

Bifurcation analysis of fractional Kirchhoff–Schrödinger–Poisson systems in \mathbb{R}^3

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Abstract. In this paper, we investigate the bifurcation results of the fractional Kirchhoff–Schrödinger–Poisson system

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi(x)u = \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0, 1)$ with $2t + 4s > 3$ and the potential function V is a continuous function. We show that the existence of components of (weak) solutions of the above equation associated with the first eigenvalue λ_1 of the problem

$$(-\Delta)^s u + V(x)u = \lambda g(x)u \quad \text{in } \mathbb{R}^3.$$

The main feature of this paper is the inclusion of a potentially degenerate Kirchhoff model, combined with the critical nonlinearity.

Keywords: Kirchhoff–Schrödinger–Poisson system, global bifurcation, first eigenvalue, fractional Laplacian, fixed point, whole space.

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1 Introduction and main results

In this paper, we investigate the bifurcation result of the fractional Kirchhoff–Schrödinger–Poisson system

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi u = \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (\mathcal{P})$$

where $s, t \in (0, 1)$ with $2t + 4s > 3$, $\lambda \in \mathbb{R}$, $p \in (0, 1)$, $g(x) \in L^{\frac{6}{5-p}}(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ is a weight function, the non-local coefficient $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ defined by $M(t) = a + bt$, where $a, b \geq 0$, and the Gagliardo semi-norm

$$[u]_s = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{1/2}.$$

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Here, we assume that $(-\Delta)^s$ is the fractional Laplacian which, up to a normalization constant, is denoted as

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

for every $u \in C_0^\infty(\mathbb{R}^3)$, where $B_\varepsilon(x)$ is the ball of \mathbb{R}^3 centered at $x \in \mathbb{R}^3$ with radius $\varepsilon > 0$.

The Kirchhoff–Schrödinger–Poisson (KSP) system, including (\mathcal{P}) as a special model, describes the interaction of a quantum particle with an electromagnetic field. The (KSP) system consisting of a Schrödinger equation coupled with a Poisson equation and a Kirchhoff function has been studied extensively in various settings, such as Euclidean spaces, fractional spaces, and Heisenberg groups, due to its strong applications in physics. Some of the main topics of interest are the existence, multiplicity, and asymptotic behavior as well as the qualitative properties of the (weak) solutions such as regularity, symmetry, and concentration. For more information and references, one can consult the following papers [2, 6, 8–11, 23].

The fractional (KSP) system is a generalization of the (KSP) system that involves fractional derivatives of order s in $(0, 1)$. The fractional part of the system introduces new challenges and difficulties involving fractional derivatives and nonlocal and nonlinear properties. Various methods and techniques have been developed to deal with these problems, such as variational methods, the Nehari manifold, Ekeland variational principle, the concentration-compactness principle, and the mountain pass theorem. We refer the readers to [12, 16, 22, 23]. Benci and Fortunato in [3] first introduced the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

to describe solitary waves with an electronic field. More recently, the authors in [16] used variational methods to obtain nonnegative solutions for an Schrödinger–Choquard–Kirchhoff type fractional p -Laplacian

$$\left(a + b \|u\|_s^{p(\xi-1)} \right) [(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \lambda f(x, u) + \left(\int_{\mathbb{R}^N} \frac{|u|^{p_{v,s}^*}}{|x-y|^\mu} dy \right) |u|^{p_{v,s}^*-2}u \quad \text{in } \mathbb{R}^N,$$

where the nonlinearity f satisfies super-linear or sub-linear growth conditions and the parameter λ is large or small enough. In particular, it can be seen as a special case of the fractional Kirchhoff–Schrödinger–Poisson system.

On the other hand, bifurcation analysis is an important method of mathematics that studies how the qualitative behavior of solutions changes as a parameter varies, and moreover, a bifurcation point may correspond to the appearance or disappearance of the solutions or a change in their stability or symmetry. For instance, He, in [7], studied the nonhomogeneous semi-linear fractional Schrödinger equation with critical growth

$$\begin{cases} (-\Delta)^s u + u = u^{2^s-1} + \lambda(f(x, u) + h(x)), & x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u(x) > 0 & x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0, 1)$, $N > 4s$ and $\lambda > 0$ is a parameter. They showed that there exists a positive bifurcation value of the parameter such that the problem has exactly two positive solutions for smaller values, no positive solutions for larger values, and a unique solution at the bifurcation value. Furthermore, many recent works investigate the bifurcation results for the fractional Kirchhoff or Schrödinger or Poisson equation under different assumptions on the

potential functions and the non-linearities. Very recently, for $p \in (1, 2)$ and λ is small, Ruiz [19] demonstrated the existence of a branch of positive radial solutions to the problem

$$\begin{cases} -\Delta u + u + \lambda \phi u = u_+^p \\ -\Delta \phi = u^2, \quad \lim_{|x| \rightarrow +\infty} \phi(x) = 0. \end{cases}$$

After that, in [24], Xu, Qin, and Chen established bifurcation results for positive solutions by using the local and global bifurcation techniques, a priori bounds for elliptic equation, and the properties of the principal eigenvalues to the Kirchhoff-type problem involving sign-changing weight functions

$$\begin{cases} -(a(x) + b(x)\|u\|^2)\Delta u = \lambda m(x)u + h(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [14], the bifurcation results and the regularity for the (weak) solutions of the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + l(x)\phi u = \lambda a(x)|u|^{p-1}u + f(\lambda, x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2, & \text{in } \mathbb{R}^3 \end{cases}$$

are proved, where a, l are weight functions and f satisfies the subcritical and critical growth condition, respectively.

Motivated by the above works, especially by [19], this paper is dedicated to investigating bifurcation results to the (weak) solutions of the (KSP) system (\mathcal{P}) , while overcoming the challenges due to the lack of compactness in critical case as well as the degenerate nature of the Kirchhoff coefficient. To our knowledge, no such general results are provided for (\mathcal{P}) .

More precisely, we put the hypotheses in the following:

(V₁) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x \in \mathbb{R}^3} V(x) \geq V_0 > 0$, where $V_0 > 0$ is a constant;

(V₂) $\text{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \leq \xi\} < +\infty$ for any $\xi \in \mathbb{R}$;

(M₁)' $M \in C(\mathbb{R}_0^+)$ satisfies that for any $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$, such that $M(t) \geq \tau$ for all $t \geq \tau$;

(g₁) $g \in L^{6/(5-p)}(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $g(x) \geq 0$ a.e. in \mathbb{R}^3 .

It is worth stressing that the degenerate case of Kirchhoff nonlinearity is included in the assumption of (M₁)'.

Before stating our main results, let us introduce some notations. Firstly, thanks to the Fourier transform, the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\widehat{u}|^2 d\xi < \infty \right\},$$

which is equipped with the standard norm and inter product

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\widehat{u}|^2 d\xi \right)^{1/2}, \quad \langle u, v \rangle = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)\widehat{u}\widehat{v} d\xi.$$

In fact, Plancherel's theorem in [5] guarantees that $\|u\|_2 = \|\widehat{u}\|_2$ and $\| |\xi|^s \widehat{u} \|_2 = \|(-\Delta)^{\frac{s}{2}} u\|_2$, and then

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) dx \right)^{1/2}, \quad \langle u, v \rangle = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv) dx.$$

Furthermore, Proposition 3.4 and Proposition 3.6 in [5] imply that

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi = \frac{1}{C(s)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy.$$

By virtue of [5, Theorem 6.5], the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, with $p \in [2, 2_s^*]$, is continuous, where 2_s^* is the fractional critical Sobolev exponent, defined as $2_s^* = 6/(3 - 2s)$. Moreover, let $D^s(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx < \infty\}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $[u]_s$. The continuous fractional Sobolev embedding $D^s(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ yields that there exists a best Sobolev constant

$$S_* = \inf_{u \in D^s(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{2/2_s^*}},$$

so that

$$\|u\|_{2_s^*} \leq c[u]_s, \quad (1.1)$$

where $c = S_*^{-1/2}$. In this paper, the main solutions spaces E is the subspace of $H^s(\mathbb{R}^3)$, considered as

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \|u\| = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)|u|^2) dx \right)^{1/2} < \infty \right\}.$$

Obviously, E is a uniformly convex Banach space, see for instance [16].

Now, we state the main results of this paper in the following theorems.

Theorem 1.1. *Suppose that $s, p \in (0, 1)$ and the hypotheses (V_1) – (V_2) , $(M_1)'$ and (g_1) hold, equation (\mathcal{P}) has the unique bifurcation point $(0, 0)$, and there exists an unbounded component \mathcal{C} of positive weak solutions emanating from $(0, 0)$.*

Notation:

- \rightarrow and \rightharpoonup denote the strong convergence and the weak convergence, respectively.
- $L^p(\mathbb{R}^3)$, $1 \leq p \leq +\infty$, denotes a Lebesgue space, and the norm in $L^p(\mathbb{R}^3)$ is denoted by $\|\cdot\|_p$.
- C, C_i are various positive constants.

2 Preliminaries

In this section, as preparation for proving the main results, we intend to introduce some fundamental notations, definitions and properties which are essential to the fractional setting.

Let $s, t \in (0, 1)$, with $2t + 4s > 3$. Then, it follows that the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ is continuous, due to the fact that $\frac{12}{3+2t} \leq \frac{6}{3-2s} = 2_s^*$. For any $u \in H^t(\mathbb{R}^3)$, we define the linear functional $I_u : D^t(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$I_u(v) = \int_{\mathbb{R}^3} u^2 v dx, \quad \text{for any } v \in D^t(\mathbb{R}^3).$$

Obviously, from the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ in the above, there exists $C_1 > 0$, such that

$$|I_u(v)| \leq \left(\int_{\mathbb{R}^3} |u|^2 |x|^{-\frac{6}{3+2t}} dx \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{\frac{6}{3-2t}} dx \right)^{\frac{3-2t}{6}} \leq cS_*^{-1/2} \|u\|_{H^t}^2 [v]_t = c_0 \|u\|_{H^t}^2 [v]_t, \quad (2.1)$$

by (1.1) and the Hölder inequality, where $c_0 = cS_*^{-1/2}$. Hence, according to the Lax–Milgram theorem, for any $u \in H^t(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^t(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx, \quad \text{for any } v \in D^t(\mathbb{R}^3), \quad (2.2)$$

which concludes ϕ_u^t is the (weak) solution of $(-\Delta)^t \phi_u^t = u^2$ in \mathbb{R}^3 . Consequently, ϕ_u^t can be represented as

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy = c_t \frac{1}{|x|^{3-2t}} * u^2, \quad x \in \mathbb{R}^3,$$

where $c_t = \Gamma(3-2t)/(\pi^{3/2} 2^{2t} \Gamma(t))$ is the t -Riesz potential. Together with (2.1), taking ϕ_u^t as a test function of (2.2), we deduce that

$$[\phi_u^t]_t^2 = \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq c_0 \|u\|_{H^t}^2 [\phi_u^t]_t, \quad \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq c_0^2 \|u\|_{H^t}^4. \quad (2.3)$$

Now, substituting ϕ_u^t in problem (P), it follows that the fractional Kirchhoff–Schrödinger–Poisson equation

$$M([u]_s^2) (-\Delta)^s u + V(x)u + \phi_u^t u = \lambda g(x) |u|^{p-1} u + |u|^{2_s^*-2} u \quad \text{in } \mathbb{R}^3.$$

Definition 2.1. We call that $u \in H^s(\mathbb{R}^3)$ is a (weak) solution of problem (P), if for any $v \in E$, there holds

$$\int_{\mathbb{R}^3} (M([u]_s^2) (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x)uv + \phi_u^t uv) dx = \lambda \int_{\mathbb{R}^3} g(x) |u|^{p-1} uv dx + \int_{\mathbb{R}^3} |u|^{2_s^*-2} uv dx.$$

Furthermore, if there exist sequences $(\lambda_n)_n \subset \mathbb{R}$ and nontrivial (weak) solutions $(u_n)_n \subset E$ of problem (P), such that $(\lambda_n, u_n)_n \rightarrow (\lambda, 0)$ as $n \rightarrow \infty$, then $(\lambda, 0)$ is a bifurcation point of problem (P).

For more information on bifurcation, see, for instance [18]. Along this paper, let $(D^s(\mathbb{R}^3))^*$ be the dual space of $D^s(\mathbb{R}^3)$ and for each $u \in D^s(\mathbb{R}^3)$, let a functional $L : D^s(\mathbb{R}^3) \rightarrow (D^s(\mathbb{R}^3))^*$ be the weak formulation, defined by

$$\langle L(u), v \rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx, \quad \text{for any } v \in E.$$

Note that, by using the Hölder inequality,

$$|\langle L(u), v \rangle| \leq [u]_s [v]_s, \quad \langle L(u), u \rangle = [u]_s^2. \quad (2.4)$$

A simple observation of (2.4) yields that L is a bounded linear operator in $D^s(\mathbb{R}^3)$. Moreover, write for brevity,

$$\langle u, v \rangle_V = \int_{\mathbb{R}^3} V(x) uv dx, \quad \|u\|_V = \left(\int_{\mathbb{R}^3} V(x) |u|^2 dx \right)^{1/2}, \quad \text{for any } u, v \in E.$$

Of course, arguing as (2.4), it follows that

$$|\langle u, v \rangle_V| \leq \|u\|_V \|v\|_V, \quad \langle u, u \rangle_V = \|u\|_V^2.$$

Now, we are in the position to state some useful lemmas.

Lemma 2.2 ([13, Proposition 1.3]). *If X is uniformly convex and (2.4) holds, then L is of type (S), i.e. every sequence $(u_j)_j \in X$ such that*

$$u_j \rightharpoonup u, \quad \langle L(u_j), u_j - u \rangle \rightarrow 0$$

has a subsequence that converges strongly to u in X .

Lemma 2.3 ([21, Lemma 2.3]). *For any $u \in H^s(\mathbb{R}^3)$, the function ϕ_u^t defined in (2.2) satisfies the next properties.*

- (i₁) ϕ_u^t is continuous with respect to u .
- (i₂) $\phi_u^t \geq 0$ in \mathbb{R}^3 and $\phi_{\xi u}^t = \xi^2 \phi_u^t$ for any $\xi > 0$.
- (i₃) If $u_n \rightharpoonup u$ in E and $u_n \rightarrow u$ in $L^p(\mathbb{R}^3)$, with $p \in [2, 2_s^*)$, as $n \rightarrow \infty$, then for any $v \in E$

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n(x) v(x) dx = \int_{\mathbb{R}^3} \phi_u^t(x) u(x) v(x) dx + o(1),$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t(x) u(x)^2 dx, \quad \text{as } n \rightarrow \infty.$$

Lemma 2.4 ([15, Lemma 1.1]). *Assume that $s \in (0, 1)$ and (V_1) – (V_2) hold. If $p \in [2, 2_s^*]$, then the embeddings*

$$E \hookrightarrow H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$$

are continuous, with $\min\{1, V_0\}[u]_s \leq \|u\|$, for all $u \in E$. Particularly, there exists a positive constant C_q , such that

$$\|u\|_q \leq C_q \|u\| \quad \text{for all } u \in E.$$

If $q \in [2, 2_s^)$, the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ is compact. Furthermore, if $q \in [1, 2_s^*)$, then the embedding $E \hookrightarrow L^q(B_R)$ is compact for any $R > 0$.*

Furthermore, to prove the main results, we need the following embedding theorem due to Lemma 2.1 in [4].

Lemma 2.5. *Let $s \in (0, 1)$ and $w \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then the embedding*

$$D^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3, w dx)$$

is continuous and compact, and $\|u\|_{2,w} \leq C_w [u]_s$, for all $u \in D^s(\mathbb{R}^3)$, with $C_w = S_^{-1/2} \|w\|_{3/2}^{1/2} > 0$.*

3 The subcritical case

In this section, we shall demonstrate the bifurcation results of the fundamental problem

$$M([u]_s^2)(-\Delta)^s u + V(x)u + \phi_u^t u = \lambda g(x)|u|^{p-1}u \quad \text{in } \mathbb{R}^3, \quad (3.1)$$

which is of significance in substantiating the proof of the main result. To this aim, let us consider the property of the first eigenvalue $\lambda_1(h)$ of the problem

$$(-\Delta)^s u + V(x)u = \lambda h(x)u, \quad (3.2)$$

where $h \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a strictly positive function.

Lemma 3.1. *The eigenvalue problem (3.2) has the first eigenpair $(\lambda_1(h), u_1)$, where*

$$0 < \lambda_1(h) = \min_{v \in E \setminus \{0\}} \frac{\|v\|^2}{\|v\|_{2,h}^2} = \min_{v \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}v|^2 + V(x)|v|^2)dx}{\int_{\mathbb{R}^3} h(x)|v|^2dx},$$

and the first eigenfunction u_1 has one sign. Furthermore, λ_1 is decreasing map with respect to h , i.e. if $0 < h_1 \leq h_2 \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $\lambda_1(h_1) \geq \lambda_1(h_2)$.

Proof. Let $(v_k)_k \subset E \setminus \{0\}$ be a minimizing sequence of $\lambda_1(h)$ in Calculus of Variations. It can be normalized so that $\int_{\mathbb{R}^3} h(x)|v_k|^2dx = 1$, and

$$\lambda_1(h) = \lim_{k \rightarrow \infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_k|^2dx + \int_{\mathbb{R}^3} V(x)|v_k|^2dx \right). \quad (3.3)$$

Moreover, the fact that $\|v\|_s \leq \|v\|_s$ for any $v \in E$ guarantees that $(|v_k|)_k$ is also a minimizing sequence, then we can further assume that v_k is positive. Since $\|v_k\|^2$ is a real convergent sequence in (3.3), we have

$$0 \leq \|v_k\|^2 \leq \lambda_1 + 1.$$

Consequently, the sequence $(v_k)_k$ is bounded in E . The reflexivity of E yields the existence of $0 \leq \hat{v} \in E$ such that $v_k \rightharpoonup \hat{v}$ in E and $v_k \rightarrow \hat{v}$ a.e. in \mathbb{R}^3 , up a subsequence if necessary. Thanks to Lemma 2.5, we obtain that

$$\int_{\mathbb{R}^3} h|v_k|^2dx \rightarrow \int_{\mathbb{R}^3} h|\hat{v}|^2dx \quad \text{as } k \rightarrow \infty. \quad (3.4)$$

Moreover, by the weak lower semi-continuity of the norm $\|\cdot\|$ and by (3.4), it follows that

$$0 \leq \|\hat{v}\| \leq \liminf_{k \rightarrow \infty} \|v_k\|.$$

Thus, $\lambda_1 = \|\hat{v}\|^2$ and \hat{v} is a critical point of $\psi(v) = \|v\|^2 / \|v\|_{2,h}^2$, i.e. for any $v \in E$

$$\begin{aligned} \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}}\hat{v} (-\Delta)^{\frac{s}{2}}v + V(x)\hat{v}v)dx \int_{\mathbb{R}^3} h(x)|\hat{v}|^2dx \\ - \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}\hat{v}|^2 + V(x)|\hat{v}|^2)dx \int_{\mathbb{R}^3} h(x)\hat{v}vdx = 0. \end{aligned}$$

In conclusion, \hat{v} is the first eigenfunction corresponding to λ_1 , provided that $\hat{v} \not\equiv 0$.

Clearly, the definition of λ_1 implies at once that $\lambda_1(h_1) \geq \lambda_1(h_2)$. \square

Proposition 3.2. *Let $P = \{v \in E^* : v \geq 0\}$ and let $f(x) \in P$. If $(M_1)'$ and (V_1) – (V_2) holds, then equation*

$$M([u]_s^2)(-\Delta)^s u + V(x)u = f(x) \quad \text{in } \mathbb{R}^3 \quad (3.5)$$

has a unique weak solution u in E . Furthermore, the operator $K : E^ \rightarrow E$, defined by $K(f) = u$, where u is the unique weak solution of (3.5), is continuous.*

Proof. Of course, if $f \equiv 0$, then $u = 0$ is a unique (weak) solution of equation (3.5). Next, put $f \not\equiv 0$ and set $R = [u]_s^2 \geq 0$. Then, the problem (3.5) becomes

$$M(R)(-\Delta)^s u + V(x)u = f(x) \quad \text{in } \mathbb{R}^3. \quad (3.6)$$

Problem (3.6) has a variational structure and $J : E \rightarrow \mathbb{R}$, denoted as

$$J(u) = \frac{1}{2}M(R)[u]_s^2 + \int_{\mathbb{R}^3} V(x)|u|^2dx - \langle f, u \rangle, \quad \text{for all } u \in E,$$

where $\langle \cdot, \cdot \rangle$ is the duality of E , is well defined and of class $C^1(E)$. It is easily deduced that the critical point of $J(u)$, defined by u_R , is a (weak) solution of (3.1). We first claim that J is coercive, bounded below, and sequentially weakly lower semi-continuous in E . Indeed, by Lemma 2.4 and $(M_1)'$, the Hölder inequality implies that

$$\begin{aligned} J(u) &\geq \frac{1}{2}M(R)[u]_s^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx - \|f\|_{E^*} \|u\| \\ &\geq \frac{1}{2} \min\{\kappa, V_0\} \|u\|^2 - C_f \|u\|. \end{aligned}$$

Consequently, $J(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ and so J is coercive in E . Now, for any minimizing sequence $(u_n)_n$ in E , with $J(u_n) \rightarrow \inf_{u \in E} J(u)$ as $n \rightarrow \infty$, the coerciveness of J guarantees that there exists $K > 0$, such that $\|u_n\| \leq K$. Thus, for all n , it follows from the Hölder inequality that

$$|J(u_n)| \leq \max \left\{ 1, \frac{1}{2}M(R) \right\} \|u_n\|^2 + C_f \|u_n\| \leq \max \left\{ 1, \frac{1}{2}M(R) \right\} K^2 + C_f K,$$

which infers that

$$\inf_{u \in E} J(u) \geq - \max \left\{ 1, \frac{1}{2}M(R) \right\} K^2 - C_f K.$$

Hence, J is bounded below. Moreover, if $v_n \rightharpoonup v$ in E , in view of the weakly lower semi-continuity of $\|\cdot\|$,

$$J(v) \leq \liminf_{n \rightarrow \infty} \left(\frac{1}{2}M(R)[v_n]_s^2 + \int_{\mathbb{R}^3} V(x)|v_n|^2 dx - \langle f, v_n \rangle \right),$$

We thus deduce that J is weakly lower semi-continuous. Consequently, it guarantees the existence of the unique global minimum u_R for the functional J in E , and moreover, u_R is obviously a (weak) solution of equation (3.6).

Next, let us turn to imply that u_R is also a (weak) solution of problem (3.5). Let $R_j \rightarrow R$ in \mathbb{R}_+ and let $(u_{R_j})_j$ be (weak) solutions of (3.5) with R replaced by R_j . Once again, by $(M_1)'$, the Hölder inequality and Lemma 2.4, we have

$$\min\{\kappa, V_0\} \|u_{R_j}\|^2 \leq M(R_j)[u_{R_j}]_s^2 + \|u_{R_j}\|_V^2 = \langle f, u_{R_j} \rangle \leq C_f \|u_{R_j}\|. \quad (3.7)$$

Thus, $\{u_{R_j}\}$ is bounded in E . The reflexivity of E , Lemmas 2.4 and 2.5 yield that, there exists $u \in E$, such that up to sequences, as $j \rightarrow \infty$,

$$(a) \ u_{R_j} \rightharpoonup u \text{ in } E; \quad (b) \ u_{R_j} \rightarrow u \text{ in } L^2(\mathbb{R}^3, w dx); \quad (c) \ u_{R_j} \rightarrow u \text{ in } L^q(\mathbb{R}^3) \text{ with } q \in [2, 2_s^*]. \quad (3.8)$$

Recalling that $R_j \rightarrow R$ and $M \in C(\mathbb{R}^3)$ in the hypothesis $(M_1)'$, one has

$$\begin{aligned} &M(R) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x) u v dx \\ &= \lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} \left(M(R_j) (-\Delta)^{\frac{s}{2}} u_{R_j} (-\Delta)^{\frac{s}{2}} v + V(x) u_{R_j} v \right) dx \\ &= \langle f, v \rangle \quad \text{for any } v \in E, \end{aligned}$$

and so u is also a weak solution of (3.6). Moreover, taking the test function $v = u_R - u$ in the weak form of (3.6) and applying the Hölder inequality, we deduce that

$$\begin{aligned} 0 &= M(R) \langle L(u) - L(u_R), u - u_R \rangle + \langle u - u_R, u - u_R \rangle_V \\ &= M(R) ([u]_s^2 - \langle L(u), u_R \rangle - \langle L(u_R), u \rangle + [u_R]_s^2) + \|u\|_V^2 - \langle u, u_R \rangle - \langle u_R, u \rangle + \|u_R\|_V^2 \\ &\geq M(R) ([u]_s^2 - 2[u]_s[u_R]_s + [u_R]_s^2) + \|u\|_V^2 - 2\|u\|_V\|u_R\|_V + \|u_R\|_V^2 \\ &= M(R) ([u]_s - [u_R]_s)^2 + (\|u\|_V - \|u_R\|_V)^2 \geq 0. \end{aligned} \quad (3.9)$$

We thus have $[u]_s^2 = [u_R]_s^2$ and $\|u\|_V = \|u_R\|_V$. Consequently,

$$\langle f, u - u_R \rangle = M(R) \langle L(u) - L(u_R), u - u_R \rangle + \langle u - u_R, u - u_R \rangle_V = 0,$$

and so $u = u_R$ a.e. in \mathbb{R}^3 due to the assumption that $f \not\equiv 0$. Hence,

$$u = u_R \quad \text{in } E, \quad (3.10)$$

and $u_{R_j} \rightharpoonup u_R$ in E due to (3.8)-(a). Now, we claim that

$$u_{R_j} \rightarrow u_R \quad \text{in } E. \quad (3.11)$$

From (3.8),

$$M(R_j) \langle L(u_{R_j}), u_{R_j} - u_R \rangle = \langle f, u_{R_j} - u_R \rangle - \int_{\mathbb{R}^3} V(x) u_{R_j} (u_{R_j} - u_R) dx \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Combining with (2.4) and the fact that $D^s(\mathbb{R}^3)$ is a uniformly space, $u_{R_j} \rightarrow u_R$ in $D^s(\mathbb{R}^3)$ by applying Lemma 2.2, and moreover $u_{R_j} \rightarrow u_R$ in E by using (3.8)-(b). Therefore, the claim holds and the (weak) solution u_R of (3.6) is continuous with respect to R .

Define $h : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h(R) = \frac{1}{M(R)} \langle f, u_R \rangle - \|u_R\|_V^2.$$

Note that, according to the continuity of mappings $R \mapsto \frac{1}{M(R)}$ by (M_1) and $R \mapsto u_R$, $h(R)$ is also a continuous mapping. Observe that $h(0) > 0$. In fact, we first claim that u_0 , with $R = 0$, is not a constant. Otherwise, $\|u_0\|_V \leq C_d [u_0]_s = 0$ for some $C_d > 0$, due to Lemma 2.4, which implies in particular that $u_0 = 0$ a.e. in \mathbb{R}^3 . Moreover, since u_0 is the (weak) solution of the problem

$$M(0)(-\Delta)^s u_0 + V(x)u_0 = f$$

and $f \not\equiv 0$, there is a contradiction with $u_0 = 0$ a.e. in \mathbb{R}^3 . For such u_0 ,

$$h(0) = \frac{1}{M(0)} \langle f, u_0 \rangle - \|u_0\|_V^2 = [u_0]_s^2 > 0.$$

Similarly, by using the same argument of (3.7) that u_R is bounded in E , there exists a positive constant C , such that

$$|h(R)| = \left| \frac{1}{M(R)} \langle f, u_R \rangle - \|u_R\|_V^2 \right| \leq \frac{1}{\kappa} C_f \|u_R\| + \|u_R\|_V^2 \leq C_{f,\kappa} \|u_R\| + \|u_R\|^2 \leq C.$$

Now, denote $h_1(R) : \mathbb{R} \rightarrow \mathbb{R}$ as $h_1 = h(R) - R$. Combining all facts in the above, there exists $R_1 > C_f$, such that

$$h_1(0) = h(0) > 0 \quad \text{and} \quad h_1(R_1) = h(R_1) - R_1 < 0.$$

The intermediate value theorem yields at once the existence of zero-point for h_1 . In other words, there exists $R > 0$, such that

$$R = h(R) = \frac{1}{M(R)} \langle f, u_R \rangle - \|u_R\|_V^2 = [u_R]_s^2.$$

Consequently, u_R is a weak solution of (3.1).

Consider the uniqueness of the (weak) solution of (3.1). Assume at first that there are distinct (weak) solutions $u_1, u_2 \in E$ of (3.1). Let $v = u_1 - u_2$ be the test function for the weak form of (3.1), which follows that

$$(a + b[u_1]_s^2) \langle L(u_1), u_1 - u_2 \rangle + \langle u_1, u_1 - u_2 \rangle_V = \int_{\mathbb{R}^3} f(u_1 - u_2) dx$$

and

$$(a + b[u_2]_s^2) \langle L(u_2), u_1 - u_2 \rangle + \langle u_2, u_1 - u_2 \rangle_V = \int_{\mathbb{R}^3} f(u_1 - u_2) dx$$

being u_1 and u_2 are the (weak) solutions of (3.1), where a, b are the constant given in the definition of Kirchhoff function M . As a consequence,

$$a \langle L(u_1) - L(u_2), u_1 - u_2 \rangle + bJ_1(u_1, u_2) + \langle u_1 - u_2, u_1 - u_2 \rangle_V = 0, \quad (3.12)$$

where

$$J_1(u_1, u_2) = [u_1]_s^2([u_1]_s^2 - \langle L(u_1), u_2 \rangle) + [u_2]_s^2([u_2]_s^2 - \langle L(u_2), u_1 \rangle).$$

By virtue of the Hölder inequality,

$$\begin{aligned} J_1(u_1, u_2) &\geq [u_1]_s^2([u_1]_s^2 - [u_1]_s[u_2]_s) + [u_2]_s^2([u_2]_s^2 - [u_2]_s[u_1]_s) \\ &\geq ([u_1]_s - [u_2]_s)([u_1]_s^3 - [u_2]_s^3) \geq 0. \end{aligned}$$

Then, clearly, by using the same argument of (3.9), from (3.12), $[u_1]_s = [u_2]_s$ and $\|u_1\|_V = \|u_2\|_V$. Similar to (3.10), it can be concluded that $u_1 = u_2$ in E .

Finally, it remains to prove that the operator K is continuous. Let $(f_j)_j \subset E^*$, $f \in E^*$ satisfy $f_j \rightarrow f$ strongly in E^* and $u_j, u \in E$ be the (weak) solutions of (3.1) corresponding to f_j and f , respectively. We only need to prove that $u_j \rightarrow u$ in E . Arguing as in the proof of (3.7) and (3.11), we conclude that $u_j \rightharpoonup u$ in E and $u_j \rightarrow u$ a.e. in $L^q(\mathbb{R}^3)$, with $q \in [2, 2_s^*)$, up to a sequence if necessary. Consequently,

$$\begin{aligned} M(u_j) \langle L(u_j), u_j - u \rangle &= \langle f_j, u_j - u \rangle - \langle u_j, u_j - u \rangle_V \\ &= \langle f_j - f, u_j \rangle + \langle f, u_j - u \rangle - \langle u_j, u_j - u \rangle_V \\ &\rightarrow 0, \quad \text{as } j \rightarrow \infty, \end{aligned}$$

which yields that $u_j \rightarrow u$ in E by Lemma 2.2. This completes the proof. \square

We next prove the bifurcation results of (3.1). For any fixed λ , first denote the operator $N_\lambda : E \rightarrow E^*$ pointwise for all $u, v \in E$ as

$$\langle N_\lambda(u), v \rangle = \int_{\mathbb{R}^3} [\lambda g(x) |u|^{p-1} u - \phi_u^t u] v dx,$$

where $\langle \cdot, \cdot \rangle$ is the duality of E . We assert that $N_\lambda(u)$ is a compact operator. Suppose that $(u_j)_j$ is a bounded sequence in E . Lemma 2.4 yields that there exist a subsequence of $(u_j)_j$ (still defined by $(u_j)_j$) and $u \in E$, such that for any $R > 0$, as $j \rightarrow \infty$,

$$(a_1) \ u_j \rightharpoonup u \quad \text{in } E \quad (a_2) \ u_j \rightarrow u \quad \text{in } L^q(\mathbb{R}^3), \text{ with } q \in [2, 2_s^*) \quad (a_3) \ u_j \rightarrow u \quad \text{a.e. in } \mathbb{R}^3. \quad (3.13)$$

By virtue of Lemma 2.3–(i₃), obviously it follows that

$$\sup_{\|v\| \leq 1} \int_{\mathbb{R}^3} (\phi_{u_j}^t u_j - \phi_u^t u) v dx \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Further, for all $R > 0$,

$$\begin{aligned} & \sup_{\|v\| \leq 1} \left| \int_{\mathbb{R}^3} g(x) (|u_j|^{p-1} u_j - |u|^{p-1} u) dx \right| \\ & \leq \sup_{\|v\| \leq 1} \left| \int_{B_R} g(x) (|u_j|^{p-1} u_j - |u|^{p-1} u) dx \right| + \sup_{\|v\| \leq 1} \left| \int_{\mathbb{R}^3 \setminus B_R} g(x) (|u_j|^{p-1} u_j - |u|^{p-1} u) dx \right|. \end{aligned} \quad (3.14)$$

Since $g \in L^{6/(5-p)}(\mathbb{R}^3)$, for any $\varepsilon > 0$, there is a constant $R > 0$ so large that

$$\begin{aligned} & \sup_{\|v\| \leq 1} \int_{\mathbb{R}^3 \setminus B_R} g(x) (|u_j|^{p-1} u_j - |u|^{p-1} u) v dx \\ & \leq \sup_{\|v\| \leq 1} \left(\int_{\mathbb{R}^3 \setminus B_R} |g(x)|^{\frac{6}{5-p}} dx \right)^{\frac{5-p}{6}} \left(\int_{\mathbb{R}^3 \setminus B_R} (|u_j|^p + |u|^p)^{\frac{6}{p}} dx \right)^{\frac{p}{6}} \|v\|_6 \\ & \leq \sup_{\|v\| \leq 1} \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_R)} 2^{\frac{6}{p}-1} (\|u_j\|_6^p + \|u\|_6^p) \|v\|_6 \\ & \leq 2^{\frac{6}{p}-1} c^{p+1} \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_R)} (\|u_j\|^p + \|u\|^p) \sup_{\|v\| \leq 1} \|v\| \\ & \leq \varepsilon/2. \end{aligned}$$

On the other hand, note that for all $R > 0$, the embedding $E \hookrightarrow L^q(B_R)$, with $q \in [1, 2_s^*)$, is compact by using Lemma 2.4. Hence, take a subsequence $(u_{j_k})_k \subset (u_j)_j$, such that $u_{j_k} \rightarrow u$ in $L^q(B_{R_\varepsilon})$ for all $q \in [1, 2_s^*)$, then up to a further subsequence, still denoted by $(u_{j_k})_k$, we have that $u_{j_k} \rightarrow u$ a.e. in B_{R_ε} . Thus, $g(x)|u_{j_k}|^{p+1} \rightarrow g(x)|u|^{p+1}$ a.e. in B_{R_ε} . Furthermore, for each measurable subset $B_E \subset B_{R_\varepsilon}$, with the help of (1.1), Lemma 2.4 and the Hölder inequality, we have

$$\int_{B_E} g(x)|u_{j_k}|^{p+1} dx \leq \|g\|_{L^{\frac{6}{5-p}}(B_E)} \|u_{j_k}\|_6^{p+1} \leq (cC)^{p+1} \|g\|_{L^{\frac{6}{5-p}}(B_E)},$$

being $(u_j)_j$ is bounded in E . Therefore, $(g(x)|u_{j_k}|^{p+1})_k$ is integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$, since $g \in L^{6/(5-p)}(\mathbb{R}^3)$ by the assumption. The Vitali convergence theorem shows that

$$\lim_{k \rightarrow \infty} \int_{B_{R_\varepsilon}} g(x)|u_{j_k}|^{p+1} dx = \int_{B_{R_\varepsilon}} g(x)|u|^{p+1} dx, \quad (3.15)$$

and so $g(x)|u_j|^{p+1} \rightarrow g(x)|u|^{p+1}$ in $L^1(B_{R_\varepsilon})$, since the sequence $(u_{j_k})_k$ is arbitrary. Therefore,

$$\sup_{\|v\| \leq 1} \left| \int_{B_R} g(x) (|u_j|^{p-1} u_j - |u|^{p-1} u) v dx \right| \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

and further (3.14) hold. Together with Proposition 3.2, we deduce that the operator $K \circ N_\lambda : E \rightarrow E$ is compact. For the fixed λ , let $K_\lambda : E \rightarrow E$, defined by

$$K_\lambda = \mathbb{I} - K \circ N_\lambda,$$

where \mathbb{I} is the identity operator. Note that the zeros of K_λ are exactly the (weak) solutions of the problem (3.1).

Having completed all necessary preparations, now, we are ready to show Theorem 3.3.

Theorem 3.3. *Let $s, p \in (0, 1)$. If (V_1) – (V_2) , $(M_1)'$ and (g_1) hold, equation (3.1) has the unique bifurcation point $(0, 0)$, and there exists an unbounded component C_0 of (weak) solutions emanating from $(0, 0)$.*

Proof. We first let $\lambda < 0$. For a fixed λ , consider the operator $H_1(r, \cdot) : E \rightarrow E$ as follows

$$H_1(r, u) = N_\lambda(r(\lambda g(x)|u|^{p-1}u - \phi_u^t u)), \quad r \in [0, 1].$$

We claim that there exists $\delta_1 > 0$, such that

$$u = H_1(r, u), \quad \text{for any } u \in B_{\delta_1}, u \neq 0 \text{ and } r \in [0, 1]. \quad (3.16)$$

Conversely, if there exists sequences $(u_n)_n$ and $(r_n)_n$, with $\|u_n\| \rightarrow 0$, $u_n \neq 0$ and $r_n \in [0, 1]$, such that $u_n = H_1(r_n, u_n)$. In other words, it follows that

$$\int_{\mathbb{R}^3} (M([u_n]_s^2)|(-\Delta)^{\frac{s}{2}}u_n|^2 + V(x)u_n^2 + r_n\phi_{u_n}^t u_n^2)dx = r_n \int_{\mathbb{R}^3} \lambda g(x)|u_n|^{p+1}dx \leq 0 \quad (3.17)$$

by the definition of λ . Thanks to (M_1) and (V_1) , we get $M([u_n]_s^2)[u_n]_s^2 + \|u_n\|_V^2 \geq 0$, and so $\|u_n\| = ([u_n]_s^2 + \|u_n\|_V^2)^{1/2} = 0$ by Lemma 2.3– (i_2) . Of course, this is a contradiction with the assumption that $u_n \neq 0$ in E and the claim is achieved. Therefore, we can choose $\varepsilon \in (0, \delta_1)$, such that

$$\deg(K_\lambda, B_\varepsilon, 0) = \deg(I - H_1(1, \cdot), B_\varepsilon, 0) = \deg(I - H_1(0, \cdot), B_\varepsilon, 0) = \deg(I, B_\varepsilon, 0) = 1 \quad (3.18)$$

by applying the homotopy invariance of H_1 .

On the other hand, let $\lambda > 0$ and let $\psi \in E$, with $\psi > 0$. For this fixed λ and for any $r \in [0, 1]$, denote $H_2(r, \cdot) : E \rightarrow E$ as

$$H_2(r, u) = N_\lambda(\lambda g(x)|u|^{p-1}u - \phi_u^t u + r\psi).$$

We claim that there exists $\delta_2 > 0$, such that $u \neq H_2(r, u)$ for any $u \in B_{\delta_2} \setminus \{0\}$ and for any $r \in [0, 1]$. Let us argue by contradiction that if there exists a sequence $(v_j)_j \subset E$, with $v_j > 0$ and $\|v_j\| \rightarrow 0$, as $j \rightarrow \infty$, such that for any $r_j \in [0, 1]$,

$$v_j = H_2(r_j, v_j), \quad (3.19)$$

which yields at once that

$$M([v_j]_s^2)(-\Delta)^s v_j + V(x)v_j + \phi_{v_j}^t v_j = \lambda g(x)|v_j|^{p-1}v_j + r_j\psi(x). \quad (3.20)$$

Moreover, there exists a positive constant C_0 , such that $\|v_j\| \leq C_0$ and $[v_j]_s \leq \|v_j\| \leq C_0$, being $\|v_j\| \rightarrow 0$ as $j \rightarrow \infty$. Furthermore, up to sequence,

$$v_j \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^3 \quad (3.21)$$

by Lemma 2.4. Consequently, $M([v_j]_s^2) \leq \max\{1, a + bC_0\} := C'_0$, and then

$$\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \leq \lambda_1((C'_0)^{-1}g(x)).$$

For any $\varepsilon > 0$, taking the test function as a first eigenfunction $w_1 > 0$, by virtue of (1.1) and the Hölder inequality, since $g \in L^{6/(5-p)}(\mathbb{R}^3)$ by the assumption (g_1) and v_j is bounded in E , there exists $R_\varepsilon > 0$ so large that for all j

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx &\leq \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|_6^p \|w_1\|_6 \\ &\leq c^{p+1} \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|^p \|w_1\| \leq \varepsilon/2. \end{aligned} \quad (3.22)$$

Thus, arguing as the proof of (3.15),

$$\int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx = o(1) \quad \text{as } j \rightarrow \infty.$$

Similarly, according to the assumption (g_2) , it is easily to see that $g(x)|v_j|w_1 \rightarrow 0$ in $L^1(B_{R_\varepsilon})$ as $j \rightarrow \infty$ and

$$\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)v_j w_1 dx \leq \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|_6 \|w_1\|_6 \leq c^2 \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\| \|w_1\| \leq \varepsilon, \quad (3.23)$$

being $g \in L^{3/2}(\mathbb{R}^3)$ by the assumption. In conclusion, from (3.21), (3.22) and (3.23), there is R_ε so large that as $j \rightarrow \infty$

$$\begin{aligned} &\lambda \int_{\mathbb{R}^3} g(x)|v_j|^p w_1 dx - \lambda_1((C'_0)^{-1}g(x)) \int_{\mathbb{R}^3} g(x)v_j w_1 dx - \int_{\mathbb{R}^3} \phi_{v_j} v_j w_1 dx \\ &= \lambda \int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx + \lambda \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx - \lambda_1((C'_0)^{-1}g(x)) \int_{B_{R_\varepsilon}} g(x)v_j w_1 dx \\ &\quad - \lambda_1((C'_0)^{-1}g(x)) \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j| w_1 dx - C \|v_j\|^3 \|w_1\| \\ &\geq \lambda \int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx - \lambda_1((C'_0)^{-1}g(x)) \int_{B_{R_\varepsilon}} g(x)|v_j| w_1 dx - C\varepsilon > 0. \end{aligned} \quad (3.24)$$

Since $\psi > 0$, (3.20) and (3.24) yield that as $n \rightarrow \infty$, we estimate

$$\begin{aligned} &\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x)v_j w_1 dx \\ &= \max\{M([v_j]_s^2), 1\} \left(\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)v_j w_1 dx \right) \\ &\geq M([v_j]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)v_j w_1 dx \\ &= \lambda \int_{\mathbb{R}^3} (g(x)|v_j|^{p-1} v_j w_1 + r_j \psi(x) w_1 - \phi_{v_j}^t v_j w_1) dx \\ &> \lambda_1((C'_0)^{-1}g(x)) \int_{\mathbb{R}^3} g(x)v_j w_1 dx, \end{aligned}$$

and so

$$\{\lambda_1(g(x) \max\{1, M([v_j]_s^2)\})^{-1}) - \lambda_1((C'_0)^{-1}g(x))\} \int_{\mathbb{R}^3} g(x)v_j w_1 dx > 0.$$

Since $\int_{\mathbb{R}^3} g(x)v_j w_1 dx > 0$, we have $\lambda_1(g(x) \max\{1, M([v_j]_s^2)\})^{-1}) > \lambda_1((C'_0)^{-1}g(x))$. This is an obvious absurdum, and we proved the claim.

Hence, choosing $\varepsilon \in (0, \delta_2)$, we can find the homotopy invariance of H_2 , i.e.

$$\deg(K_\lambda, B_\varepsilon, 0) = \deg(I - H_2(0, \cdot), B_\varepsilon, 0) = \deg(I - H_2(1, \cdot), B_\varepsilon, 0) = 0. \quad (3.25)$$

It follows from (3.18) and (3.25) that $(0, 0)$ is a bifurcation point of (\mathcal{P}) .

Now, it is sufficient to prove the existence of the unbounded component of (weak) solutions of (3.1). It is important to note that while the classical global bifurcation theorem [17, Theorem 1.3] is relevant to our argument, we cannot apply it directly because the operator K_λ lacks the differentiability at $u = 0$ and of odd-multiplicity eigenvalue. However, by modifying the global bifurcation theorem in Proposition 3.5 of [1] and replacing these conditions with the topological degree proofs for (3.18) and (3.25), we can derive an efficient version of [17, Theorem 1.3] for the assertion below.

For $\lambda_0 \neq 0$, we claim that $(\lambda_0, 0)$ is an isolated (weak) solution of (3.1). Set $\lambda < 0$. Similar to the analysis of (3.17), there are no nontrivial (weak) solutions of equation (3.1). Let $\lambda > 0$. Assume that there exists a sequence of (weak) solutions $(\lambda_n, u_n)_n \subset \mathbb{R} \times E$ of (3.1), such that $\lambda_n \rightarrow \lambda_0$ and $\|u_n\| \rightarrow 0$, as $n \rightarrow \infty$. Hence, arguing as (3.24), for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, such that for any $n \geq N(\varepsilon)$,

$$\begin{aligned} & \lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x)v_j w_1 dx \\ & \geq M([v_j]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)v_j w_1 dx \\ & \geq (\lambda_0 - \varepsilon) \int_{\mathbb{R}^3} g(x)|v_j|^{p-1} v_j w_1 dx - \int_{\mathbb{R}^3} \phi_{v_j}^t v_j w_1 dx \\ & > \lambda_1((C'_0)^{-1}g(x)) \int_{\mathbb{R}^3} g(x)v_j w_1 dx, \end{aligned}$$

which yields an absurdum $\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) > \lambda_1((C'_0)^{-1}g(x))$. Therefore, $(0, 0)$ is a unique bifurcation point of equation (3.1).

Furthermore, if \mathcal{C}_0 is bounded in $\mathbb{R} \times E$, by [17, Lemma 1.2] there is a bounded open set $\mathcal{O} \subset \mathbb{R} \times E$ such that $(0, 0) \in \mathcal{O}$ and \mathcal{O} contains nontrivial solution other than those in $B_\varepsilon \subset E$, with $\varepsilon > 0$ sufficiently small.

Now, we can argue as (1.11) of [17] to conclude that the existence of $\varepsilon > 0$ and values $\underline{\lambda}$ and $\bar{\lambda}$, such that $-\varepsilon < \underline{\lambda} < 0 < \bar{\lambda} < \varepsilon$ and $i(K_{\underline{\lambda}}, 0) = i(K_{\bar{\lambda}}, 0)$. Therefore, owing to (3.18) and (3.25), we have

$$1 = i(K_{\underline{\lambda}}, 0) = i(K_{\bar{\lambda}}, 0) = 0,$$

which is an obvious contradiction. Then, \mathcal{C}_0 is an unbounded component. \square

4 Main result

To determine the bifurcation results of problem (\mathcal{P}) , for any fixed λ , we define pointwise for $u, v \in E$, $T_\lambda : E \rightarrow E^*$ by

$$\langle T_\lambda(u), v \rangle = \int_{\mathbb{R}^3} \left\{ \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u - \phi_u^t u \right\} v dx.$$

Suppose that $(u_n)_n \subset E$ is a bounded sequence in E . Then up to a subsequence, (3.13) also holds for some $u \in E$ by the reflexivity of E . Recalling the compactness result for the operator

N_λ , as shown in section 3, it remains to prove that for any $v \in E$,

$$\int_{\mathbb{R}^3} (|u_n|^{2_s^*-2}u_n - |u|^{2_s^*-2}u)v dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.1)$$

Since $|u_n|^{2_s^*-2}u_n \in L^{(2_s^*)'}(\mathbb{R}^3)$, $v \in E$ and $E \subset L^{2_s^*}(\mathbb{R}^3)$, the definition of weak convergence yields at once that (4.1) is achieved. In conclusion, the operator $K \circ T_\lambda$ is also compact using Proposition 3.2.

Proof of Theorem 1.1. Let $H_\lambda : E \rightarrow E$ be defined as $H_\lambda(u) = K \circ T_\lambda(u)$, where K is the operator introduced by Proposition 3.2. Clearly, Theorem 3.3 guarantees the existence of the positive constants ε and δ , such that

$$\deg(K_\lambda, B_\delta, 0) = \begin{cases} 1, & \lambda \in (-\varepsilon, 0), \\ 0, & \lambda \in (0, \varepsilon). \end{cases}$$

We claim that for any λ , with $0 < \lambda < \varepsilon$, there exist δ_1 , such that for any $r \in [0, 1]$ and for the operator, defined by

$$\langle T_\lambda^r(u), v \rangle = \int_{\mathbb{R}^3} \left\{ \lambda g(x)|u|^{p-1}u - \phi_{u_n}^t u + r|u|^{2_s^*-2}u \right\} v dx,$$

the problem

$$u - K \circ T_\lambda^r(u) = 0 \quad (4.2)$$

has no (weak) solutions with $\|u\| = \delta_1$. Otherwise, if there exists a sequence of nontrivial (weak) solutions $(u_n)_n$ of (4.2), with $\|u_n\| \rightarrow 0$ and $u_n > 0$, then it yields that

$$M([u_n]_s^2)[u_n]_s^2 + \|u_n\|_V^2 + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3} \left\{ \lambda g(x)|u_n|^p + r|u_n|^{2_s^*} \right\} dx.$$

Thanks to (3.24), taking the test function as the first eigenvalue w_1 , we have

$$\begin{aligned} & \lambda_1(g(x)(\max\{1, M([u_n]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x)u_n w_1 dx \\ &= M([u_n]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)u_n w_1 dx \\ &= \int_{\mathbb{R}^3} (\lambda g(x)|u_n|^{p-1}u_n w_1 - \phi_{u_n}^t u_n w_1 + |u_n|^{2_s^*-1}w_1) dx \\ &> \lambda_1((C'_0)^{-1}g(x)) \int_{\mathbb{R}^3} g(x)u_n w_1 dx, \end{aligned}$$

which implies an absurdum that $\lambda_1(g(x)(\max\{1, M([u_n]_s^2)\})^{-1}) > \lambda_1((C'_0)^{-1}g(x))$. The claim holds. Hence, the homotopy invariance of the topological degree shows that for any $\lambda \in (0, \varepsilon)$ and $R \in (0, \delta_1)$

$$\deg(I - H_\lambda, B_R, 0) = \deg(K_\lambda, B_R, 0) = 0. \quad (4.3)$$

Fix $\lambda < 0$. Applying the same argument of (3.24), it follows that

$$\int_{\mathbb{R}^3} \left\{ \lambda g(x)|u|^{p+1} + |u|^{2_s^*} \right\} dx \leq 0.$$

Now similar to the analysis of (3.17), there are no nontrivial (weak) solutions of (4.2). Consequently, there exist $\varepsilon > 0$ and $\delta > 0$, with $\varepsilon \leq \varepsilon_1$ and $\delta \leq \delta_1$, such that for any $\lambda \leq \varepsilon$ and for any $R \leq \delta$

$$\deg(I - H_\lambda, B_R, 0) = \deg(K_\lambda, B_R, 0) = 1. \quad (4.4)$$

By utilizing (4.3) and (4.4), we get $(0, 0)$ is a bifurcation point of equation (\mathcal{P}) . Moreover, similar to the argument in Theorem 3.3, we imply that the existence of an unbounded component \mathcal{C} of weak solutions of (\mathcal{P}) . \square

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