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Existence of solutions for asymptotically periodic quasilinear Schrödinger equations with local nonlinearities

Jian-Xin Han , **Yu-Ting Wei** and **Yan-Fang Xue** ✉

School of Mathematics and Statistics, Xinyang Normal University, Xinyang, 464000, China

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Abstract. This paper is concerned with the existence of positive solutions for asymptotically periodic quasilinear Schrödinger equations. By using a Nehari-type constraint and Moser iteration, we get the existence results which is a complement to the ones in Chu and Liu [*Nonlinear Anal. Real World Appl.* **44**(2018), 118–127]. Moreover, we consider a new reformative asymptotic processes of the potential function and the nonlinearity term is only locally defined.

Keywords: quasilinear Schrödinger equation, L^∞-estimate, asymptotically periodic, Nehari manifold.

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1 Introduction and main results

We are concerned with the existence of solutions for the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = \lambda h(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

which models the self-channeling of a high-power ultrashort laser in matter (see [2]).

The main mathematical difficulty with problem (1.1) is caused by the quasilinear term $\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2})$, the natural functional corresponding to problem (1.1) maybe not well defined for all $u \in H^1(\mathbb{R}^N)$. To overcome this difficulty, various arguments have been developed, such as a change of variables (see [1,4,5,11,15,17]) and a perturbation method (see [3]). Chu and Liu [1] proved that (1.1) has a positive solution by using the monotonicity trick and a priori estimate in the radial space. It is a little surprising that no condition is assumed on the nonlinear term $h(u)$ near infinity. For the periodic potential, there are references [4,5], they discussed the following equation

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = h(x,u), \tag{1.2}$$

✉ Corresponding author. Email: xueyanfang2015@163.com

where α is a parameter. Jalilian [4] considered equation (1.2) with $1.36 < \alpha \leq 2$ and proved that (1.2) had infinitely many geometrically distinct solutions. Then, Li [5] extended the results to $1 \leq \alpha \leq 2$ and proved the existence of a ground state solution for equation (1.2). Shen and Wang [11] studied the well potential and got the standing wave solutions for (1.1) with subcritical or critical growth by using Resonance Theorem and Hahn–Banach Theorem. For the steep potential well, one can see [15], the authors obtained the existence of a ground state solution by using the Mountain Pass Theorem, and considered the concentration behavior of the solution. In [17], the authors considered the constant potential and obtained the existence and multiplicity of radial and nonradial normalized solutions for problem (1.1) when h satisfies the well-known Berestycki–Lions condition. As far as we know, there are no results concerning problem (1.1) with the asymptotically periodic potential except [16].

However, the related semilinear equation with the asymptotically periodic condition has been extensively studied, see [6,8,13,18] and their references. We would like to point out that in reference [6,8], they discussed the asymptotically periodic potential and given reformative conditions which unify the asymptotic processes of V, h at infinity. The asymptotic processes is weaker than those in [13,18].

In the present paper, we borrow an idea from [1,6] to discuss problem (1.1) with the asymptotically periodic potential. Denote

$$\mathcal{F}_0 := \left\{ k(x) : \forall \epsilon > 0,\ \lim_{|y| \to \infty} \text{meas}\{x \in B_1(y) : |k(x)| \geq \epsilon\} = 0 \right\}.$$

Then, we give some assumptions on the potential $V(x)$ and the nonlinear term $h(s)$.

(V) $0 \leq V(x) \leq V_0(x) \in L^\infty(\mathbb{R}^N)$, $V(x) - V_0(x) \in \mathcal{F}_0$, $\inf_{x \in \mathbb{R}^N} V_0(x) > 0$ and $V_0(x)$ satisfies $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

The function $h \in C(\mathbb{R}, \mathbb{R})$ satisfies

(h_1) there exist $p > 2$, $\delta \in (0,1)$ such that the function $s \mapsto \frac{h(s)}{s^{p-1}}$ is nondecreasing and $h(s) > 0$ on $(0, \delta]$.

(h_2) there exists $q \in (2, 2^*)$ such that $\liminf_{s \to 0^+} \frac{H(s)}{s^q} > 0$, where $H(s) = \int_0^s h(t)dt$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

Now we state our main result.

Theorem 1.1. *Suppose that conditions (V) and (h_1), (h_2) are satisfied, then there is $\lambda_1 > 0$ such that problem (1.1) possesses a positive solution for $\lambda \geq \lambda_1$.*

Remark 1.2. (1) We emphasize that no condition is assumed on the nonlinear term $h(u)$ near infinity in Theorem 1.1. In all these previous works for problem (1.1), among other assumptions, the authors always assume that the nonlinear term $h(u)$ has growth conditions near infinity except [1]. However, Chu and Liu [1] investigated quasi-linear Schrödinger equations in the radial space. They had the compactness and got certain solutions easily. In our cases, we do not have compact embedding. Due to the lack of compact embedding, the existence of ground states of problem (1.1) becomes rather complicated. we borrow an idea from [6] to overcome this difficulty.

(2) Our results also can be seen as the extension of semilinear poroblem in [6] to the quasilinear one.

(3) For simplicity, we will abbreviate $\int_{\mathbb{R}^N} k(x)dx$ as $\int_{\mathbb{R}^N} k(x)$.

Notation: In this paper, we use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right).$$

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s, \quad \forall s \in [1, +\infty).$$

- $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|$ denotes the usual norm in $L^\infty(\mathbb{R}^N)$.
- $E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < \infty\}$ is endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right).$$

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $C, C_1, C_2, \ldots$ denote various positive (possibly different) constants.

2 Some preliminary results

We note that the solutions of problem (1.1) are the critical points of the functional

$$J_h(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[\left(1 + \frac{u^2}{2(1+u^2)}\right)|\nabla u|^2\right] + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \lambda \int_{\mathbb{R}^N} H(u).$$

Variational methods cannot be applied directly to find weak solutions of problem (1.1), since the natural associated functional $J_h(u)$ is not well defined in general in the space E. To overcome this difficulty, we borrow an idea from Shen and Wang [10].

Let $F(u) := \int_0^u f(t)dt$, where f is defined by

$$f(t) = \sqrt{1 + \frac{t^2}{2(1+t^2)}}. \tag{2.1}$$

After the change of variables $u = F^{-1}(v)$ from J, we get a new variational functional

$$I_h(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|F^{-1}(v)|^2) - \lambda \int_{\mathbb{R}^N} H(F^{-1}(v)).$$

Since f is a nondecreasing positive function, we obtain $|F^{-1}(v)| \le \frac{|v|}{f(0)} = |v|$. From this and the conditions of h, it is clear that I_h is well defined in E and $I_h \in C^1(E, \mathbb{R})$ (see [2, 10, 11] for details). Now, we give another equation

$$-\operatorname{div}\left[\left(1 + \frac{u^2}{2(1+u^2)}\right)\nabla u\right] + V(x)u + \frac{u}{2(1+u^2)^2}|\nabla u|^2 = \lambda h(u), \tag{2.2}$$

which is equivalent to (1.1). In fact, we only need to show that

$$-\text{div}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{u}{2(1+u^2)^2}|\nabla u|^2=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).$$

By a direct calculation, we obtain

$$\begin{aligned}
&-\text{div}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\text{div}\nabla u-\frac{u^2}{2(1+u^2)}\text{div}\nabla u-\nabla u\cdot\nabla\frac{u^2}{2(1+u^2)}+\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{u^2}{2(1+u^2)}\Delta u-\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\left(\frac{u}{\sqrt{1+u^2}}\text{div}\nabla u+\nabla u\cdot\nabla\frac{u}{\sqrt{1+u^2}}\right)\\
&=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).
\end{aligned}$$

If u is a weak solution of problem (1.1), then it is also a weak solution of (2.2) and should satisfy

$$\int_{\mathbb{R}^N}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\cdot\nabla\varphi+\frac{u}{2(1+u^2)^2}|\nabla u|^2\varphi+V(x)u\varphi-\lambda h(u)\varphi\right]=0, \tag{2.3}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Let $\varphi=\frac{\psi}{f(u)}$, then, it can be checked that (2.3) is equivalent to the following equality

$$\int_{\mathbb{R}^N}\left(\nabla v\cdot\nabla\psi+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}\psi-\lambda\frac{h(F^{-1}(v))}{f(F^{-1}(v))}\psi\right)=0. \tag{2.4}$$

Therefore, in order to find the solutions of problem (1.1), it suffices to study the existence of solutions of the following equation

$$-\triangle v+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}=\lambda\frac{h(F^{-1}(v))}{f(F^{-1}(v))},\qquad x\in\mathbb{R}^N. \tag{2.5}$$

Now, we summarize the properties of F^{-1}, f.

Lemma 2.1. *The functions F^{-1}, f satisfy the following properties:*

(1) $1\le f(t)\le\sqrt{\frac{3}{2}}$ *for all* $t\in\mathbb{R}$;

(2) $1\le\frac{F^{-1}(t)f(F^{-1}(t))}{t}\le 6-2\sqrt{6}$ *for all* $t\in\mathbb{R}, t\neq 0$;

(3) $\sqrt{\frac{2}{3}}|t|\le|F^{-1}(t)|\le|t|$ *for all* $t\in\mathbb{R}$;

(4) $\frac{F^{-1}(t)}{t}\to 1$ *as* $t\to 0$;

(5) $\frac{F^{-1}(t)}{t}\to\sqrt{\frac{2}{3}}$ *as* $t\to\infty$;

(6) $0\le\frac{f'(t)t}{f(t)}\le 5-2\sqrt{6}$ *for all* $t\in\mathbb{R}$;

(7) *The function* $\frac{t}{f(t)F(t)}$ *is strictly decreasing for all* $t \geq 0$;

(8) *The function* $\frac{t^\mu}{f(t)F(t)}$, $\mu \in (2, p)$ *is strictly increasing for all* $t \geq 0$.

Proof. The proof of the items (1)–(6) have been proved in [11], we only need to prove items (7)(8). Let $l_1(t) = \frac{t}{f(t)F(t)}$. Since $f(t)$ is strictly increasing in $(0, +\infty)$, one has

$$0 \leq F(t) = \int_0^t f(s)ds < tf(t). \tag{2.6}$$

Then using item (6) and (2.6), we obtain

$$l'(t) = \frac{F(t) - tf(t) - \frac{f'(t)t}{f(t)}F(t)}{f(t)F^2(t)} \leq \frac{F(t) - tf(t)}{f(t)F^2(t)} < 0.$$

The above inequality proves item (7).

Let $l_\mu(t) = \frac{t^\mu}{f(t)F(t)}$, $l_0(t) = [\mu - (5 - 2\sqrt{6})]F(t) - tf(t)$. It is following from item (6) that

$$l_0'(t) = (\mu F(t) - tf(t))' = f(t)\left(\mu - (6 - 2\sqrt{6}) - \frac{f'(t)t}{f(t)}\right) > 0.$$

We can get that $l_0(t)$ is strictly increasing in $(0, +\infty)$ and $l_0(t) > l_0(0) = 0$ for $t > 0$. Then, using item (6) again, we obtain

$$l_\mu'(t) = \frac{t^{\mu-1}}{f(t)F^2(t)}\left[\mu F(t) - tf(t) - \frac{f'(t)t}{f(t)}F(t)\right] \geq \frac{t^{\mu-1}}{f(t)F^2(t)}l_0(t) > 0.$$

The above inequality proves item (8). □

Lemma 2.2 ([8])**.** *Suppose that condition* (V) *is satisfied. Then, the norms* $\|\cdot\|_H$ *and* $\|\cdot\|$ *are equivalent in the space* E *and the embedding* $E \hookrightarrow L^\alpha(\mathbb{R}^N)$ *is continuous for any* $\alpha \in [2, 2^*]$.

Lemma 2.3 ([12])**.** *Let* E *be a real Banach space and* $I \in C^1(E, \mathbb{R})$. *Let* S *be a closed subset of* E *which disconnects* E *in distinct connected components* E_1, E_2. *Suppose further that* $I(0) = 0$ *and*

(1) $0 \in E_1$ *and there is* $\alpha > 0$ *such that* $I|_S \geq \alpha > 0$.

(2) *there is* $\rho > 0$, $e \in E_2$, $\|e\| > \rho$, *such that* $I(e) < 0$.

Then I *possesses a sequence* $\{u_n\} \subset E$ *satisfying*

$$I(u_n) \to c \geq \alpha, \qquad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \tag{2.7}$$

where $c \geq \alpha > 0$ *given by*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \qquad \Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

We call the sequence $\{u_n\}$ that satisfies (2.7) ia a $(C)_c$ sequence of the functional I.

Lemma 2.4. *Assume that condition* (V) *holds. If* $\{u_n\}$ *is bounded in* E *and* $u_n \to 0$ *in* $L^\alpha_{\text{loc}}(\mathbb{R}^N)$ *for* $\alpha \in [2, 2^*)$, *one has*

$$A_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x))\,|F^{-1}(u_n)|^2 = o_n(1).$$

Proof. When $k(x) \in \mathcal{F}_0$, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{|k(x)|\geq\epsilon} u^2 \leq C_0 \int_{B_{R_\epsilon+1}(0)} u^2 + C_1\epsilon^{2/N}\|u\|_H^2, \qquad \forall u \in E, \tag{2.8}$$

where C_0, C_1 are positive constants and independent on ϵ. Inequality (2.8) has already been proved in [8], we omit it here.

Let $k(x) := V(x) - V_0(x) \in \mathcal{F}_0$, then, $|k(x)| \leq 2|V_0(x)| \leq 2\|V_0\|_\infty$, by using Lemma 2.1-(3) and (2.8), we have

$$\begin{aligned} |A_{n1}| &\leq \int_{\mathbb{R}^N} |k(x)||F^{-1}(u_n)|^2 \leq \int_{\mathbb{R}^N} |k(x)u_n^2| \\ &= \int_{|k(x)|\geq\epsilon} |k(x)u_n^2| + \int_{|k(x)|<\epsilon} |k(x)u_n^2| \\ &\leq 2\|V_0\|_\infty \left[C_0 \int_{B_{R_\epsilon+1}(0)} u_n^2 + C_1\epsilon^{\frac{2}{N}}\|u_n\|_H^2\right] + \epsilon \int_{\mathbb{R}^N} |u_n|^2 \\ &= o_n(1) + C_2\epsilon^{\frac{2}{N}} + C_3\epsilon. \end{aligned}$$

Let $\epsilon \to 0$, Lemma 2.4 holds. □

Lemma 2.5. *Assume that condition (V) holds, $\{u_n\} \subset E$ is bounded, $|z_n| \to +\infty$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has*

$$B_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x)) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) = o_n(1).$$

Proof. Since $\varphi \in C_0^\infty(\mathbb{R}^N)$, we get that

$$\int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 = o_n(1). \tag{2.9}$$

Let $k(x) := V(x) - V_0(x) \in \mathcal{F}_0$, by using Lemma 2.1-(3), (2.8), (2.9) and the Hölder inequality, we have

$$\begin{aligned} |B_{n1}| &\leq \int_{|k|\geq\epsilon} \left|\frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)\right| + \int_{|k|<\epsilon} \left|\frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)\right| \\ &\leq 2\|V_0\|_\infty \int_{|k|\geq\epsilon} |u_n\varphi(x - z_n)| + \epsilon \int_{|k|<\epsilon} |u_n\varphi(x - z_n)| \\ &\leq 2\|V_0\|_\infty\|u_n\|_2 \left(\int_{|k|\geq\epsilon} |\varphi(x - z_n)|^2\right)^{1/2} + \epsilon\|u_n\|_2\|\varphi\|_2 \\ &\leq C_4 \left(C_0 \int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 + C_1\epsilon^{2/N}\|\varphi\|_H^2\right)^{1/2} + C_5\epsilon \\ &= o_n(1) + C_6\epsilon^{1/N} + C_5\epsilon. \end{aligned}$$

Let $\epsilon \to 0$, Lemma 2.5 is proved. □

3 Proof of Theorem 1.1

By assumptions (h_1) and (h_2), we get that $p \leq q$ and

$$|h(s)s| \leq C|s|^p, \qquad \forall |s| \leq \delta.$$

Then, we need to modify $h(u)$ to prove our main results. Set

$$g(s) := \begin{cases} 0, & s \le 0, \\ h(s), & 0 < s \le \delta, \\ C_1 s^{p-1}, & s > \delta. \end{cases}$$

We can fix $C_1 > 0$ such that $g \in C(\mathbb{R}, \mathbb{R}^+)$. According to the definition of g, and we can get the following lemma easily.

Lemma 3.1. *Suppose that* (h_1) *is satisfied. Then*

(1) $\lim_{s\to+\infty} \frac{G(s)}{s^2} = +\infty$, *where* $G(s) = \int_0^s g(t)dt$.

(2) *there exists* $C > 0$ *such that* $|g(s)s| \le C|s|^p$ *and* $|G(s)| \le C|s|^p$ *for all* $s \in \mathbb{R}$.

(3) *there exists* $\mu \in (2, p)$ *such that the function* $s \mapsto \frac{g(s)}{s^{\mu-1}}$ *is strictly increasing on* $(0, +\infty)$.

Let us consider the modified equation of problem (2.5) given by

$$-\triangle v + V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))} = \lambda \frac{g(F^{-1}(v))}{f(F^{-1}(v))}, \qquad x \in \mathbb{R}^N. \tag{3.1}$$

We note that the solutions of problem (3.1) are the critical points of the functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|F^{-1}(v)|^2) - \lambda \int_{\mathbb{R}^N} G(F^{-1}(v)).$$

In order to prove our results, we need the periodic problem as follow

$$-\triangle v + V_0(x)\frac{F^{-1}(v)}{f(F^{-1}(v))} = \lambda \frac{g(F^{-1}(v))}{f(F^{-1}(v))}, \qquad x \in \mathbb{R}^N, \tag{3.2}$$

whose corresponding energy functional is denoted as

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V_0(x)|F^{-1}(v)|^2\right] - \int_{\mathbb{R}^N} G(x, F^{-1}(v)).$$

Define

$$\mathcal{N} = \{u \in E : \langle I'(u), u\rangle = 0, u \ne 0\}, \qquad \mathcal{N}_0 = \{u \in E : \langle I_0'(u), u\rangle = 0, u \ne 0\},$$

$$c = \inf_{u\in\mathcal{N}} I(u), \qquad c_0 = \inf_{u\in\mathcal{N}_0} I_0(u).$$

Then we can deduce the following lemma.

Lemma 3.2. *Suppose that conditions (V) and* $(h_1), (h_2)$ *hold, then for each* $u \in E,\ u \ne 0$, *there is a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}$. *Moreover, the maximum of* $I(tu)$ *for* $t \ge 0$ *is achieved at* t_u.

Proof. By Lemma 2.1-(3), Lemma 3.1-(2) and the Sobolev inequality, one has

$$\int_{\mathbb{R}^N} G(F^{-1}(tu)) \le C\int_{\mathbb{R}^N} |F^{-1}(tu)|^p \le Ct^p \int_{\mathbb{R}^N} u^p \le Ct^p\|u\|^p. \tag{3.3}$$

It follows from Lemma 2.1-(3), (3.3) and Lemma 2.2 that

$$\begin{aligned}\Psi(t) := I(tu) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla(tu)|^2 + V(x)|F^{-1}(tu)|^2\right] - \lambda\int_{\mathbb{R}^N} G(F^{-1}(tu))\\ &\geq \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{t^2}{3}\int_{\mathbb{R}^N} V(x)u^2 - \lambda C t^p\|u\|^p\\ &\geq \frac{t^2}{3}\|u\|^2 - \lambda C t^p \|u\|^p.\end{aligned}$$

Therefore, we can get $\Psi(t) > 0$ whenever $t > 0$ is small enough.

Let $\Omega = \{x \in \mathbb{R}^N : u(x) > 0\}$, then thanks to Lemma 3.1-(1), Lemma 2.1-(3)(5) and the Fatou Lemma, we can deduce that

$$\limsup_{t\to\infty}\frac{\Psi(t)}{t^2} \leq \frac{1}{2}\|u\|^2 - \lambda\liminf_{t\to\infty}\int_\Omega \frac{G(F^{-1}(tu))}{|F^{-1}(tu)|^2}\cdot\frac{|F^{-1}(tu)|^2}{(tu)^2}\cdot u^2 = -\infty.$$

Hence, $\Psi(t) \to -\infty$ as $t \to \infty$ and Ψ has a positive maximum.

The condition $\Psi'(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N}|\nabla u|^2 = \int_{\mathbb{R}^N}\left[\frac{\lambda g(F^{-1}(tu))}{tuf(F^{-1}(tu))} - \frac{V(x)F^{-1}(tu)}{f(F^{-1}(tu))tu}\right]u^2.$$

Let

$$Z(s) := \frac{g(s)}{f(s)F(s)} - \frac{V(x)s}{f(s)F(s)}.$$

By Lemma 3.1-(3) and Lemma 2.1-(7)(8), $s \mapsto Z(s)$ is strictly increasing for $s > 0$, so there is a unique $t_u > 0$ such that $\Psi'(t_u) = 0$. The conclusion is true since $\Psi'(t) = t^{-1}\langle I'(tu), tu\rangle$. □

Lemma 3.3. *Suppose that (V) and (h_1), (h_2) hold. Then*

(i) there exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}$.

(ii) the functional I is bounded from below on $\mathcal{N}$ by a positive constant.

Proof. (i) For any $u \in \mathcal{N}$, By Lemma 3.1-(1)(2), Lemma 2.2-(1)(3) and the Sobolev inequality, we have

$$\begin{aligned}\frac{2}{3}\|u\|^2 &\leq \int_{\mathbb{R}^N}|\nabla u|^2 + \int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}u = \lambda\int_{\mathbb{R}^N}\frac{g(F^{-1}(u))}{f(F^{-1}(u))}u\\ &\leq \lambda C\int_{\mathbb{R}^N} u^p \leq \lambda C\|u\|^p.\end{aligned}$$

Hence, there exists $\rho > 0$ independent of u such that $\|u\| \geq \rho$.

(ii) It follows from (3.3) and Lemma 2.1-(3) that

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(u)|^2 - \lambda\int_{\mathbb{R}^N} G(F^{-1}(u))\\ &\geq \frac{1}{3}\|u\|^2 - \lambda C\|u\|^p.\end{aligned}$$

Since $p > 2$, there exists $\sigma > 0$ such that $I(u) \geq \frac{\sigma^2}{4} > 0$ for $\|u\| = \sigma > 0$. For any $v \in \mathcal{N}$, there exists $t_1 > 0$ such that $t_1\|v\| = \sigma$. By Lemma 3.1-(1)(2), we obtain

$$I(v) \geq I(t_1 v) \geq \frac{\sigma^2}{4}.$$

This completes the proof. □

Lemma 3.4. *Suppose that conditions (V) and $(h_1), (h_2)$ are satisfied. If $u \in \mathcal{N}$ and $I(u) = c$, then u is a ground state solution of problem (3.1) (see [8, 16]).*

It follows from [16] that the periodic problem (3.2) has a positive ground state solution u. From Lemma 3.2, there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t \geq 0$ is achieved at t_u. Thanks to $V(x) \leq V_0(x)$, we obtain

$$c \leq I(t_u u) \leq I_0(t_u u) \leq I_0(u) = c_0, \tag{3.4}$$

hence $c \leq c_0$. Thanks to Lemma 3.3-(ii), we can also get $c > 0$.

As the argument in [14, Theorem 4.2], we obtain the following lemma due to Lemmas 3.1–3.3.

Lemma 3.5. *Suppose that (V) holds, h satisfies $(h_1), (h_2)$, then*

$$c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in E} \max_{t > 0} I(tu) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(t)) < 0\}$.

The above lemma is also valid for functional I_0.

Next, we will give the boundedness of the Cerami sequences.

Lemma 3.6. *Suppose that conditions (V) and $(h_1), (h_2)$ hold. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence for the functional I. Then $\{u_n\}$ is bounded in E .*

Proof. Suppose by contradiction that $\{u_n\} \subset E$ be a sequence such that $\|u_n\| \to \infty$, $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. Set $v_n := \frac{u_n}{\|u_n\|}$, then, there is a $v \in E$ such that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$.

If $v \neq 0$, let $\Omega_* = \{x \in \mathbb{R}^N : v(x) > 0\}$, then meas $\Omega_* > 0$. For a.e. $x \in \Omega_*$, one has

$$u_n(x) \to +\infty \quad \text{as } \|u_n\| \to +\infty,$$

since $v_n(x) = \frac{u_n(x)}{\|u_n\|} \to v(x) > 0$ for a.e. $x \in \Omega_*$, from Lemma 2.1-(5) and the fact that $F^{-1}(t)$ is strictly increasing, we can deduce that for a.e. $x \in \Omega_*$,

$$F^{-1}(u_n) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

It follows from Lemma 2.1-(3)(5) and Lemma 3.1-(1) that

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\ &\leq \limsup_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n))}{\|u_n\|^2} \\ &= \frac{1}{2} - \lambda \liminf_{n\to\infty} \int_{\Omega_*} \left(\frac{G(F^{-1}(u_n))}{|F^{-1}(u_n)|^2} \cdot \frac{|F^{-1}(u_n)|^2}{u_n^2} \cdot v_n^2 \right) \\ &= -\infty. \end{aligned}$$

A contradiction, thus $v = 0$. Define

$$\beta := \limsup_{n\to\infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} v_n^2 dx.$$

If $\beta = 0$, by the Lions lemma [14, Lemma 1.21], we get $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. It follows from Lemma 3.1-(2) and Lemma 2.1-(3) that

$$\int_{\mathbb{R}^N} G(F^{-1}(tv_n)) \leq C \int_{\mathbb{R}^N} |F^{-1}(tv_n)|^p \leq Ct^p \int_{\mathbb{R}^N} |v_n|^p = o_n(1), \tag{3.5}$$

for any $t \geq 0$. Especially, set $t = 4\sqrt{c}$, we obtain

$$\int_{\mathbb{R}^N} G(F^{-1}(4\sqrt{c}v_n)) = o_n(1). \tag{3.6}$$

By Lemma 2.1-(4), one has $F^{-1}(4\sqrt{c}v_n) \to 4\sqrt{c}v_n$, since $4\sqrt{c}v_n \to 0$ a.e. in $\mathbb{R}^N$. Then, we can deduce that

$$\int_{\mathbb{R}^N} V(x) \left[(4\sqrt{c}v_n)^2 - [F^{-1}(4\sqrt{c}v_n)]^2\right] = o_n(1). \tag{3.7}$$

Setting

$$k(x,s) = \lambda \frac{g(F^{-1}(s))}{f(F^{-1}(s))} - V(x)\frac{F^{-1}(s)}{f(F^{-1}(s))} + V(x)s,$$

and

$$K(x,s) := \int_0^s k(x,t)dt = \lambda G(F^{-1}(s)) - \frac{1}{2}V(x)|F^{-1}(s)|^2 + \frac{1}{2}V(x)s^2.$$

Then,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] - \int_{\mathbb{R}^N} K(x,u). \tag{3.8}$$

Thanks to (3.6) and (3.7), we can obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n) = \lambda \int_{\mathbb{R}^N} G(F^{-1}(4\sqrt{c}v_n)) \\ + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[(4\sqrt{c}v_n)^2 - [F^{-1}(4\sqrt{c}v_n)]^2\right] = o_n(1).\end{aligned}$$

By the continuity of I, there exists $t_n \in [0,1]$ such that $I(t_n u_n) = \max_{0\leq t\leq 1} I(tu_n)$. Since $\|u_n\| \to \infty$, we have $\frac{4\sqrt{c}}{\|u_n\|} \leq 1$ when n is large enough. Hence, one has

$$\begin{aligned}I(t_n u_n) + o_n(1) &\geq I\left(\frac{4\sqrt{c}}{\|u_n\|}u_n\right) + o_n(1) = I(4\sqrt{c}v_n) + o_n(1) \\ &= 8c\|v_n\|^2 - \int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n) + o_n(1) \\ &= 8c + o_n(1).\end{aligned}$$

Note that $I(u_n) \to c$, so $0 < t_n < 1$ and $\langle I'(t_n u_n), t_n u_n\rangle = 0$ when n is large enough. By Lemma 3.1-(3) and Lemma 2.1-(7)(8), the function

$$\frac{k(x,s)}{s} = \frac{\lambda g(F^{-1}(s))}{f(F^{-1}(s))s} - V(x)\frac{F^{-1}(s)}{f(F^{-1}(s))s} + V(x)$$

is strictly increasing for $s > 0$. Since $\{u_n\}$ is a Cerami sequence of I and the monotonicity of $\frac{k(x,s)}{s}$, we can conclude

$$\begin{aligned}
c &= I(u_n) + o_n(1) \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,u_n)u_n - K(x,u_n)\right) + o_n(1) \\
&\geq \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,t_nu_n)t_nu_n - K(x,t_nu_n)\right) + o_n(1) \\
&= I(t_nu_n) - \frac{1}{2}\langle I'(t_nu_n), t_nu_n\rangle + o_n(1) \\
&= I(t_nu_n) + o_n(1) \\
&\geq 8c + o_n(1),
\end{aligned}$$

which is a contradiction for $c > 0$.

If $\beta > 0$, by the definition of β, there is $z_n \in \mathbb{R}^N$ such that

$$\frac{\beta}{2} < \int_{B_1(z_n)} v_n^2.$$

If z_n is bounded, there exists $R > 0$ such that

$$\frac{\beta}{2} < \int_{B_R(0)} v_n^2,$$

which is a contradiction with $v_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.

If z_n is unbounded, up to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := v_n(x+z_n) = \frac{u_n(x+z_n)}{\|u_n\|}$, we have

$$\frac{\beta}{2} < \int_{B_1(0)} w_n^2. \tag{3.9}$$

There is a function $w \in E$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Moreover, by (3.9), one has $w(x) \neq 0$. Define $\Omega_{**} = \{x \in \mathbb{R}^N : w(x) > 0\}$, then $meas\Omega_{**} > 0$ and for a.e. $x \in \Omega_{**}$, we have

$$u_n(x+z_n) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

Since $F^{-1}(t)$ is strictly increasing for $t \geq 0$, by Lemma 2.1-(5), we can conclude that for a.e. $x \in \Omega_{**}$,

$$F^{-1}(u_n(x+z_n)) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

Then, from Lemma 3.1-(1) and Lemma 2.1-(5), one has

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(F^{-1}(u_n))}{\|u_n\|^2} \\
&\quad= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(F^{-1}(u_n(x+z_n)))}{\|u_n\|^2} \\
&\quad\geq \liminf_{n\to\infty} \int_{\Omega_{**}} \frac{G(F^{-1}(u_n(x+z_n)))}{|F^{-1}(u_n(x+z_n))|^2} \frac{|F^{-1}(u_n(x+z_n))|^2}{(u_n(x+z_n))^2} w_n^2 \\
&\quad= +\infty.
\end{aligned}$$

Combining the above inequality with Lemma 2.1-(3), we have

$$\begin{aligned}0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2}\\ &\le \frac{1}{2} - \lambda \liminf_{n\to\infty} \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} G(F^{-1}(u_n))\\ &= -\infty,\end{aligned}$$

this contradiction finished the proof. □

Lemma 3.7. *Suppose that conditions (V) and $(h_1), (h_2)$ hold. Then problem (3.1) has a positive ground state solution.*

Proof. It follows from Lemma 3.3-(ii) and (3.44) that

$$0 < c \le c_0.$$

If $c = c_0$, we can get from (3.4) that

$$c_0 = c \le I(t_u u) \le I_0(t_u u) \le I_0(u) = c_0.$$

Then $t_u u$ is a positive ground solution of problem (3.1).

If $0 < c < c_0$, we see that I satisfies the mountain pass geometry from the proof of Lemma 3.2. Then, we can get a Cerami sequence $\{u_n\}$ on level c due to Lemma 2.3. Applying Lemma 3.6, the $(C)_c$ sequence is bounded. Then, we may get, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By using the Lebesgue dominated convergence theorem, through the standard discussion, we can get that

$$0 = \langle I'(u_n), \phi\rangle + o_n(1) = \langle I'(u), \phi\rangle,$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$, i.e. u is a weak solution of problem (3.1).

(i) The case $u \ne 0$. Since u is a weak solution of problem (3.1), $I(u) \ge c$ and $u \in \mathcal{N}$. By (3.8), the monotonicity of $\frac{k(x,s)}{s}$ and the Fatou lemma, one has

$$\begin{aligned}c &= I(u_n) + o_n(1)\\ &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1)\\ &= \int_{\mathbb{R}^N}\left(\frac{1}{2}k(x,u_n)u_n - K(x,u_n)\right) + o_n(1)\\ &\ge \int_{\mathbb{R}^N}\left(\frac{1}{2}k(x,u)u - K(x,u)\right) + o_n(1)\\ &= I(u) - \frac{1}{2}\langle I'(u), u\rangle\\ &= I(u).\end{aligned}$$

Hence, $I(u) = c$ and $I'(u) = 0$, which implies that u is a ground state solution of problem (3.1). Moreover, we could deduce that u is a positive solution by applying the strongly maximum principle.

(ii) The case $u = 0$. Define

$$\beta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} u_n^2.$$

If $\beta = 0$, by the Lions lemma [14, Lemma 1.21], we get $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$. It is similar to the proof of (3.5), we can deduce

$$\int_{\mathbb{R}^N} G(F^{-1}(u_n)) \leq o_n(1). \tag{3.10}$$

Combining (3.10) with Lemma 2.1-(3), we obtain

$$\begin{aligned} c &= I(u_n) + o_n(1) \\ &\leq \frac{1}{2}\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n)) = o_n(1). \end{aligned}$$

A contradiction, thus $\beta > 0$. By the definition of β, up to a subsequence, there exist $R > 0$ and $z_n \in \mathbb{Z}^N$ such that

$$\int_{B_R(0)} u_n^2(x + z_n) = \int_{B_R(z_n)} u_n^2(x) > \frac{\beta}{2}.$$

If z_n is bounded, there is $R' > 0$ such that

$$\int_{B_{R'}(0)} u_n^2 \geq \int_{B_R(z_n)} u_n^2 > \frac{\beta}{2},$$

which contradicts with $u_n \to u = 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Thus, z_n is unbounded, going if necessary to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := u_n(x + z_n)$, then there exists a function $w \in E \backslash \{0\}$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. It follows from Lemma 2.5 that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} 0 &= \langle I'(u_n), \varphi(x - z_n)\rangle + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x - z_n) + \int_{\mathbb{R}^N} V(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \\ &\quad - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{f(F^{-1}(u_n))} \varphi(x - z_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x - z_n) + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \\ &\quad - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{f(F^{-1}(u_n))} \varphi(x - z_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \varphi + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(w_n)}{f(F^{-1}(w_n))} \varphi - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(w_n))}{f(F^{-1}(w_n))} \varphi + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(w)}{f(F^{-1}(w))} \varphi - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(w))}{f(F^{-1}(w))} \varphi \\ &= \langle I_0'(w), \varphi\rangle, \end{aligned}$$

i.e. w is a weak solution of the periodic problem (3.2).

On the one hand, it follows from Lemmas 2.4–2.5 that

$$\begin{aligned} c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(u_n)|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} u_n \\ &\quad + \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{2f(F^{-1}(u_n))} u_n - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n)) + o_n(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)|F^{-1}(u_n)|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n \\
&\quad + \lambda\int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{2f(F^{-1}(u_n))}u_n - \int_{\mathbb{R}^N} G(F^{-1}(u_n)) + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w_n)|^2 - \frac{F^{-1}(w_n)}{f(F^{-1}(w_n))}w_n\right] \\
&\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{g(F^{-1}(w_n))}{2f(F^{-1}(w_n))}w_n - G(F^{-1}(w_n))\right] + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w)|^2 - \frac{F^{-1}(w)}{f(F^{-1}(w))}w\right] \\
&\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{g(F^{-1}(w))}{2f(F^{-1}(w))}w - G(F^{-1}(w))\right] \\
&= I_0(w) - \frac{1}{2}\langle I_0'(w), w\rangle \\
&= I_0(w) \geq c_0,
\end{aligned}
$$

which is a contradiction with $c \leq c_0$. Hence, the case $u = 0$ cannot happen, this completes the proof. □

Lemma 3.8. *Suppose that (V) and (h_1) hold. If u is a critical point of I, then $u \in L^\infty(\mathbb{R}^N)$. Moreover, there is a constant $C > 0$ independent of λ such that*

$$
\|u\|_\infty \leq C\lambda^{\frac{1}{2^*-p}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{2^*-2}{2(2^*-p)}}.
$$

Proof. For all $k > 0$, we set

$$
u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k, \\ \pm k, & \text{if } \pm u(x) > k. \end{cases}
$$

We use $\varphi_k = |u_k|^{2(\beta-1)}u$ with $\beta > 1$ as a test function and calculate $\langle I'(u), \varphi_k\rangle = 0$, namely,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u_k|^{2(\beta-1)}|\nabla u|^2 + 2(\beta-1)\int_{\mathbb{R}^N}|u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k \\
&\qquad + \int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u = \lambda\int_{\mathbb{R}^N}\frac{g(F^{-1}(u))}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u. \quad (3.11)
\end{aligned}
$$

According to the facts that $u^2|\nabla u_k|^2 \leq u_k^2|\nabla u|^2$, $\beta > 1$, and the Sobolev inequality, we obtain

$$
\begin{aligned}
&\beta^2\int_{\mathbb{R}^N}\left(|u_k|^{2(\beta-1)}|\nabla u|^2 + 2(\beta-1)|u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k\right) \\
&\quad \geq \int_{\mathbb{R}^N}|u_k|^{2(\beta-1)}|\nabla u|^2 + \int_{\mathbb{R}^N}(\beta-1)^2|u_k|^{2(\beta-2)}u^2|\nabla u_k|^2 \\
&\qquad + \int_{\mathbb{R}^N}2(\beta-1)|u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k \\
&\quad = \int_{\mathbb{R}^N}\left|\nabla\left(|u_k|^{\beta-1}u\right)\right|^2 \\
&\quad \geq C\left(\int_{\mathbb{R}^N}\left||u_k|^{\beta-1}u\right|^{2^*}\right)^{\frac{2}{2^*}}, \quad (3.12)
\end{aligned}
$$

By Lemma 2.1-(1)(3) and Lemma 3.1-(2), we have

$$\int_{\mathbb{R}^N} \frac{g(F^{-1}(u))}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u \leq \int_{\mathbb{R}^N} g(F^{-1}(u))|u_k|^{2(\beta-1)}u \leq C\int_{\mathbb{R}^N} |u|^p|u_k|^{2(\beta-1)}. \tag{3.13}$$

Using Lemma 2.1-(1)(3) again, we can obtain

$$\int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u \geq \frac{2}{3}\int_{\mathbb{R}^N} V(x)|u_k|^{2(\beta-1)}u^2 \geq 0. \tag{3.14}$$

By (3.11)–(3.14) and the Hölder inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left||u_k|^{\beta-1}u\right|^{2^*}\right)^{\frac{2}{2^*}} \\
&\quad \leq C\beta^2\lambda\int_{\mathbb{R}^N}(|u|^{p-2}|u_k|^{2(\beta-1)}u^2) \\
&\quad \leq C\beta^2\lambda\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{p-2}{2^*}}\left(\int_{\mathbb{R}^N}\left||u_k|^{2(\beta-1)}u^2\right|^{\frac{2^*}{2^*-p+2}}\right)^{\frac{2^*-p+2}{2^*}}.
\end{aligned}$$

Then, let $k\to\infty$, we obtain

$$\|u\|_{\beta\cdot 2^*} \leq (C\beta^2\lambda)^{\frac{1}{2\beta}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta}}\|u\|_{\frac{2\cdot 2^*\beta}{2^*-p+2}}. \tag{3.15}$$

Set

$$\beta_m = \left(\frac{2^*-p+2}{2}\right)^{m+1}, \qquad m = 0,1,\ldots$$

Then we get

$$\frac{2\cdot 2^*\beta_m}{2^*-p+2} = 2^*\beta_{m-1}.$$

It follows from (3.15) that

$$\begin{aligned}
\|u\|_{\beta_m\cdot 2^*} &\leq (C\beta_m^2\lambda)^{\frac{1}{2\beta_m}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta_m}}\|u\|_{\frac{2\cdot 2^*\beta_m}{2^*-p+2}} \\
&= (C\lambda)^{\frac{1}{2\beta_m}}\beta_m^{\frac{1}{\beta_m}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta_m}}\|u\|_{\beta_{m-1}\cdot 2^*}.
\end{aligned}$$

According to the Moser iteration, we obtain

$$\|u\|_{\beta_m\cdot 2^*} \leq (C\lambda)^{\sum_{i=0}^m\frac{1}{2\beta_i}}\prod_{i=0}^m\beta_i^{\frac{1}{\beta_i}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4}\sum_{i=0}^m\frac{1}{\beta_i}}\|u\|_{2^*}. \tag{3.16}$$

Since $\beta_0 = (\frac{2^*-p+2}{2}) > 1$ and $\beta_i = \beta_0^{i+1}$, we get

$$\sum_{i=0}^m\frac{1}{\beta_i} = \sum_{i=0}^m\frac{1}{\beta_0^{i+1}}, \qquad \prod_{i=0}^m\beta_i^{\frac{1}{\beta_i}} = \prod_{i=0}^m(\beta_0^{i+1})^{\frac{1}{\beta_0^{i+1}}} = (\beta_0)^{\sum_{i=0}^m\frac{i+1}{\beta_0^{i+1}}}.$$

We can see

$$\sum_{i=0}^\infty\frac{i+1}{\beta_0^{i+1}} = \beta^* < +\infty, \quad \sum_{i=0}^\infty\frac{1}{\beta_0^{i+1}} = \frac{2}{2^*-p}.$$

Then, letting $m \to \infty$ in (3.16), we obtain that $u \in L^\infty(\mathbb{R}^N)$ and

$$\begin{aligned}\|u\|_\infty &\le C\lambda^{\frac{1}{2^*-p}}\beta_0^{\beta_*}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{2(2^*-p)}}\|u\|_{2^*}\\ &\le C\lambda^{\frac{1}{2^*-p}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{2^*-2}{2(2^*-p)}}. \end{aligned} \tag{3.17}$$

This lemma is proved. □

Proof of Theorem 1.1. According to Lemma 3.7, equation (3.1) has a ground state solution u and $u \in \mathcal{N}$. By (3.8), Lemma 3.3-(i) and the Sobolev embedding, we have

$$\begin{aligned} c &= I(u) - \frac{1}{\mu}\langle I'(u), u\rangle\\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2) + \int_{\mathbb{R}^N}[\frac{1}{\mu}k(x,u)u - K(x,u)]\\ &\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u\|^2. \end{aligned} \tag{3.18}$$

We can choose $v \in E \cap L^\infty(\mathbb{R}^N)$ such that $\|v\|_\infty < 1$. By (h_2) and (3) of Lemma 2.1, there exists a positive constant C_1 independent of λ such that

$$G(F^{-1}(tv)) \ge C_1|F^{-1}(tv)|^q \ge C|tv|^q, \quad t \in [0,1].$$

Meanwhile, there exists $\lambda_0 > 0$ such that $I(v) < 0$ for $\lambda \ge \lambda_0$. It follows from the definition of c, Lemma 3.1-(2) and Lemma 2.1-(3) that

$$\begin{aligned} c &\le \max_{t\in[0,1]} I(tv)\\ &\le \max_{t\in[0,1]} \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla v|^2dx + V(x)v^2) - \lambda\int_{\mathbb{R}^N}G(F^{-1}(tv))\\ &\le \max_{t\in[0,1]}\frac{t^2}{2}\|v\|^2 - Ct^q\lambda\int_{\mathbb{R}^N}|v|^q\\ &\le C\lambda^{-\frac{2}{q-2}}. \end{aligned} \tag{3.19}$$

Combining (3.17), (3.18) with (3.19), one has

$$\|u\|_\infty \le C\lambda^{\frac{1}{2^*-p}}\|u\|^{\frac{2^*-2}{2^*-p}} \le C\lambda^{\frac{1}{2^*-p}}\lambda^{\frac{1}{2-q}\cdot\frac{2^*-2}{2^*-p}}.$$

Since $p, q \in (2, 2^*)$, there exists $\lambda_1 \ge \lambda_0$ such that

$$\|u\|_\infty \le C\lambda_1^{\frac{(2^*-q)}{(2^*-p)(2-q)}} \le \delta.$$

Therefore, by the definition of g, we can obtain that u is also a positive solution of equation (2.5) for $\lambda \ge \lambda_1$. This ends the proof. □

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