] \subset \{y = 0\}$ (Fig. 3.1). We suppose that $[\psi_-(s), \psi_+(s)] \subset [\eta_-, \eta_+]$ for all $s \sim 0$. In [25] Γ_s is called a canard cycle. From the chosen parameterization of S it follows that $\psi_-'(s) > 0$ and $\psi_+'(s) < 0$. Following [25, Section 3], to study the number of limit cycles of (3.3) produced by Γ_s for $(\epsilon, \lambda) \sim (0, \lambda_0)$ one can use the slow divergence integral associated to the segment $[\psi_-(s), \psi_+(s)]$:

$$I(s) = \int_{\psi_-(s)}^{\psi_+(s)} \frac{(Y_{\lambda_0}^+ - Y_{\lambda_0}^-)^2(x,0)}{-\det Z_{\lambda_0}(x)} \phi' \left(\phi^{-1} \left(\frac{-Y_{\lambda_0}^-}{Y_{\lambda_0}^+ - Y_{\lambda_0}^-}(x,0) \right) \right) dx. \tag{3.4}$$

Remark 3.2. In (3.4) we use Definition 2.7 and Remark 2.9. Note that

$$I(s) = I(m_{\lambda_0}) + I(\tilde{m}_{\lambda_0})$$

where $m_{\lambda_0} = [0, \psi_+(s)]$ and $\tilde{m}_{\lambda_0} = [\psi_-(s), 0]$.

Remark 3.3. We suppose that Assumptions A through D are satisfied and write

$$\lambda = \lambda_0 + \epsilon\widetilde{\lambda}$$

with $\widetilde{\lambda} \sim 0$. We say that the cyclicity of the canard cycle Γ_0 inside (3.3) is bounded by $N \in \mathbb{N}$ if there exist $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $\mathcal{U}$ of 0 in the $\widetilde{\lambda}$-space such that (3.3) has at most N limit cycles, each lying within Hausdorff distance δ_0 of Γ_0, for all $(\epsilon, \widetilde{\lambda}) \in]0, \epsilon_0] \times \mathcal{U}$. We call the smallest N with this property the cyclicity of Γ_0 and denote it by $\text{Cycl}(\Gamma_0)$.

If the slow divergence integral I in (3.4) has a simple zero at $s = 0$, then $\text{Cycl}(\Gamma_0) = 2$ and, for each small $\epsilon > 0$, the $\widetilde{\lambda}$-family in (3.3) undergoes a saddle-node bifurcation of limit cycles near Γ_0 when we vary $\widetilde{\lambda} \sim 0$. Under the same assumption on I, there is a smooth function $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$, $\widetilde{\lambda}(0) = 0$, such that (3.3) with $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}(\epsilon)$ has a unique (hyperbolic) limit cycle Hausdorff close to Γ_0 for each small $\epsilon > 0$.

If I has a zero of multiplicity $m \geq 1$ at $s = 0$, then $\text{Cycl}(\Gamma_0) \leq m + 1$. When $I(0) < 0$ (resp. $I(0) > 0$), then $\text{Cycl}(\Gamma_0) = 1$, and the limit cycle is hyperbolic and attracting (resp. repelling).

We refer the reader to [25, Theorem 3.1] and [25, Remark 3.4].

We say that the canard cycle Γ_0 is balanced if $s = 0$ is a zero of $I(s)$ defined in (3.4) ($I(0) = 0$). If Γ_0 is balanced, then there exists a unique function $G(s)$ satisfying $G(0) = 0$, $G'(0) > 0$ and

$$\int_{\psi_-(s)}^{\psi_+(G(s))} \frac{(Y_{\lambda_0}^+ - Y_{\lambda_0}^-)^2(x,0)}{-\det Z_{\lambda_0}(x)} \phi' \left(\phi^{-1} \left(\frac{-Y_{\lambda_0}^-}{Y_{\lambda_0}^+ - Y_{\lambda_0}^-}(x,0) \right) \right) dx = 0 \tag{3.5}$$

for $s \sim 0$. This follows from the Implicit Function Theorem because $I(0) = 0$, $\psi_-'(s) > 0$, $\psi_+'(s) < 0$ and the integrand in (3.4) is negative for $x < 0$ and positive for $x > 0$ (see Assumptions A and D). We call G defined by (3.5) the slow relation function.

Assumption E. We suppose that Γ_0 is balanced and that $s = 0$ is an isolated zero of $I(s)$, meaning that $s = 0$ has a small neighborhood $]-\tilde{s}, \tilde{s}[$ ($\tilde{s} > 0$) that does not contain any other zero of $I(s)$.

Assumption E implies that I is either negative or positive for $s > 0$ (I is continuous). Using the above mentioned property of the integrand in (3.4) it can be easily seen that $0 < G(s) < s$ for $s > 0$ when I is negative and $G(s) > s$ for $s > 0$ when I is positive. Let $s_0 > 0$ be small and fixed. Thus, if I is negative (resp. positive), then the orbit of s_0

$$U_0 = \{s_0, s_1, s_2, \dots\} \tag{3.6}$$

defined by $s_{n+1} = G(s_n)$ (resp. $s_{n+1} = G^{-1}(s_n)$), $n \geq 0$, tends monotonically to the fixed point $s = 0$ of G. We want to study the Minkowski dimension of U_0.

Let us first define the notion of Minkowski (or box) dimension (see [14,34] and references therein). Let $U \subset \mathbb{R}^N$ be a bounded set. We define the δ-neighborhood of U:

$$U_\delta = \{x \in \mathbb{R}^N \mid d(x, U) \leq \delta\},$$

and denote by $|U_\delta|$ the Lebesgue measure of U_δ. The lower u-dimensional Minkowski content of U, for $u \geq 0$, is defined by

$$\mathcal{M}_*^u(U) = \liminf_{\delta \to 0} \frac{|U_\delta|}{\delta^{N-u}},$$

and analogously the upper u-dimensional Minkowski content $\mathcal{M}^{*u}(U)$ (we replace $\liminf_{\delta \to 0}$ with $\limsup_{\delta \to 0}$). We define lower and upper Minkowski dimensions of U:

$$\underline{\dim}_B U = \inf\{u \geq 0 \mid \mathcal{M}_*^u(U) = 0\}, \qquad \overline{\dim}_B U = \inf\{u \geq 0 \mid \mathcal{M}^{*u}(U) = 0\}.$$

We have $\underline{\dim}_B U \leq \overline{\dim}_B U$ and, if $\underline{\dim}_B U = \overline{\dim}_B U$, we call it the Minkowski dimension of U, and denote it by $\dim_B U$.

The upper Minkowski dimension is finitely stable. More precisely,

$$\overline{\dim}_B(U_1 \cup U_2) = \max\{\overline{\dim}_B U_1, \overline{\dim}_B U_2\}, \ U_1, U_2 \subset \mathbb{R}^N.$$

If $U_1 \subset U_2$, then $\underline{\dim}_B U_1 \leq \underline{\dim}_B U_2$ and $\overline{\dim}_B U_1 \leq \overline{\dim}_B U_2$ ($\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic).

Furthermore, if $0 < \mathcal{M}_*^d(U) \leq \mathcal{M}^{*d}(U) < \infty$ for some d, then we say that U is Minkowski nondegenerate. In this case we have necessarily that $d = \dim_B U$. Recall also that the notion of being Minkowski nondegenerate is invariant under bi-Lipschitz maps. Namely, if Φ is a bi-Lipschitz map and U is Minkowski nondegenerate, then $\Phi(U)$ is also Minkowski nondegenerate (see [36, Theorem 4.1]).

We use these properties in Section 4.2.

Following [12], $\dim_B U_0$ exists, it is independent of the choice of $s_0 > 0$ and can take only the following discrete set of values: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1$ (see also Theorem 4.1). The set U_0 is defined in (3.6).

3.2 Statement of results

In this section we consider the family (3.3) that satisfies Assumptions A through E and assume that $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}$ with $\widetilde{\lambda} \sim 0$.

Theorem 3.4. *Let $s_0 > 0$ be small and fixed and let U_0 be the orbit of s_0 defined in (3.6). If $\dim_B U_0 = 0$, then the following statements hold.*

1. *(λ unbroken) There exists a smooth function $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$, $\widetilde{\lambda}(0) = 0$, such that (3.3) with $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}(\epsilon)$ has a unique (hyperbolic) limit cycle Hausdorff close to Γ_0 for each small $\epsilon > 0$.*

2. *(λ broken) We have that $\text{Cycl}(\Gamma_0) = 2$ and, for every small $\epsilon > 0$, the $\widetilde{\lambda}$-family (3.3) undergoes a saddle-node bifurcation of limit cycles Hausdorff close to Γ_0.*

Theorem 3.4 will be proved in Section 4.1.

Theorem 3.5. *Let U_0 be the orbit of s_0 defined in (3.6), for a small $s_0 > 0$. If $\dim_B U_0 < 1$, then $\text{Cycl}(\Gamma_0) \leq \frac{2-\dim_B U_0}{1-\dim_B U_0}$.*

Theorem 3.5 will be proved in Section 4.1.

Theorem 3.6. *Let U_0 be the orbit of s_0 defined in (3.6), for a small $s_0 > 0$, and $\dim_B U_0 = 0$. The following statements are true.*

1. *For $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$ given in Theorem 3.4.1 and for each small $\epsilon > 0$, the Minkowski dimension of any spiral trajectory accumulating (in forward or backward time) on the unique limit cycle of (3.3) near Γ_0 is equal to 1.*

2. *For each small $\epsilon > 0$, the Minkowski dimension of any spiral trajectory accumulating (in forward or backward time) on the limit cycle of multiplicity 2 of (3.3), born in a saddle-node bifurcation of limit cycles Hausdorff close to Γ_0, is equal to $\frac{3}{2}$ and moreover, the spiral is Minkowski nondegenerate.*

Theorem 3.6 will be proved in Section 4.2. A small (Hausdorff) neighborhood of Γ_0 in which we consider spiral trajectories in Theorem 3.6.1 or Theorem 3.6.2 does not shrink to Γ_0 as $\epsilon \to 0$ (see Section 4.2).

4 Proof of Theorems 3.4–3.6

4.1 Proof of Theorems 3.4–3.5

Let $\tilde{s} > 0$ be small and fixed. Suppose that F is a smooth function on $[0, \tilde{s}[$, $F(0) = 0$ and $0 < F(s) < s$ for all $s \in]0, \tilde{s}[$. We define $H(s) := s - F(s)$ and the orbit of $s_0 \in]0, \tilde{s}[$ by H:

$$U := \{s_n = H^n(s_0) \mid n = 0, 1, \dots\}$$

where H^n denotes n-fold composition of H. It is clear that s_n tends monotonically to zero. We say that the multiplicity of the fixed point $s = 0$ of H is equal to m if $s = 0$ is a zero of multiplicity m of F ($F(0) = \cdots = F^{(m-1)}(0) = 0$ and $F^{(m)}(0) \neq 0$). If $F^{(n)}(0) = 0$ for each $n = 0, 1, \dots$, then we say that the multiplicity of $s = 0$ of H is ∞.

Theorem 4.1 ([12])**.** *Let F be a smooth function on $[0, \tilde{s}[$, $F(0) = 0$ and $0 < F(s) < s$ for each $s \in]0, \tilde{s}[$. Let $H = \text{id} - F$ and let U be the orbit of $s_0 \in]0, \tilde{s}[$ by H. Then $\dim_B U$ is independent of the initial point s_0 and, for $1 \leq m \leq \infty$, the following bijective correspondence holds:*

$$m = \frac{1}{1 - \dim_B U} \tag{4.1}$$

where m is the multiplicity of $s = 0$ of H (if $m = \infty$, then $\dim_B U = 1$).

If we denote by Φ the integrand in (3.4) and (3.5), then we have

$$I(s) = \int_{\psi_-(s)}^{\psi_+(s)} \Phi(x)dx = \int_{\psi_-(s)}^{\psi_+(G(s))} \Phi(x)dx + \int_{\psi_+(G(s))}^{\psi_+(s)} \Phi(x)dx = \int_{\psi_+(G(s))}^{\psi_+(s)} \Phi(x)dx$$

where in the last step we use (3.5). From $\psi_+'(s) < 0$, $\Phi(x) > 0$ for $x > 0$ and The Fundamental Theorem of Calculus it follows that there exists a negative smooth function $\Psi(s)$ such that

$$I(s) = \Psi(s)(s - G(s)).$$

This implies that $s = 0$ is a zero of multiplicity m of $I(s)$ if and only if $s = 0$ is a zero of multiplicity m of $s - G(s)$.

We will first suppose that the orbit U_0 in (3.6) is generated by the slow relation function G. If $\dim_B U_0 = 0$, then Theorem 4.1, with $H = G$, implies that the multiplicity of the fixed point $s = 0$ of G is 1. Thus, we have that $s = 0$ is a simple zero of I and Theorem 3.4.1 (resp. Theorem 3.4.2) follows directly from [25, Theorem 3.1] (resp. [25, Remark 3.4]). See also Remark 3.3. If $\dim_B U_0 < 1$, then the multiplicity of $s = 0$ of G is equal to $\frac{1}{1 - \dim_B U_0}$ (see (4.1)). Thus, $s = 0$ is a zero of multiplicity $\frac{1}{1 - \dim_B U_0}$ of I and [25, Remark 3.4]) implies that

$$\text{Cycl}(\Gamma_0) \leq 1 + \frac{1}{1 - \dim_B U_0} = \frac{2 - \dim_B U_0}{1 - \dim_B U_0}.$$

This completes the proof of Theorem 3.5.

If U_0 is generated by G^{-1}, Theorem 3.4 and Theorem 3.5 can be proved in the same way as above (we use Theorem 4.1 with $H = G^{-1}$ and the fact that G and G^{-1} have the same multiplicity of the fixed point $s = 0$).

4.2 Proof of Theorem 3.6

The Minkowski dimension of spiral trajectories accumulating on a hyperbolic or non-hyperbolic limit cycle of planar vector fields (without parameters) has been studied in [35,37]. We prove Theorem 3.6 for spiral trajectories in a Hausdorff neighborhood of the canard cycle Γ_0 that does not shrink to Γ_0 as $\epsilon \to 0$.

We will first prove Theorem 3.6.1. We assume that $\dim_B U_0 = 0$ and $\widetilde{\lambda}(\epsilon)$ is given in Theorem 3.4.1. Let $\bar{V}$ be a fixed neighborhood of Γ_0. Then the unique limit cycle of (3.3) with $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$ is located in $\bar{V}$ for each $\epsilon > 0$ small enough (see [25]). For such fixed $\epsilon > 0$, let Γ be any spiral trajectory in $\bar{V}$ accumulating on the limit cycle (in the forward time if the limit cycle is attracting or in the backward time if the limit cycle is repelling). We write $\Gamma = \tilde{\Gamma} \cup \bar{\Gamma}$ where $\bar{\Gamma}$ is the part of Γ sufficiently close to the limit cycle (we can apply the results of [35,37]) and $\tilde{\Gamma}$ is the rest of Γ (of finite length). It is clear that $\dim_B \tilde{\Gamma} = 1$ and $\overline{\dim}_B \bar{\Gamma} \geq 1$. Since $\underline{\dim}_B \leq \overline{\dim}_B$, $\underline{\dim}_B$ is monotonic and $\overline{\dim}_B$ is finitely stable (see Section 3.1), we have

$$\underline{\dim}_B \bar{\Gamma} \leq \underline{\dim}_B (\tilde{\Gamma} \cup \bar{\Gamma}) \leq \overline{\dim}_B (\tilde{\Gamma} \cup \bar{\Gamma}) = \max\{\overline{\dim}_B \tilde{\Gamma}, \overline{\dim}_B \bar{\Gamma}\} = \overline{\dim}_B \bar{\Gamma}. \tag{4.2}$$

Since the limit cycle is hyperbolic (see Theorem 3.4.1), [35, Theorem 10] implies that $\dim_B \bar{\Gamma} = \underline{\dim}_B \bar{\Gamma} = \overline{\dim}_B \bar{\Gamma} = 1$. Using (4.2) we obtain $\dim_B \Gamma = 1$. This completes the proof of Theorem 3.6.1.

The first part of Theorem 3.6.2 can be proved in the same way as Theorem 3.6.1. Since the non-hyperbolic limit cycle is generated by a saddle-node bifurcation of limit cycles we have $\dim_B \bar{\Gamma} = \underline{\dim}_B \bar{\Gamma} = \overline{\dim}_B \bar{\Gamma} = \frac{3}{2}$ (see [35, Theorem 10] and [37, Theorem 1]). To prove the claim about Minkowski nondegeneracy; first observe that $\mathcal{M}^{3/2}(\tilde{\Gamma}) = 0$ since $\dim_B(\tilde{\Gamma}) = 1 < 3/2$ so that this part does not affect the upper and lower Minkowski content of $\Gamma = \tilde{\Gamma} \cup \bar{\Gamma}$; hence, it is enough to show that $\bar{\Gamma}$ is Minkowski nondegenerate. To see this, we observe that $\bar{\Gamma}$ can be partitioned into finitely many pieces $\bar{\Gamma}_i$; $i = 1, \ldots, k$ such that each $\bar{\Gamma}_i$ is bi-Lipschitz equivalent to $[0, 1[\times U$ by the Lipschitz flow-box Theorem [3]. Note also that $\dim_B U = 1/2$ and it is Minkowski nondegenerate which implies that $[0, 1[\times U$ is also Minkowski nondegenerate; see the proof of [37, Theorem 4(b)]. Finally, the finite stability of Minkowski dimension and of Minkowski nondegeneracy now complete the proof exactly as in the proof of [37, Theorem 4(b)].

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare that they have no conflict of interest.

Authors' contributions All authors conceived of the presented idea, developed the theory, performed the computations and contributed to the final manuscript.

Funding The research of R. Huzak and G. Radunović was supported by: Croatian Science Foundation (HRZZ) grant PZS-2019-02-3055 from "Research Cooperability" program funded by the European Social Fund. Additionally, the research of G. Radunović was partially supported by the HRZZ grant UIP-2017-05-1020.

Availability of data and materials Not applicable.

References

[1] C. Bonet-Reves, J. Larrosa, T. M-Seara, Regularization around a generic codimension one fold-fold singularity, *J. Differential Equations* **265**(2018), No. 5, 1761–1838. https://doi.org/10.1016/j.jde.2018.04.047; MR3800102; Zbl 1393.34029

[2] D. C. Braga, L. F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane, *Nonlinear Dyn.* **73**(2013), No. 3, 1283–1288. https://doi.org/10.1007/s11071-013-0862-3; MR3083780; Zbl 1281.34037

[3] C. Calcaterra, A. Boldt, Lipschitz Flow-box Theorem, *J. Math. Anal. Appl.* **338**(2008), No. 2, 1108–1115. https://doi.org/10.1016/j.jmaa.2007.06.001; MR2386485; Zbl 1155.34033

[4] V. Carmona, F. Fernández-Sánchez, D. D. Novaes, Uniform upper bound for the number of limit cycles of planar piecewise linear differential systems with two zones separated by a straight line, *Appl. Math. Lett.* **137**(2023), 108501. https://doi.org/10.1016/j.aml.2022.108501; MR4513619; Zbl 1512.34071

[5] P. De Maesschalck, F. Dumortier, Classical Liénard equations of degree $n \geq 6$ can have $[\frac{n-1}{2}] + 2$ limit cycles, *J. Differential Equations* **250**(2011), No. 4, 2162–2176. https://doi.org/10.1016/j.jde.2010.12.003; MR2763568; Zbl 1215.37038

[6] P. De Maesschalck, F. Dumortier, R. Roussarie, , *Canard cycles—from birth to transition*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vol. 73, Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-79233-6; MR4304039; Zbl 1515.34002

[7] P. De Maesschalck, R. Huzak, Slow divergence integrals in classical Liénard equations near centers, *J. Dynam. Differential Equations* **27**(2015), No. 1, 177–185. https://doi.org/10.1007/s10884-014-9358-1; MR3317395; Zbl 1325.34040

[8] P. De Maesschalck, R. Huzak, A. Janssens, G. Radunović, Fractal codimension of nilpotent contact points in two-dimensional slow–fast systems, *J. Differential Equations* **355**(2023), 162–192. https://doi.org/10.1016/j.jde.2023.01.030; MR4543564; Zbl 1516.34087

[9] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, *Piecewise-smooth dynamical systems: theory and applications*, Applied Mathematical Sciences, Vol. 163, Springer-Verlag, 2008. https://doi.org/10.1007/978-1-84628-708-4; Zbl 1146.37003

[10] F. Dumortier, D. Panazzolo, R. Roussarie, More limit cycles than expected in Liénard equations, *Proc. Amer. Math. Soc.* **135**(2007), No. 6, 1895–1904. https://doi.org/10.1090/S0002-9939-07-08688-1; MR2286102; Zbl 1130.34018

[11] F. Dumortier, R. Roussarie, Canard cycles and center manifolds, *Mem. Amer. Math. Soc.* **121**(1996), No. 577, x+100. MR1327208; Zbl 0851.34057

[12] N. Elezović, V. Županović, D. Žubrinić, Box dimension of trajectories of some discrete dynamical systems, *Chaos Solitons Fractals* **34**(2007), No. 2, 244–252. https://doi.org/10.1016/j.chaos.2006.03.060; MR2327406; Zbl 1133.37007

[13] M. Esteban, J. Llibre, C. Valls, The 16th Hilbert problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line, *Chaos* **31**(2021), No. 4, 043112. https://doi.org/10.1063/5.0023055; MR4241107; Zbl 1470.34085

[14] K. Falconer, *Fractal geometry: mathematical foundations and applications*, John Wiley and Sons, Ltd., Chichester, 1990. MR1102677; Zbl 0689.28003

[15] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations* **31**(1979), No. 1, 53–98. https://doi.org/10.1016/0022-0396(79)90152-9; MR0524817; Zbl 0476.34034

[16] A. F. Filippov, *Differential equations with discontinuous righthand sides*, Mathematics and its Applications, Kluwer Academic Publishers, 1988. MR1028776; Zbl 0664.34001

[17] E. Freire, E. Ponce, F. Rodrigo, F. Torres, Bifurcation sets of continuous piecewise linear systems with two zones, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8**(1998), No. 11, 2073–2097. https://doi.org/10.1142/S0218127498001728; MR1681463; Zbl 0996.37065

[18] A. Gasull, J. Torregrosa, X. Zhang, Piecewise linear differential systems with an algebraic line of separation, *Electron. J. Differential Equations* **2020**(2020), No. 19, 1–14. MR4072179; Zbl 1440.34032

[19] M. Guardia, T. M. Seara, M. A. Teixeira, Generic bifurcations of low codimension of planar Filippov systems, *J. Differential Equations* **250**(2011), No. 4, 1967–2023. https://doi.org/10.1016/j.jde.2010.11.016; MR2763562; Zbl 1225.34046

[20] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, *J. Differential Equations* **248**(2010), No. 9, 2399–2416. https://doi.org/10.1016/j.jde.2009.10.002; MR2595726; Zbl 1198.34059

[21] S. M. Huan, X. S. Yang, On the number of limit cycles in general planar piecewise linear systems, *Discrete Contin. Dyn. Syst.* **32**(2012), No. 6, 2147–2164. https://doi.org/10.3934/dcds.2012.32.2147; MR2885803; Zbl 1248.34033

[22] R. Huzak, Box dimension and cyclicity of canard cycles, *Qual. Theory Dyn. Syst.* **17**(2018), No. 2, 475–493. https://doi.org/10.1007/s12346-017-0248-x; MR3810561; Zbl 1402.34063

[23] R. Huzak, V. Crnković, D. Vlah, Fractal dimensions and two-dimensional slow–fast systems, *J. Math. Anal. Appl.* **501**(2021), No. 2, 21 p. https://doi.org/10.1016/j.jmaa.2021.125212; MR4242049; Zbl 1470.34150

[24] R. Huzak, K. Uldall Kristiansen, General results on sliding cycles in regularized piecewise linear systems, in progress.

[25] R. Huzak, K. Uldall Kristiansen, The number of limit cycles for regularized piecewise polynomial systems is unbounded, *J. Differential Equations* **342**(2023), 34–62. https://doi.org/10.1016/j.jde.2022.09.028; MR4493144; Zbl 1512.34073

[26] R. Huzak, D. Vlah, Fractal analysis of canard cycles with two breaking parameters and applications, *Commun. Pure Appl. Anal.* **18**(2019), No. 2, 959–975. https://doi.org/10.3934/cpaa.2019047; MR3917688; Zbl 1411.34083

[27] K. Uldall Kristiansen, S. J. Hogan, Regularizations of two-fold bifurcations in planar piecewise smooth systems using blowup, *SIAM J. Appl. Dyn. Syst.* **14**(2015), No. 4, 1731–1786. https://doi.org/10.1137/15M1009731; MR3404676; Zbl 1353.37100

[28] Yu. A. Kuznetsov, S. Rinaldi, A. Gragnani, One parameter bifurcations in planar Filippov systems, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13**(2003), No. 8, 2157–2188. https://doi.org/10.1142/S0218127403007874; MR2012652; Zbl 1079.34029

[29] T. Li, J. Llibre, On the 16th Hilbert problem for discontinuous piecewise polynomial Hamiltonian systems, *J. Dynam. Differential Equations* **35**(2023), No. 1, 87–102. https://doi.org/10.1007/s10884-021-09967-3; MR4549811; Zbl 7735757

[30] J. Llibre, M. Ordóñez, E. Ponce, On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry, *Nonlinear Anal. Real World Appl.* **14**(2013), No. 5, 2002–2012. https://doi.org/10.1016/j.nonrwa.2013.02.004; MR3043136; Zbl 1293.34047

[31] J. Llibre, E. Ponce, Three nested limit cycles in discontinuous piecewise linear differential systems with two zones, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms* **19**(2012), No. 3, 325–335. MR2963277; Zbl 1268.34061

[32] J. Llibre, M. A. Teixeira, J. Torregrosa, Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **23**(2013), No. 4, 1350066. https://doi.org/10.1142/S0218127413500661; MR3063363; Zbl 1270.34018

[33] J. Sotomayor, M. A. Teixeira, Regularization of discontinuous vector fields, in: *International Conference on Differential Equations. Papers from the conference, EQUADIFF 95, Lisboa, Portugal, July 24–29, 1995*, pp. 207–223. MR1639359; Zbl 0957.37015

[34] C. Tricot, *Curves and fractal dimension,* With a foreword by Michel Mendès France, translated from the 1993 French original, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4612-4170-6; MR1302173; Zbl 0847.28004

[35] D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, *Bull. Sci. Math.* **129**(2005), No. 6, 457–485. https://doi.org/10.1016/j.bulsci.2004.11.007; MR2142893; Zbl 1076.37015

[36] D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some vector fields in $\mathbb{R}^3$, *C. R. Math. Acad. Sci. Paris* **342**(2006), No. 12, 959–963. https://doi.org/10.1016/j.crma.2006.04.021; MR2235618; Zbl 1096.37010

[37] D. Žubrinić, V. Županović, Poincaré map in fractal analysis of spiral trajectories of planar vector fields, *Bull. Belg. Math. Soc. Simon Stevin* **15**(2008), No. 5, 947–960. MR2484143; Zbl 153.37011