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Weighted Lorentz estimates for subquadratic quasilinear elliptic equations with measure data

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Abstract. In this work we mainly prove the following interior gradient estimates in weighted Lorentz spaces

$$g^{-1}\left[\mathcal{M}_1(\mu)\right] \in L^{q,r}_{w,loc}(\Omega) \implies |Du| \in L^{q,r}_{w,loc}(\Omega),$$

where $g(t) = ta(t)$ for $t \geq 0$ and $\mathcal{M}_1(\mu)(x)$ is the first-order fractional maximal function

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|},$$

for a class of non-homogeneous divergence quasilinear elliptic equations with measure data in the subquadratic case

$$-\operatorname{div}\left[a\left((ADu\cdot Du)^{\frac{1}{2}}\right)ADu\right] = \mu \quad \text{in } \Omega,$$

whose model cases are the classical elliptic p-Laplacian equation with measure data

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \quad \text{for } 1<p<2$$

and the elliptic p-Laplacian equation with the logarithmic term and measure data

$$-\operatorname{div}\left(|Du|^{p-2}\log\left(1+|Du|\right)Du\right) = \mu \quad \text{for } 1<p<2.$$

It deserves to be specially noted that the subquadratic case is a little different from the superquadratic case since as an example, the modulus of ellipticity in the above-mentioned special cases tends to infinity when $|Du| \to 0$ for $1<p<2$.

Keywords: weighted, Lorentz, gradient, subquadratic, quasilinear elliptic, measure, data.

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1 Introduction

In this paper we mainly study the local gradient estimates in weighted Lorentz spaces for the following non-homogeneous quasilinear elliptic equations with right-hand side measure in divergence form

$$-\operatorname{div}\left[a\left((ADu\cdot Du)^{\frac{1}{2}}\right)ADu\right]=\mu \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n\geq 2$, μ is a Borel measure with finite mass and $a:[0,\infty)\to[0,\infty)\in C^1[0,\infty)$ satisfies

$$-1<i_a:=\inf_{t>0}\frac{ta'(t)}{a(t)}\leq\sup_{t>0}\frac{ta'(t)}{a(t)}=:s_a<0 \quad \text{for any } t>0. \tag{1.2}$$

Moreover, the symmetric matrix $A(x)=\{a_{ij}(x)\}$ satisfies the following uniformly elliptic condition

$$\Lambda^{-1}|\xi|^2\leq A(x)\xi\cdot\xi\leq\Lambda|\xi|^2 \tag{1.3}$$

for every $\xi\in\mathbb{R}^n$, *a.e.* $x\in\mathbb{R}^n$ and some constant $\Lambda>0$. We remark that if $a(t)=t^{p-2}$ and A is the identity matrix I, then $i_a=s_a=p-2$ for $1<p<2$ and (1.1) is reduced to the classical elliptic p-Laplacian equation with right-hand side measure in divergence form

$$-\operatorname{div}\left(|Du|^{p-2}Du\right)=\mu \quad \text{for } 1<p<2. \tag{1.4}$$

It may be worthwhile to remark that another two natural examples of the functions a are $a(t)=t^{p-2}\log(1+t)$ for $1<p<2$, which makes (1.1) for $A=I$ is equal to

$$-\operatorname{div}\left(|Du|^{p-2}\log\left(1+|Du|\right)Du\right)=\mu,$$

and a more general example (see page 600 in [9] and page 314 in [46]), which is related to (p,q)-growth condition given by appropriate gluing of the monomials.

Define

$$g(t):=ta(t) \tag{1.5}$$

and

$$G(t):=\int_0^t g(\tau)\,d\tau=\int_0^t \tau a(\tau)\,d\tau \quad \text{for } t\geq 0. \tag{1.6}$$

From (1.2) we know that

$$g(t) \text{ is strictly increasing and continuous over } [0,+\infty), \tag{1.7}$$

and then

$$G(t) \text{ is increasing over } [0,+\infty) \text{ and strictly convex with } G(0)=0. \tag{1.8}$$

The partial differential equations involving measure data allow to consider various mathematical models in many interesting phenomena such as the blood flow in the heart [58] and state-constrained optimal control problems [23, 24]. The pointwise estimates of solutions to elliptic PDEs via suitable linear and nonlinear potentials of the right-hand side measure μ

were first investigated by Kilpeläinen & Malý [39, 40], in which they obtained the pointwise estimates for u in terms of nonlinear Wolff potentials $W^{\mu}_{\beta,p}$ defined by

$$W^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|\left(B(x,\varrho)\right)}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \text{for } \beta \in \left(0, \frac{n}{p}\right],$$

where

$$|\mu|(B(x,\varrho)) := \int_{B(x,\varrho)} |\mu(y)| dy.$$

Remarkably, such estimates played an essential role in the nonlinear potential theory (see [38, 60]). In more specific terms, Kilpeläinen & Malý [39, 40] proved the following estimate

$$|u(x_0)| \leq C(n,p)\left[W^{\mu}_{1,p}(x,R) + \left(\fint_{B(x,R)} |u|^{\gamma} dx\right)^{\frac{1}{\gamma}}\right], \quad \gamma > p-1 \tag{1.9}$$

with $B(x,R) \subseteq \Omega$ for solution to the p-Laplacian equation with right-hand side measure (1.4). Afterwards, Trudinger & Wang [64] used a different approach to prove the pointwise estimate via the nonlinear Wolff potential for the p-Laplacian operators. Later, Duzaar & Mingione [35, 51] extended (1.9) to the pointwise estimate at the gradient level

$$|Du(x_0)| \leq C(n,p)\left[\fint_{B(x_0,2R)} |Du| dx + W^{\mu}_{1/p,p}(x,2R)\right]$$

for solutions to the elliptic p-Laplacian equation (1.4) and more general case. In the subsequent papers, for the case $p \geq 2$ Kuusi & Mingione [44, 45] made a deep study of the pointwise estimates for gradient

$$|Du(x_0)| \leq C(n,p)\left[\fint_{B(x_0,2R)} |Du| dx + C\left(I^{|\mu|}_1(x_0,2R)\right)^{\frac{1}{p-1}}\right]$$

of solutions to (1.4) and more general case in terms of the linear Riesz potential of the right-hand side $I^{|\mu|}_1(x,R)$ which is defined by

$$I^{|\mu|}_1(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}} \frac{d\varrho}{\varrho}.$$

In particular, we mention here that Duzaar & Mingione [33] obtained gradient estimates via linear Riesz potentials

$$|Du(x_0)| \leq C\fint_{B(x_0,2R)} |Du| dx + C\left[I^{|\mu|}_1(x_0,2R)\right]^{\frac{1}{p-1}}$$

for solutions of the general case of the elliptic p-Laplacian equation for $2 - 1/n < p < 2$. We remark that the lower bound $2 - 1/n$ on the exponent p is to ensure $W^{1,1}$-solutions (see [33]). It deserves to be specially noted that Dong, Nguyen, Phuc & Zhu [32, 55, 57] also studied the local and global pointwise gradient estimates for solutions to the quasilinear elliptic equation with measure data $-\operatorname{div} A(x,Du) = \mu$ in the case $1 < p \leq 2 - 1/n$, whose prototype is given by the elliptic p-Laplace equation (1.4). Moreover, an extension of the previous results to a class of general elliptic equations

$$-\operatorname{div}\left[a\left(|Du|\right)Du\right] = \mu$$

including the p-Laplacian equation has been recently given by Baroni [7], in which the author proved the following pointwise gradient estimates via the linear Riesz potential

$$g(|Du(x_0)|) \leq Cg\left(\fint_{B(x_0,2R)} |Du| dx\right) + CI_1^{|\mu|}(x_0, 2R).$$

Actually, Cianchi & Maz'ya [26–28] have proved Lipschitz regularity and sharp estimates for weak solutions of

$$-\operatorname{div}\left(a\left(|Du|\right) Du\right) = f, \tag{1.10}$$

which is first introduced and studied by Lieberman [46] as the most natural and best generalization of the p-Laplacian equation. In the meanwhile, the authors [5, 6, 10, 21, 25, 30, 31, 52, 65] also studied regularity estimates of weak solutions for the quasilinear elliptic equations (1.10).

In a general way we call w belongs to the class of the Muckenhoupt weights A_p for some $p > 1$ if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ almost everywhere satisfies

$$\left(\fint_{B_r} w(x) dx\right)\left(\fint_{B_r} w(x)^{\frac{-1}{p-1}} dx\right)^{p-1} \leq C$$

for any ball B_r in $\mathbb{R}^n$. Moreover, we denote

$$A_\infty := \bigcup_{1<p<\infty} A_p \quad \text{and} \quad w(B_r) := \int_{B_r} w(x) dx.$$

Furthermore, the corresponding weighted Lebesgue space $L^p_w(B_r)$ consists of all functions h which satisfy

$$\|h\|_{L^p_w(B_r)} := \left(\int_{B_r} |h|^p \, w(x) dx\right)^{1/p} < \infty.$$

Now we give the following definition of weighted Lorentz spaces.

Definition 1.1. The weighted Lorentz space $L^{q,r}_w(\Omega)$ for any $0 < q < \infty$ and $0 < r \leq \infty$ is the set of all measurable functions h satisfying

$$\|h\|_{L^{q,r}_w(\Omega)} < \infty,$$

where

$$\|h\|_{L^{q,r}_w(\Omega)} := \begin{cases} \left[q \displaystyle\int_0^\infty \lambda^{r-1} w\left(\{x \in \Omega : |h(x)| > \lambda\}\right)^{\frac{r}{q}} d\lambda\right]^{\frac{1}{r}} & \text{for } r < \infty, \\ \displaystyle\sup_{\lambda>0} \lambda \, w\left(\{x \in \Omega : |h(x)| > \lambda\}\right)^{\frac{1}{q}} & \text{for } r = \infty. \end{cases}$$

Actually, the weighted Lebesgue space $L^q_w(\Omega) = L^{q,q}_w(\Omega)$ and Marcinkiewicz space $\mathcal{M}^q(\Omega) = L^{q,\infty}(\Omega)$.

Lemma 1.2 (see [16, 19, 47, 62, 63]). *Assume that $w \in A_p$ for some $p > 1$. Then there exists a small positive constant $\sigma > 0$ such that*

$$C_1 \left(\frac{|B_r|}{|B_R|}\right)^p \leq \frac{w(B_r)}{w(B_R)} \leq C_2 \left(\frac{|B_r|}{|B_R|}\right)^\sigma$$

for any balls $B_r \subset B_R \subset \mathbb{R}^n$, where $C_2 > 1$ and $C_1 > 0$.

There are various kinds of Calderón–Zygmund type estimates for the elliptic equations of p-Laplacican type (see, for example, [3, 8, 17, 29, 41, 47, 48] and the references therein). More to the point, Mingione [50] first proved the local sharp estimates in Lorentz spaces for the solutions to the following p-Laplacian type elliptic equation with measure data

$$-\operatorname{div} \mathbf{a}(x, Du) = \mu \quad \text{in } \Omega. \tag{1.11}$$

Furthermore, Phuc [59] obtained the following global weighted norm inequality in Lorentz spaces for gradients of solutions to (1.11)

$$(\mathcal{M}_1(\mu))^{\frac{1}{p-1}} \in L^{q,r}_w(\Omega) \Longrightarrow |Du| \in L^{q,r}_w(\Omega)$$

for $2 - 1/n < p \leq n$, any $q \in (0, +\infty)$ and $r \in (0, +\infty]$, where $\mathcal{M}_1(\mu)(x)$ is the first-order fractional maximal function

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|}, \quad x \in \mathbb{R}^n.$$

Subsequently, Nguyen & Phuc [54, 56] obtained existence and global regularity estimates for gradients of solutions to quasilinear elliptic equations with measure data, whose prototypes are of the form $-\operatorname{div}\left(|Du|^{p-2}Du\right) = \delta|Du|^q + \mu$ for $1 < p \leq 2 - 1/n$. In the meanwhile, Byun, Ok & Park [18] established the corresponding Calderón–Zygmund type estimates for quasilinear elliptic equations (1.11) with variable $p(x)$-growth involving measure data. Moreover, Byun, Cho & Youn [14] studied the existence of distributional solutions and the global Calderón–Zygmund type estimates to nonlinear elliptic problems (1.1) and more general case with the right-hand side Radon measure. Moreover, Avelin, Kuusi & Mingione [4] have investigated a limiting case of Calderón–Zygmund theory for a class of nonlinear elliptic equations modeled on the elliptic p-Laplacian equation with right-hand side measure (1.4). Motivated by the works mentioned above, our purpose of this paper is to establish the local weighted Lorentz gradient estimates for weak solutions of the problem (1.1) with the condition (1.2) in the case $-1/n < i_a \leq s_a < 0$. More precisely, we shall prove that

$$g^{-1}\left[\mathcal{M}_1(\mu)\right] \in L^{q,r}_{w,loc}(\Omega) \implies |Du| \in L^{q,r}_{w,loc}(\Omega).$$

We now state the definition of weak solutions.

Definition 1.3. A function $u \in W^{1,G}_{loc}(\Omega)$ (see Definition 2.4) is a local weak solution of (1.1) if for any $\varphi \in W^{1,G}_0(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega a\left((ADu \cdot Du)^{\frac{1}{2}}\right) ADu \cdot D\varphi dx = \int_\Omega \varphi d\mu.$$

In this work we shall assume that the coefficients of $A = \{a_{ij}\}$ are in the BMO space and their semi-norms are small enough. Higher integrability of solutions to various kinds of elliptic/parabolic PDEs with discontinuous coefficients of VMO/BMO type has been extensively studied by many authors (see [2, 15, 20, 41, 43]). We would like to point out that a function satisfies the small BMO condition if it satisfies the VMO condition. More precisely, we use the following small BMO condition.

Definition 1.4. We say that the matrix A of coefficients is (δ, R)-vanishing if

$$\sup_{0<r\leq R} \sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A(y) - \overline{A}_{B_r(x)}\right| dy \leq \delta,$$

where

$$\overline{A}_{B_r(x)} = \fint_{B_r(x)} A(y)\, dy.$$

The main result of this work is stated as follows. First of all, we remark that the following conclusion is stated as a priori estimate for weak solutions. Actually, solutions to measure data problems (very weak solutions) are usually found by approximation procedures. So, they are often called SOLA (Solutions Obtained by Limiting Approximation). We can refer to the relevant existence theory in the papers [11–13, 37, 40]. In the following we shall mention a space $W^{1,f}(\Omega)$, where

$$f(t) := \int_0^t \frac{g(s)}{s} ds,$$

whose definition is just like Section 3.2 in [7]. More precisely, the exact definition of SOLA is given as follows: a function $u \in W^{1,f}_{loc}(\Omega)$ is a local SOLA of (1.1) if

$$\int_\Omega a\left((ADu \cdot Du)^{\frac{1}{2}}\right) ADu \cdot D\varphi dx = \int_\Omega \varphi d\mu$$

holds for any $\varphi \in C_0^\infty(\Omega)$, and moreover there exists a sequence of weak solutions $\{u_k\} \in W^{1,G}_{loc}(\Omega)$ of

$$-\operatorname{div}\left(a\left((ADu_k \cdot Du_k)^{\frac{1}{2}}\right) ADu_k\right) = \mu_k \quad \text{in } \Omega, \tag{1.12}$$

such that $u_k \to u$ in $W^{1,f}_{loc}(\Omega)$, where $\{\mu_k\} \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure. In particular, we shall assume that $-1/n < i_a \le s_a < 0$ in the theorem below just like in the paper of Duzaar & Mingione [33], in which they supposed that $p > 2 - 1/n$ for the elliptic p-Laplacian equations and general case.

Now we shall give a concrete conclusion of this paper.

Theorem 1.5. *Suppose that $\mu \in L^\infty(\Omega)$ and $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) in $\Omega \supset B_2$ for $-1/n < i_a \le s_a < 0$. Then we have*

$$g^{-1}\left[\mathcal{M}_1(\mu)\right] \in L^{q,r}_{w,loc}(\Omega) \Longrightarrow |Du| \in L^{q,r}_{w,loc}(\Omega)$$

for any $q \in (1, \infty)$ and $r \in (0, \infty]$, with the estimate

$$\|Du\|_{L^{q,r}_{w,loc}(B_1)} \le C \int_{B_2} (|Du| + 1)\, dx + C \|g^{-1}\left[\mathcal{M}_1(\mu)\right)\|_{L^{q,r}_{w,loc}(B_2)},$$

where C is independent of u and μ.

2 Proof of the main result

In this section we shall finish the proof of the main result in this work, Theorem 1.5. First of all, we shall give some definitions on the general Orlicz spaces, which have been extensively studied in the area of analysis (see [1, 53]) and play a crucial role in many fields of mathematics including geometric, probability theory, stochastic analysis, Fourier analysis, partial differential equations and so on (see [61]).

Definition 2.1. A function G belongs to Φ, which consists of all increasing and convex functions $G : [0, +\infty) \to [0, +\infty)$, is said to be a Young function if

$$\lim_{t \to 0+} \frac{G(t)}{t} = \lim_{t \to +\infty} \frac{t}{G(t)} = 0.$$

Additionally, a Young function G is said to $G \in \Delta_2$ if there exists $M > 0$ such that

$$G(2t) \leq MG(t) \quad \text{for any } t > 0. \tag{2.1}$$

Moreover, we call a Young function $G \in \nabla_2$ if there exists a number $a > 1$ such that

$$G(t) \leq \frac{G(at)}{2a} \quad \text{for any } t > 0. \tag{2.2}$$

Example 2.2.

(1) $G_1(t) = (1+t)\log(1+t) - t \in \Delta_2$, but $G_1(t) \notin \nabla_2$.

(2) $G_2(t) = e^t - t - 1 \in \nabla_2$, but $G_2(t) \notin \Delta_2$.

(3) $G_3(t) = t^p \log(1+t) \in \Delta_2 \cap \nabla_2$ for $p > 1$.

Remark 2.3. Actually, if $G \in \Delta_2 \cap \nabla_2$, then we have

$$G(\theta_1 t) \leq K\theta_1^{\beta_1} G(t) \quad \text{and} \quad G(\theta_2 t) \leq 2a\theta_2^{\beta_2} G(t) \tag{2.3}$$

for every $t > 0$ and $0 < \theta_2 \leq 1 \leq \theta_1 < \infty$, where $\beta_1 = \log_2 M \geq \beta_2 = \log_a 2 + 1 > 1$.

Definition 2.4. Assume that G is a Young function. Then the Orlicz class $K^G(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} G(|f|)\, dx < \infty.$$

The Orlicz space $L^G(\mathbb{R}^n)$ is the linear hull of $K^G(\mathbb{R}^n)$ and $W^{1,G}(\mathbb{R}^n) := \{f \in L^G(\mathbb{R}^n) \mid Df \in L^G(\mathbb{R}^n)\}$.

Moreover, in this work we need the following crucial lemmas, which will be used in the subsequent proof.

Lemma 2.5 ([1]). *Let G be a Young function satisfying $G \in \Delta_2 \cap \nabla_2$. Then*

(1) $K^G(\Omega) = L^G(\Omega)$.

(2) $C_0^\infty(\Omega)$ *is dense in* $L^G(\Omega)$.

(3) $L^{\beta_1}(\Omega) \subset L^G(\Omega) \subset L^{\beta_2}(\Omega) \subset L^1(\Omega)$ *with* $\beta_1 \geq \beta_2 > 1$ *as in* (2.3).

(4) If $f \in L^G(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} G(|f|)\, dx = \int_0^\infty |\{x \in \mathbb{R}^n : |f| > \mu\}|\; d\,[G(\mu)]\,.$$

(5)

$$st \leq \epsilon \widetilde{G}(s) + C(\epsilon) G(t) \quad \text{for any } s, t \geq 0 \text{ and } \epsilon > 0,$$

where $\widetilde{G}$ *is the conjugate function of* G

$$\widetilde{G}(t) := \sup_{s \geq 0} \{st - G(s)\} \quad \text{for any } t \geq 0.$$

Now we shall recall the following results, which can be derived from Proposition 2.9 of [26], Lemma 1.9 and Lemma 2.4 of [65] and the change of variable.

Lemma 2.6. *Assume that $a(t)$ satisfies (1.2) for $s_a < 0$ and $G(t) = \int_0^t \tau a(\tau)\,d\tau$ for $t \geq 0$ is defined in (1.6).*

1. *For any $t > 0$ we find that*
$$\theta^{i_a} \leq \frac{a(\theta t)}{a(t)} \leq \theta^{s_a} \quad \text{and} \quad \theta^{2+i_a} \leq \frac{G(\theta t)}{G(t)} \leq \theta^{2+s_a} \quad \text{for any } \theta \geq 1. \tag{2.4}$$
2. *$G(t) \in \nabla_2 \cap \Delta_2$ and $\widetilde{G}(g(t)) \leq CG(t)$ for $t \geq 0$.*
3. *There exist $C = C(n, i_a, s_a) > 0$ and $\epsilon_0 = \epsilon_0(n, i_a, s_a) > 0$ we have*
$$G\left(|\xi - \eta|\right) \leq C(\epsilon) \left[a\left((A\xi \cdot \xi)^{\frac{1}{2}} \right) A\xi - a\left((A\eta \cdot \eta)^{\frac{1}{2}} \right) A\eta \right] \cdot (\xi - \eta) + \epsilon G\left(|\eta|\right)$$
for any $\xi, \eta \in \mathbb{R}^n$ and small positive constant $\epsilon \in (0, \epsilon_0)$.

Next, we can obtain the following important results for $s_a < 0$.

Lemma 2.7. *Assume that $a(t)$ satisfies (1.2) and $s_a < 0$, $G(t)$ is defined in (1.6) and*
$$V(z) = \sqrt{a\left(|z|\right)}z. \tag{2.5}$$
Then for any $\xi, \eta \in \mathbb{R}^n$ there exists $C = C(n, i_a, s_a) > 0$ we have
$$Ca\left(|\xi| + |\eta|\right) |\xi - \eta|^2 \leq |V(\xi) - V(\eta)|^2 \leq Ca\left(|\xi| + |\eta|\right) |\xi - \eta|^2, \tag{2.6}$$
$$C\sqrt{a\left(|\xi| + |\eta|\right)}\, |\xi - \eta|^2 \leq \left[V\left(\xi\right) - V\left(\eta\right)\right] \cdot \left(\xi - \eta\right) \leq C\sqrt{a\left(|\xi| + |\eta|\right)}\, |\xi - \eta|^2 \tag{2.7}$$
and
$$\left[a\left((A\xi \cdot \xi)^{\frac{1}{2}} \right) A\xi - a\left((A\eta \cdot \eta)^{\frac{1}{2}} \right) A\eta \right] \cdot (\xi - \eta) \geq C\left|V(\xi) - V(\eta)\right|^2. \tag{2.8}$$

Proof. We first find that
$$\begin{aligned}
&V\left(\xi\right) - V\left(\eta\right) \\
&= \sqrt{a\left(|\xi|\right)}\xi - \sqrt{a\left(|\eta|\right)}\eta \\
&= \left(\xi - \eta\right) \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)}\, ds \\
&\quad + \frac{1}{2} \int_0^1 \frac{a'\left(|s\xi + (1-s)\eta|\right)}{|s\xi + (1-s)\eta|} \frac{1}{\sqrt{a\left(|s\xi + (1-s)\eta|\right)}} \left(s\xi + (1-s)\eta\right) \left[s\xi + (1-s)\eta\right] \cdot \left(\xi - \eta\right) ds.
\end{aligned}$$
Then from (1.2) we deduce that
$$\begin{aligned}
\left|V\left(\xi\right) - V\left(\eta\right)\right| &\leq |\xi - \eta| \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)} ds - \frac{i_a}{2} |\xi - \eta| \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)} ds \\
&= \left(1 - \frac{i_a}{2}\right) |\xi - \eta| \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)}\, ds.
\end{aligned} \tag{2.9}$$
Similarly, we have
$$\left|V\left(\xi\right) - V\left(\eta\right)\right| \geq \left(1 + \frac{i_a}{2}\right) |\xi - \eta| \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)}\, ds, \tag{2.10}$$

$$[V(\xi) - V(\eta)] \cdot (\xi - \eta) \geq \left(1 + \frac{i_a}{2}\right) |\xi - \eta|^2 \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)}\, ds \tag{2.11}$$

and

$$[V(\xi) - V(\eta)] \cdot (\xi - \eta) \leq \left(1 - \frac{i_a}{2}\right) |\xi - \eta|^2 \int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)}\, ds. \tag{2.12}$$

In view of the facts that $a(t)$ is strictly decreasing and $|s\xi + (1-s)\eta| \leq |\xi| + |\eta|$ for any $0 \leq s \leq 1$, we find that

$$\int_0^1 \sqrt{a\left(|s\xi + (1-s)\eta|\right)} ds \geq \int_0^1 \sqrt{a\left(|\xi| + |\eta|\right)} ds = \sqrt{a\left(|\xi| + |\eta|\right)}, \tag{2.13}$$

which implies that the left-hand inequalities of (2.6) and (2.7) hold true. On the other hand, we define

$$s_0 := \frac{|\xi - \eta_0|}{|\xi - \eta|},$$

where η_0 is the minimum norm point on the line through ξ and η. Without loss of generality we may as well assume that $|\xi| \geq |\eta| > 0$. It is easy to check that $s_0 \geq \frac{1}{2}$. The following two cases shall be considered separably.

Case 1: $s_0 \geq 1$. Then $|s\eta + (1-s)\xi| \geq |s\eta_0 + (1-s)\xi| \geq |s0 + (1-s)\xi| = (1-s)\,|\xi| \geq \frac{(1-s)}{2}\,(|\xi| + |\eta|)$ for any $s \in [0,1]$ and $|\xi| \geq |\eta| > 0$. Furthermore, from Lemma 2.6 (1) and the decreasing property of $a(t)$ we conclude that

$$\begin{aligned}\int_0^1 \sqrt{a\left(|s\eta + (1-s)\xi|\right)} ds &\leq \int_0^1 \sqrt{a\left(\frac{(1-s)}{2}\,(|\xi| + |\eta|)\right)} ds \\ &\leq C\sqrt{a\left(|\xi| + |\eta|\right)} \int_0^1 (1-s)^{\frac{i_a}{2}} ds \\ &\leq C\sqrt{a\left(|\xi| + |\eta|\right)}. \end{aligned} \tag{2.14}$$

Case 2: $\frac{1}{2} \leq s_0 < 1$. Recalling the definition of η_0 and choosing $s = \theta s_0$, we have

$$\begin{aligned}\int_0^1 \sqrt{a(|s\eta + (1-s)\xi|)} \mathrm{d}s &\leq 2\int_0^{s_0} \sqrt{a(|s\eta + (1-s)\xi|)} \mathrm{d}s \\ &\leq C\int_0^1 \sqrt{a(|\theta\eta_0 + (1-\theta)\xi|)} \mathrm{d}\theta \\ &\leq C\int_0^1 \sqrt{a\left((1-\theta)\,|\xi|\right)} d\theta, \end{aligned}$$

in view of the facts that $|\theta\eta_0 + (1-\theta)\xi| \geq |\theta 0 + (1-\theta)\xi| = (1-\theta)\,|\xi|$ for any $\theta \in [0,1]$ and $a(t)$ is decreasing. Similarly to Case 1, we find that

$$\int_0^1 \sqrt{a\left(|s\eta + (1-s)\xi|\right)} ds \leq C\sqrt{a\left(|\xi| + |\eta|\right)}. \tag{2.15}$$

Therefore, from (2.9)–(2.15) we can conclude that the right-hand inequalities of (2.6) and (2.7) are true.

For the sake of clarity and brevity, we may as well assume that $A = I$ in the following proof. First of all, we can compute as follows

$$\xi a(|\xi|) - \eta a(|\eta|) = (\xi - \eta) \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + \int_0^1 \frac{a'(|s\xi + (1-s)\eta|)}{|s\xi + (1-s)\eta|} (s\xi + (1-s)\eta) [s\xi + (1-s)\eta] \cdot (\xi - \eta)\, ds,$$

which implies that

$$\begin{aligned} &[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \\ &\geq |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + i_a \int_0^1 a(|s\xi + (1-s)\eta|) \left| \frac{[s\xi + (1-s)\eta] \cdot (\xi - \eta)}{|s\xi + (1-s)\eta|} \right|^2 ds \\ &\geq (1 + i_a) |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds \end{aligned}$$

in view of (1.2). Then similarly to (2.13), we find that

$$\int_0^1 a(|s\xi + (1-s)\eta|)\, ds \geq a(|\xi| + |\eta|),$$

which implies that

$$[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \geq C a(|\xi| + |\eta|) |\xi - \eta|^2. \tag{2.16}$$

Thus, from (2.6) and (2.16) we can obtain (2.8) and then finish the proof. □

For a locally integrable function f in $\mathbb{R}^n$, we define its Hardy–Littlewood maximal function $\mathcal{M}(f)(x)$ as

$$\mathcal{M}(f)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

If f is not defined outside a bounded domain Ω, then we let f be zero in the above definition if x leaves Ω. Moreover, we can obtain the following basic properties for the Hardy–Littlewood maximal functions.

Lemma 2.8 (see [42]).

1. *If $f \in L^1(\Omega)$, then we have the weak* 1-1 *estimate*

$$|\{x \in \Omega : (\mathcal{M}f)(x) > \lambda\}| \leq \frac{C_3}{\lambda} \int_\Omega |f(x)|\, dx \quad \text{for some constant } C_3 > 0. \tag{2.17}$$

2. *If $f \in L^G(\Omega)$ for $G \in \Delta_2 \cap \nabla_2$, then we have $\mathcal{M}f \in L^G(\Omega)$ with the estimates*

$$\frac{1}{C} \int_\Omega G(|f|)\, dx \leq \int_\Omega G(\mathcal{M}f)\, dx \leq C \int_\Omega G(|f|)\, dx.$$

In this paper we shall use the following version of the weighted Vitali covering lemma, which will be a crucial ingredient in obtaining our main result.

Lemma 2.9 ([59, Lemma 3.4]). *Assume that E and F are measurable sets, $E \subset F \subset B_1$, and that there exists an $\epsilon > 0$ such that $w(E) < \epsilon w(B_1)$ and that for all $x \in B_1$ and for all $r \in (0,1]$ with $w(E \cap B_r(x)) \geq \epsilon w(B_r(x))$ we have $B_r(x) \cap B_1 \subset F$. Then, we have*

$$w(E) \leq C\epsilon w(F).$$

Moreover, we shall also use the following standard arguments of measure theory.

Lemma 2.10 (see [22,59]). *Assume that $r \in (0, +\infty)$ and f is a nonnegative and measurable function in Ω. Let $m > 1$ be a constant. Then for $0 < q < \infty$ we have*

$$f \in L_w^{q,r}(\Omega) \ \text{ iff } S := \sum_{i \geq 1} m^{ir} \left[w\left(\left\{ x \in \Omega : f(x) > m^i \right\} \right) \right]^{\frac{r}{q}} < \infty$$

and

$$\frac{1}{C} S \leq \|f\|_{L_w^{q,r}(\Omega)}^r \leq C \left[(w(\Omega))^{\frac{r}{q}} + S \right],$$

where $C > 0$ is a constant depending only on m and w.

Furthermore, we shall prove the following important result, which involves a delicate argument and a new scaling procedure in the subquadratic case $s_a < 0$.

Lemma 2.11. *Assume that $u \in W_{loc}^{1,G}(\Omega)$ is a local weak solution of (1.1) with (1.2) and $B_{2R} \subset \Omega$. If $v \in W^{1,G}(B_{2R})$ is the weak solution of*

$$\begin{cases} \operatorname{div} \left[a\left((ADv \cdot Dv)^{\frac{1}{2}} \right) ADv \right] = 0 & \text{in } B_{2R}, \\ v = u & \text{on } \partial B_{2R}, \end{cases} \tag{2.18}$$

then for any $\epsilon_1 > 0$ there exists a constant $C = C(n, i_a, s_a, \epsilon_1) > 1$ such that

$$\fint_{B_{2R}} |Du - Dv| \, dx \leq C g^{-1} \left(\frac{1}{\epsilon_1} \frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right) + \epsilon_1 \fint_{B_{2R}} |Du| \, dx.$$

Proof. Without loss of generality we may as well assume that $R = 1$ by defining

$$\tilde{u}(x) = R^{-1} u(Rx), \quad \tilde{v}(x) = R^{-1} v(Rx) \quad \text{and} \quad \tilde{\mu}(x) = R\mu(Rx).$$

For $k \geq 1$ we define the following truncation operators (see [33,34,44,49])

$$T_k(s) := \max\{-k, \min\{k, s\}\} \quad \text{and} \quad \Phi_k(s) := T_1(s - T_k(s)), \ s \in \mathbb{R}.$$

Since u and v are weak solutions of (1.1) and (2.18) respectively, then we have

$$\int_{B_2} \left[a\left((ADu \cdot Du)^{\frac{1}{2}} \right) ADu - a\left((ADv \cdot Dv)^{\frac{1}{2}} \right) ADv \right] \cdot D\varphi dx = \int_{B_2} \varphi d\mu \tag{2.19}$$

for any $\varphi \in L^\infty(B_2) \cap W_0^{1,G}(B_2)$. Without loss of generality we may as well assume that

$$|\mu|(B_2) \leq \epsilon_1 \quad \text{and} \quad \fint_{B_2} |Du| dx \leq \frac{1}{\epsilon_1} \tag{2.20}$$

for any small constant $\epsilon_1 \in (0, 1)$. If not, we can define

$$\tilde{u}(x) = \frac{u(x)}{\lambda}, \qquad \tilde{v}(x) = \frac{v(x)}{\lambda}, \qquad \tilde{\mu}(x) = \frac{\mu(x)}{g(\lambda)},$$

$$\tilde{a}(t) = \frac{a(\lambda t)}{a(\lambda)} \quad \text{and} \quad \widetilde{G}(t) = \frac{G(\lambda t)}{G(\lambda)},$$

where

$$\lambda = g^{-1}\left(\frac{1}{\epsilon_1}|\mu|(B_2)\right) + \epsilon_1 \fint_{B_2} |Du|dx.$$

Then, $\tilde{a}(t)$ satisfies (1.2) and $\tilde{u}(x) \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of

$$-\mathrm{div}\left[\tilde{a}\left((AD\tilde{u}\cdot D\tilde{u})^{\frac{1}{2}}\right)AD\tilde{u}\right] = \tilde{\mu}.$$

Therefore, it is sufficient to prove the following inequality

$$\fint_{B_2} |Du - Dv|\, dx \le C \tag{2.21}$$

under the condition (2.20), where C is independent of ϵ_1. By choosing $\varphi = \Phi_k(u-v) \in L^\infty(B_2) \cap W^{1,G}_0(B_2)$ in (2.19) and using (2.8), we find that

$$\int_{C_k} |V(Du) - V(Dv)|^2\, dx \le C\int_{B_2} |\mu| dx \le C, \tag{2.22}$$

where $C_k := \{x \in B_2 : k < |u(x) - v(x)| \le k+1\}$. In the meantime, from (2.4), (2.6) and Young's inequality we find that

$$\begin{aligned}
|Du - Dv| &\le Ca^{-\frac{1}{2}}\left(|Du| + |Dv|\right)|V(Du) - V(Dv)| \\
&\le Ca^{-\frac{1}{2}}\left(|Du| + |Dv| + 1\right)|V(Du) - V(Dv)| \\
&\le C\left(|Du| + |Dv| + 1\right)^{-\frac{i_a}{2}}|V(Du) - V(Dv)| \\
&\le C\left(|Du - Dv|^{-\frac{i_a}{2}} + |Du|^{-\frac{i_a}{2}} + 1\right)|V(Du) - V(Dv)| \\
&\le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + \frac{1}{2}|Du - Dv| \\
&\quad + C|Du|^{-\frac{i_a}{2}}|V(Du) - V(Dv)| + |V(Du) - V(Dv)| \\
&\le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + \frac{1}{2}|Du - Dv| + C|Du|^{-\frac{i_a}{2}}|V(Du) - V(Dv)| + 1
\end{aligned}$$

for $i_a \in (-1/n, 0)$, which implies that

$$|Du - Dv| \le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + C|Du|^{-\frac{i_a}{2}}|V(Du) - V(Dv)| + 1$$

and then

$$\begin{aligned}
\int_{B_2} |Du - Dv|\, dx &\le C\int_{B_2} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + 1 dx \\
&\quad + C\left(\int_{B_2} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\, dx\right)^{\frac{2+i_a}{2}}\left(\int_{B_2} |Du| dx\right)^{-\frac{i_a}{2}}
\end{aligned} \tag{2.23}$$

by using Hölder's inequality. Moreover, from Hölder's inequality, (2.22) and the definition of C_k we find that

$$\begin{aligned}
\int_{C_k} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\, dx &\le C\,|C_k|^{1-\frac{1}{2+i_a}}\left(\int_{C_k} |V(Du) - V(Dv)|^2\, dx\right)^{\frac{1}{2+i_a}} \\
&\le C\,|C_k|^{1-\frac{1}{2+i_a}}\left[|\mu|(B_2)\right]^{\frac{1}{2+i_a}} \\
&\le C\left[|\mu|(B_2)\right]^{\frac{1}{2+i_a}}\frac{1}{k^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})}}\left(\int_{C_k} |u - v|^{\frac{n}{n-\sigma}}\, dx\right)^{1-\frac{1}{2+i_a}}
\end{aligned}$$

for some $\sigma \in (-ni_a, 1) \subset (0,1)$. Therefore, we conclude that

$$\begin{aligned}
&\int_{B_2} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\, dx \\
&\le \int_{C_0} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\, dx + \sum_{k=1}^{\infty} \int_{C_k} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\, dx \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{k^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})}} \left(\int_{C_k} |u-v|^{\frac{n}{n-\sigma}}\, dx \right)^{1-\frac{1}{2+i_a}} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-\sigma}}} \right]^{\frac{1}{2+i_a}} \left(\sum_{k=1}^{\infty} \int_{C_k} |u-v|^{\frac{n}{n-\sigma}}\, dx \right)^{1-\frac{1}{2+i_a}} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |u-v|^{\frac{n}{n-\sigma}}\, dx \right)^{(1-\frac{1}{2+i_a})} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\}
\end{aligned} \tag{2.24}$$

by Sobolev's inequality and the fact that

$$\frac{n(1+i_a)}{n-\sigma} > 1,$$

since $\sigma \in (-ni_a, 1)$. Furthermore, from (2.20), (2.23), (2.24) and Young's inequality we obtain

$$\begin{aligned}
\int_{B_2} |Du-Dv|\, dx &\le C + C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\} \\
&\quad + C\,[|\mu|(B_2)]^{\frac{1}{2}} \left(\int_{B_2} |Du| dx \right)^{-\frac{i_a}{2}} \left[1 + \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right]^{\frac{2+i_a}{2}} \\
&\le C + C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\} \\
&\quad + C \left[[|\mu|(B_2)]^{\frac{1}{2}} \left(\int_{B_2} |Du| dx \right)^{-\frac{i_a}{2}} \right]^{\frac{2}{-i_a}} \\
&\quad + C \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \\
&\le C + C\epsilon_1^{\frac{1+i_a}{-i_a}} + C \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \\
&\le C + C \left(\int_{B_2} |Du-Dv|\, dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})},
\end{aligned}$$

which implies that (2.21) is true since

$$\frac{n}{n-\sigma}\left(1 - \frac{1}{2+i_a}\right) < 1 \quad \text{for } i_a \in \left(-\frac{1}{n}, 0\right).$$

Thus, we finish the proof. □

We now switch to another comparison estimate for solutions to (1.1) and the homogeneous constant coefficient problem.

Lemma 2.12. *Assume that $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with $B_R \subset \Omega$ and (1.2). If $w \in W^{1,G}(B_R)$ is the weak solution of*

$$\begin{cases} \operatorname{div}\left[a\left((\overline{A}_{B_R}Dw \cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\right] = 0 & \text{in } B_R, \\ w = v & \text{on } \partial B_R, \end{cases} \tag{2.25}$$

then for any $\epsilon_1 > 0$ there exists a constant $C = C(n, i_a, s_a, \epsilon_1) > 1$ such that

$$\fint_{B_R} |Du - Dw|\,dx \le Cg^{-1}\left(\frac{1}{\epsilon_1}\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + \epsilon_1 \fint_{B_{2R}} |Du|\,dx. \tag{2.26}$$

Proof. If we select the test function $\varphi = v - w$, then after a direct calculation we can show the resulting expression as

$$I_1 := \int_{B_R} a\left((\overline{A}_{B_R}Dw \cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw \cdot Dwdx = \int_{B_R} a\left((\overline{A}_{B_R}Dw \cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw \cdot Dvdx =: I_2.$$

Using (1.3) and Lemmas 2.5–2.6, we find that

$$C\int_{B_R} G(|Dw|)\,dx \le I_1 = I_2 \le \tau \int_{B_R} G(|Dw|)\,dx + C(\tau)\int_{B_R} G(|Dv|)\,dx,$$

which implies that

$$\int_{B_R} G(|Dw|)\,dx \le C\int_{B_R} G(|Dv|)\,dx \tag{2.27}$$

by choosing τ small enough. Moreover, we apply Gehring's lemma (see Theorem 6.7 in [36]) to obtain the reverse Hölder type inequality

$$\left[\fint_{B_R} [G(|Dw|)]^{1+\delta_0}\,dx\right]^{\frac{1}{1+\delta_0}} \le C\fint_{B_{2R}} G(|Dw|)\,dx \tag{2.28}$$

for some positive constant $\delta_0 > 0$. On the other hand, we can also calculate the result of the expression $I_3 = I_4$, where

$$\begin{aligned} I_3 &:= \int_{B_R}\left[a\left((ADv \cdot Dv)^{\frac{1}{2}}\right)ADu - a\left((ADw \cdot Dw)^{\frac{1}{2}}\right)ADw\right] \cdot (Dv - Dw)dx, \\ I_4 &:= -\int_{B_R}\left[a\left((ADw \cdot Dw)^{\frac{1}{2}}\right)ADw - a\left((\overline{A}_{B_R}Dw \cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\right] \cdot (Dv - Dw)dx. \end{aligned}$$

From Lemma 2.6 we find that

$$\epsilon\int_{B_R} G(|Dv|)\,dx + I_3 \ge C\int_{B_R} G(|Dv - Dw|)\,dx.$$

Moreover, we first discover

$$\begin{aligned} |I_4| &\le \int_{B_R} a\left((ADw \cdot Dw)^{\frac{1}{2}}\right)|A - \overline{A}_{B_R}|\,|Dw|\,|Dw - Dv|\,dx \\ &\quad + \int_{B_R}\left|a\left((ADw \cdot Dw)^{\frac{1}{2}}\right) - a\left((\overline{A}_{B_R}Dw \cdot Dw)^{\frac{1}{2}}\right)\right|\,|\overline{A}_{B_R}Dw|\,|Dw - Dv|\,dx \\ &=: I_{41} + I_{42}. \end{aligned}$$

Estimate of I_{41}. From (1.3), Lemma 2.6, Young's inequality and Hölder's inequality we find that

$$\begin{aligned} I_{41} &\le C\int_{B_R} a(|Dw|)\,|Dw|\,\left|A-\overline{A}_{B_R}\right|\,|Dw-Dv|\,dx \\ &\le \frac{\epsilon}{2\Lambda}\int_{B_R} G(|Dw-Dv|)\left|A-\overline{A}_{B_R}\right|dx + C(\epsilon)\int_{B_R}\widetilde{G}\left(a(|Dw|)\,|Dw|\right)\left|A-\overline{A}_{B_R}\right|dx \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\int_{B_R} G(|Dw|)\left|A-\overline{A}_{B_R}\right|dx \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\left\{\int_{B_R}\left[G(|Dw|)\right]^{1+\delta_0}dx\right\}^{\frac{1}{1+\delta_0}}\left[\int_{B_R}\left|A-\overline{A}_{B_R}\right|^{\frac{1+\delta_0}{\delta_0}}dx\right]^{\frac{\delta_0}{1+\delta_0}} \end{aligned}$$

for any $\epsilon > 0$, which implies that

$$\begin{aligned} I_{41} &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\int_{B_{2R}} G(|Dw|)\,dx\left[\int_{B_R}\left|A-\overline{A}_{B_R}\right|dx\right]^{\frac{\delta_0}{1+\delta_0}} \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx, \end{aligned}$$

where we used Definition 1.4 and (2.27)–(2.28).

Estimate of I_{42}. (1.2), (1.3), Lemma 2.6 and Lagrange's mean value theorem yield the bound

$$\begin{aligned} I_{42} &\le C\int_{B_R}\left|a'(\zeta)\right|\left|(ADw\cdot Dw)^{\frac{1}{2}} - \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}\right|\,|Dw|\,|Dw-Dv|\,dx \\ &\le C\int_{B_R}\frac{|a(\zeta)|}{\zeta}\,\frac{\left|A-\overline{A}_{B_R}\right|\,|Dw|^2}{(ADw\cdot Dw)^{\frac{1}{2}} + \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}}\,|Dw|\,|Dw-Dv|\,dx \\ &\le C\int_{B_R} a(|Dw|)\,|Dw|\,\left|A-\overline{A}_{B_R}\right|\,|Dw-Dv|\,dx, \end{aligned}$$

where ζ is between $(ADw\cdot Dw)^{\frac{1}{2}}$ and $\left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}$ satisfying

$$\Lambda^{-\frac{1}{2}}|Dw| \le \zeta, (ADw\cdot Dw)^{\frac{1}{2}}, \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}} \le \Lambda^{\frac{1}{2}}|Dw|.$$

And then, we have

$$I_{42} \le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx$$

for any $\epsilon > 0$, whose proof is totally similar to that of I_{41}. Thus, we choose ϵ small enough and combine the estimates of I_3 and I_4 to conclude that

$$\begin{aligned} \int_{B_R} G(|Dv-Dw|)\,dx &\le \epsilon\int_{B_R} G(|Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx \\ &\le \epsilon_1^{2+s_a}\int_{B_{2R}} G(|Dv|)\,dx \end{aligned}$$

by selecting ϵ, δ small enough satisfying the last inequality. Since $\theta^{2+i_a}G(t) \le G(\theta t) \le \theta^{2+s_a}G(t)$ for any $\theta \ge 1$ and $t \ge 0$ by (2.4), we find that

$$\theta^{2+i_a}t \le G\left(\theta G^{-1}(t)\right) \le \theta^{2+s_a}t \quad \text{for any } \theta \ge 1,$$

which implies that

$$G^{-1}\left(\theta^{2+i_a}t\right) \leq \theta G^{-1}(t) \leq G^{-1}\left(\theta^{2+s_a}t\right) \quad \text{for any } \theta \geq 1.$$

In other words, we conclude that

$$\theta^{\frac{1}{2+s_a}} \leq \frac{G^{-1}(\theta t)}{G^{-1}(t)} \leq \theta^{\frac{1}{2+i_a}} \quad \text{for any } \theta \geq 1. \tag{2.29}$$

From Jensen's inequality and the reverse Hölder's inequality (see Lemma 4.2 in [7]) we deduce that

$$\begin{aligned} G\left(\fint_{B_R} |Dv - Dw| dx\right) &\leq C \fint_{B_R} G\left(|Dv - Dw|\right) dx \\ &\leq C\epsilon_1^{2+s_a} \fint_{B_R} G\left(|Dv|\right) dx \\ &\leq C\epsilon_1^{2+s_a} G\left(\fint_{B_{2R}} |Dv| dx\right), \end{aligned}$$

which implies that

$$\fint_{B_R} |Dv - Dw| dx \leq C\epsilon_1 \fint_{B_{2R}} |Dv| dx$$

by using (2.29). Finally, by using Lemma 2.11 and the above inequality we obtain

$$\begin{aligned} \fint_{B_R} |Du - Dw| dx &\leq \fint_{B_R} |Du - Dv| dx + \fint_{B_R} |Dv - Dw| dx \\ &\leq Cg^{-1}\left(\frac{1}{\epsilon_1} \frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_R} |Du| dx + C\epsilon_1 \int_{B_{2R}} |Dv| dx \\ &\leq Cg^{-1}\left(\frac{1}{\epsilon_1} \frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_{2R}} |Du| dx + C\epsilon_1 \int_{B_{2R}} |Dv - Du| dx \\ &\leq Cg^{-1}\left(\frac{1}{\epsilon_1} \frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_{2R}} |Du| dx \end{aligned}$$

and then finish the proof. □

Additionally, we can get the following local Lipschitz regularity for the homogeneous constant coefficient problem.

Lemma 2.13 (see [7, Lemma 4.1]). *Let $w \in W^{1,G}(\Omega)$ be a weak solution to*

$$\operatorname{div}\left[a\left(\left(\overline{A}_{B_R} Dw \cdot Dw\right)^{\frac{1}{2}}\right) \overline{A}_{B_R} Dw\right] = 0 \quad \text{in } B_R \subset \mathbb{R}^n.$$

Then we can obtain the following De Giorgi type estimate

$$\sup_{B_{R/2}} |Dw| \leq C \fint_{B_R} |Dw|\, dx.$$

The following crucial lemma, which shows how the upper level sets of $|Du|$ decay, allows us to build the interior gradient estimates.

Lemma 2.14. *Assume that $\lambda > 0$. There is a constant $N = N(n, i_a, s_a) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) in $B_{6r} \subset \Omega$ for $r \in (0, 1]$ with*

$$B_r \cap \{x \in B_1 : \mathcal{M}(|Du|)(x) \leq \lambda\} \cap \{x \in B_1 : g^{-1}[\mathcal{M}_1(\mu)](x) \leq \delta\lambda\} \neq \emptyset, \tag{2.30}$$

then we have

$$w\left(\{x \in B_r : \mathcal{M}(|Du|)(x) > N\lambda\}\right) < \epsilon w(B_r). \tag{2.31}$$

Proof. From (2.30), there exists a point $x_0 \in B_r$ such that

$$\fint_{B_\rho(x_0)} |Du|\, dx \leq \lambda \quad \text{and} \quad g^{-1}\left(\rho \fint_{B_\rho(x_0)} d|\mu|\right) \leq \delta\lambda \tag{2.32}$$

for all $\rho > 0$. Since $B_{4r} \subset B_{5r}(x_0)$, it follows from (2.32) that

$$\fint_{B_{4r}} |Du|\, dx \leq \frac{|B_{5r}(x_0)|}{|B_{4r}|} \cdot \frac{1}{|B_{5r}(x_0)|} \int_{B_{5r}(x_0)} |Du|\, dx \leq 2^n\lambda. \tag{2.33}$$

Since $t^{i_a+1}g(1) \leq g(t) \leq t^{s_a+1}g(1)$ for any $t \geq 1$ by Lemma 2.6, we know that $t^{\frac{1}{s_a+1}} \lesssim g^{-1}(t) \lesssim t^{\frac{1}{i_a+1}}$ for any $t \geq 1$. Similarly, we also see that $t^{\frac{1}{i_a+1}} \lesssim g^{-1}(t) \lesssim t^{\frac{1}{s_a+1}}$ for any $0 < t < 1$. In the same way, we also have

$$g^{-1}\left(4r \fint_{B_{4r}} d|\mu|\right) \leq g^{-1}\left(\frac{4r}{5r} \cdot \frac{|B_{5r}(x_0)|}{|B_{4r}|} \cdot 5r \fint_{B_{5r}(x_0)} d|\mu|\right) \leq C\delta\lambda.$$

Then we apply Lemma 2.12 to deduce that

$$\begin{aligned} \fint_{B_{3r}} |Du - Dw|\, dx &\leq Cg^{-1}\left(\frac{4r}{\epsilon_1} \fint_{B_{4r}} d|\mu|\right) + \epsilon_1 \fint_{B_{4r}} |Du|\, dx \\ &\leq C\frac{\delta\lambda}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1\lambda \end{aligned}$$

by choosing δ, ϵ_1 small enough satisfying $C\frac{\delta\lambda}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1\lambda \leq \lambda$ in advance and then

$$\begin{aligned} \|Dw\|_{L^\infty(B_{3r})} &\leq C \fint_{B_{4r}} |Dw|\, dx \\ &\leq C \fint_{B_{4r}} |Du|\, dx + C \fint_{B_{4r}} |Du - Dw|\, dx \\ &\leq N_1\lambda \end{aligned}$$

by Lemma 2.13 and (2.33), for some positive constant $N_1 \geq 1$. Now we shall claim that

$$\{x \in B_r : \mathcal{M}(|Du|)(x) > N\lambda\} \subset \{x \in B_r : \mathcal{M}(|Du - Dw|)(x) > N_1\lambda\} \tag{2.34}$$

for $N := \max\{3^n, 2N_1\}$. Actually, we take $x_1 \in \{x \in B_r : \mathcal{M}(|Du - Dw|)(x) \leq N_1\lambda\}$. If $0 < \rho < r$, then we find that $B_\rho(x_1) \subset B_{2r}$ and so

$$\begin{aligned} \fint_{B_\rho(x_1)} |Du|\, dx &\leq \fint_{B_\rho(x_1)} (|Dw| + |Du - Dw|)\, dx \\ &\leq \mathcal{M}(|Du - Dw|)(x_1) + N_1\lambda \\ &\leq 2N_1\lambda. \end{aligned}$$

On the other hand, if $\rho \geq r$, then $B_\rho(x_1) \subset B_{3\rho}(x_0)$. From (2.32), we deduce that

$$\fint_{B_\rho(x_1)} |Du|\,dx \leq \frac{|B_{3\rho}(x_0)|}{|B_\rho(x_1)|} \fint_{B_{3\rho}(x_0)} |Du|\,dx \leq 3^n \lambda \leq N\lambda.$$

Thus, the claim (2.34) is true. Then from Lemma 2.8 we estimate

$$\begin{aligned}\frac{1}{|B_r|}\left|\{x \in B_r : \mathcal{M}(|Du|) > N\lambda\}\right| &\leq \frac{1}{|B_r|}\left|\{x \in B_r : \mathcal{M}(|Du - Dw|) > N_1\lambda\}\right| \\ &\leq \frac{C}{N_1\lambda} \fint_{B_{3r}} |Du - Dw|\,dx \\ &\leq C\frac{\delta}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1,\end{aligned}$$

which implies that

$$w\left(\{x \in B_r : \mathcal{M}(|Du|) > N\lambda\}\right) \leq C\left(\frac{\delta}{\epsilon_1^{\frac{1}{i_a+1}}} + \epsilon_1\right)^\sigma w(B_r) < \epsilon w(B_r)$$

by Lemma 1.2 and choosing δ, ϵ_1 small enough satisfying the last inequality. Therefore, we finish the final proof of this lemma. □

Now we are ready to finish the proof of the main result, Theorem 1.5.

Proof. Let $u \in W^{1,G}_{loc}(\Omega)$ be the local weak solution of (1.1),

$$E = \{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}$$

and

$$F = \{x \in B_1 : \mathcal{M}(|Du|) > \lambda\lambda_0\} \cup \left\{x \in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\right\} \quad \text{for any } \lambda \geq 1,$$

where

$$\lambda_0 = \frac{C_3}{N|B_1|}\left(\frac{C_2}{\epsilon}\right)^{\frac{1}{\sigma}} \int_{B_2} |Du| + 1dx. \tag{2.35}$$

It follows from the weak 1-1 estimate that

$$|\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}| \leq \frac{C_3}{N\lambda\lambda_0} \int_{B_1} |Du|dx < \left(\frac{\epsilon}{C_2}\right)^{\frac{1}{\sigma}} |B_1|,$$

which implies that

$$w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right) < \epsilon w(B_1)$$

by Lemma 1.2. Therefore, we apply Lemma 2.9 and Lemma 2.14 to have

$$w\left(E\right) \leq C\epsilon w\left(F\right). \tag{2.36}$$

Next, we divide into two cases.

Case 1: $r = +\infty$. From (2.36) we conclude that

$$\begin{aligned}\left[w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} &\leq C\epsilon^{\frac{1}{q}}\left[w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\quad + C\epsilon^{\frac{1}{q}}\left[w\left(\{x \in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\}\right)\right]^{\frac{1}{q}}\end{aligned}$$

for any $\lambda \geq 1$, which implies that

$$\begin{aligned}\|\mathcal{M}(|Du|)\|_{L_w^{q,\infty}(B_1)} &:= \sup_{\lambda>0} N\lambda\lambda_0 \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\le CN\epsilon^{\frac{1}{q}} \sup_{\lambda\ge 1} \lambda\lambda_0 \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{1}{q}} + C\lambda_0 \\ &\quad + \frac{CN\epsilon^{\frac{1}{q}}}{\delta} \sup_{\lambda\ge 1} \delta\lambda\lambda_0 \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\le C_4\epsilon^{\frac{1}{q}} \|\mathcal{M}(|Du|)\|_{L_w^{q,\infty}(B_1)} + C(\delta,\epsilon)\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L_w^{q,\infty}(B_1)} + C\lambda_0.\end{aligned}$$

Then, by selecting ϵ small enough such that $C_4\epsilon^{\frac{1}{q}} = 1/2$ and using an approximation argument by choosing $|\nabla u|_k := \min\{\nabla u, k\}$ like the one in [8], we deduce that

$$\begin{aligned}\|Du\|_{L_w^{q,\infty}(B_1)} &\le \|\mathcal{M}(|Du|)\|_{L_w^{q,\infty}(B_1)} \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L_w^{q,\infty}(B_1)} + C\lambda_0 \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L_w^{q,\infty}(B_1)} + C\int_{B_2} |Du| + 1dx.\end{aligned}$$

Case 2: $0 < r < +\infty$. From (2.36) we find that

$$\begin{aligned}&\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &= \left[w(E)\right]^{\frac{r}{q}} \le C\epsilon^{\frac{r}{q}} \left[w(F)\right]^{\frac{r}{q}} \\ &\le C\epsilon^{\frac{r}{q}} \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\epsilon^{\frac{r}{q}} \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\}\right)\right]^{\frac{r}{q}}.\end{aligned}$$

Actually, by applying an iteration procedure we can also prove

$$\begin{aligned}&\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N^m\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\le C\epsilon^{\frac{mr}{q}} \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\sum_{i=1}^{m} \epsilon^{\frac{ir}{q}} \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}}.\end{aligned} \tag{2.37}$$

Now we select ϵ small enough satisfying $N^r\epsilon^{\frac{r}{q}} < 1$ and then apply Lemma 2.10 to observe that

$$\begin{aligned}&\sum_{m=1}^{\infty} N^{mr}\lambda_0^r \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N^m\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\le C\sum_{m=1}^{\infty} N^{mr}\lambda_0^r \sum_{i=1}^{m} \epsilon^{\frac{ir}{q}} \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\sum_{m=1}^{\infty} N^{mr}\epsilon^{\frac{mr}{q}}\lambda_0^r \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\le \frac{C}{\delta^r}\sum_{i=1}^{\infty} \epsilon^{\frac{ir}{q}} N^{ir} \sum_{m=i}^{\infty} N^{(m-i)r}\delta^r\lambda_0^r \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\lambda_0^r \sum_{m=1}^{\infty} N^{mr}\epsilon^{\frac{mr}{q}} \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L_w^{q,r}(B_1)}^r + C\lambda_0^r < +\infty,\end{aligned}$$

which implies that

$$\|Du\|_{L_w^{q,r}(B_1)} \leq \|\mathcal{M}(|Du|)\|_{L_w^{q,r}(B_1)} \leq C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L_w^{q,r}(B_1)} + C\int_{B_2} |Du| + 1dx.$$

Thus, this finishes the proof of the main result in this work. □

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