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Linearized instability for differential equations with dependence on the past derivative

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Abstract. We provide a criterion for instability of equilibria of equations in the form $\dot{x}(t) = g(x_t', x_t)$, which includes neutral delay equations with state-dependent delay. The criterion is based on a lower bound $\Delta > 0$ for the delay in the neutral terms, on regularity assumptions of the functions in the equation, and on spectral assumptions on a semigroup used for approximation. The spectral conditions can be verified studying the associated characteristic equation. Estimates in the C^1-norm, a manifold containing the state space X_2 of the equation and another manifold contained in X_2, and an invariant cone method are used for the proof. We also give mostly self-contained proofs for the necessary prerequisites from the constant delay case, and conclude with an application to a mechanical example.

Keywords: neutral delay equations, dependence on past derivative, state-dependent delay, linearized instability.

2020 Mathematics Subject Classification: 34K40, 34K43, 34K20.

1 Introduction

Functional differential equations with constant delays, distributed delays, time-dependent delays, and state-dependent delays are all special cases of a dependence of the present derivative $\dot{x}(t)$ on the past history $\dot{x}|_{(-\infty,t]}$. (Some models also include dependence on the future.) A basic theory for equations with a general past dependence, following a generally familiar dynamical systems framework, is still in development, see e.g. the work of Nishiguchi [41]. The present paper is a contribution in this sense. We consider neutral equations $\dot{x}(t) = g(\dot{x}_t, x_t)$ with dependence on a bounded past interval, and with a lower bound of the delay in the derivative terms on the rght hand side. This includes neutral equations with a state-dependent point delay.

Equations with state-dependent retarded and advanced terms appear already in work of Poisson [44] from 1806. Papers by Driver [10] going back to the 1960s (on the particularly

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difficult case of the two-body problem of electrodynamics), or by Grimm [18] from 1971 are among the earliest that consider models with state-dependent time shift. But it seems that a systematic treatment by a larger number of authors started not earlier than in the late 1980s, for example, Jackiewicz 1987 [31], Mallet-Paret, Nussbaum and Paraskevopoulos [40], Jackiewicz 1995 [32], Hartung, Herdman and Turi [24], Krisztin [36] and Walther [49]. The article [25] gives an impression of the history of the subject. In models for real-world phenomena, state-dependent time shifts arise from position-dependent signal (or force, in the electrodynamics problem) propagation times, or from threshold conditions in mathematical biology. The resulting time shifts are sometimes implicitly defined via properties of the system state, and then a solution theory has to take the solvability of these implicit equations into account.

Neutral differential equations (i.e., the time derivative of the solution appears also on the right hand side of the equation $\dot{x}(t) = \dots$) arise in the famous two-body problem of electrodynamics, as well as in models of biological and mechanical systems, see for example [37], Chapter 9, [52] and [38]. Constant delays in such models certainly result from simplification, so it seems desirable to have a basic theory that covers also state-dependent variable delays, or, more general forms of dependence on the past derivative.

We introduce some notational conventions: Let $n \in \mathbb{N}$ and $h > 0$ be given. We assume that all delays are bounded above by h, so that the system state at time t is given by the segment $x_t \in (C^0[-h,0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h,0]$. By C^0 we briefly denote the Banach space of continuous functions $C^0([-h,0], \mathbb{R}^n)$ with the norm given by $|\phi|_{C^0} = \max_{-h \le t \le 0} |\phi(t)|$, here $|\ |$ is the 1-norm given by $|z| = \max_{j=1,\dots,n} |z_j|$ on $\mathbb{C}^n$, which also induces the 1-norm on $\mathbb{R}^n$. More generally, C^k denotes the Banach space of k-times continuously differentiable functions $\phi\colon [-h,0] \to \mathbb{R}^n$, with the C^k-norm given by $|\phi|_{C^k} = |\phi|_{C^0} + |\phi'|_{C^0} + \dots + |\phi^{(k)}|_{C^0}$. We write $C^k_{\mathbb{C}}$ for the complexified spaces, which we identify with $C^k([-h,0], \mathbb{C}^n)$. For functions defined on a different domain, e.g., an interval of the form $[-h, -\Delta]$, the corresponding notation is used. Sometimes balls are indexed with the intended norm, for example $B_{|\ |_{C^2}}(0,\delta) = \left\{ \psi \in C^2 \;\middle|\; |\psi|_{C^2} < \delta \right\}$. We also use the index $\mathbb{C}$ for canonical complexifications of linear operators, in particular, for semigroups and their generators.

In the present paper we adopt the framework of equations of the form

$$\dot{x}(t) = g(x_t', x_t) \tag{1.1}$$

introduced by Walther in [55]. We use the notation ψ', ψ'' etc. (instead of $\partial\psi, \partial\partial\psi$ etc. as in [55]), and we write a dot for derivatives at specific times. Note that $(x')_t = (x_t)'$ if $x|_{[t-h,t]}$ is of class C^1.

In eq. (1.1), the functional $g : W \subset C^0 \times C^1 \to \mathbb{R}^n$ is continuous on an open neighborhood W of zero in the product space $C^0 \times C^1$ (with $|\ |_{C^0}$ in the first and $|\ |_{C^1}$ in the second factor), and with an equilibrium at zero: $g(0,0) = 0$. For real numbers t_0, T with $t_0 < T$, a function $x : [t_0 - h, T) \to \mathbb{R}^n$ is a solution of equation (1.1) if it is of class C^1, satisfies $(x_t', x_t) \in W$ for $t \in [t_0, T)$, and (1.1) is true for $t \in [t_0, T)$.

This setting includes state dependent point delays of the form $\tau(x_t)$ as a special case. One main assumption is that the dependence of g on the first argument (the derivative history) has a minimal delay $\Delta > 0$, meaning that one stays in safe distance to implicit differential equations. Similar conditions were used in [31](p. 10, before Section 2), [45](condition (H), Section 4, p. 3980), and [23](condition (A4), p. 6), but, for example, not in [32]. This property (and also the presence of delayed, but not advanced terms) excludes the classical electrodynamics problem, as also remarked at the end of the introduction to [55]. A typical example class that does fit our framework is the $\mathbb{R}^n$ valued version

$$\dot{x}(t) = A[x'(t - d(x(t)))] + f[x(t - r(x(t)))], \tag{1.2}$$

of the example class from [54] (details in Section 2).

The purpose of the present work is to complement the linearized stability results from the papers [53] and [54], and also those from [2] and [22], with a linearized instability result. As in [54], one difficulty lies in the fact that the 'obvious candidate' for linearization at the zero solution, given by a semigroup $\{S^0(t)\}_{t\geq 0}$ of linear operators, does not have the usual quality of approximation for the full nonlinear equation.

The further organization of the paper is as follows: Section 2 lists the essential assumptions and gives a class of typical examples where they hold. Then Section 3 studies the linearization equation, associated semigroups on spaces of C^0 and C^1 functions, and spectral properties of generators and semigroups. The second part of Section 3 then prepares the study of the nonlinear equation, in particular, by a variation of constants formula from [53]. Properties related to the minimal value Δ of the delay in the derivative become important here. In Section 4, we introduce the nonlinear semiflow and preparatory estimates for solutions in the C^1-norm. The state space X_2 of the semiflow is contained in a manifold $\mathcal{M}_2$, which is tangent at zero to the state space of a semigroup obtained from linearization.

In Section 5 we use the manifold $\mathcal{M}_2$ to obtain a splitting of solutions into three terms, the first of which is given by a linear semigroup on a space of C^1 functions (namely, the so-called extended tangent space of $\mathcal{M}_2$ at zero), the second corresponds to the deviation between $\mathcal{M}_2$ and its tangent space, and the third to the nonlinear part of the equation.

For each of the three terms, C^1-estimates are possible for short time. In Lemma 5.4 we obtain the decisive estimate that expresses smallness of the nonlinear effects w.r.t. to the C^1-norm. In the following part of Section 5 we employ the additional smoothness condition $(\mathbf{D^2 g_2})$ to construct a manifold $\mathcal{M}_4$ contained in X_2, and describe its tangent space at to zero. A part of this manifold is then used to provide initial functions which will have an unstable evolution under the semiflow.

Section 6 contains the main theorem. Based on spectral assumptions, the estimate on the deviation between linear approximation and 'remainder terms', and the presence of suitable initial functions, an appropriate invariant cone method allows to prove the 'linearized instability' result.

Finally, in Section 7, we consider an example from [38] which models mechanical systems coupled to computer simulations. We show that generalizations of the equations considered in [38], in the sense of equations with state-dependent delay and nonlinear dependence of the delayed derivative, fit in our framework. The linearization at zero and its characteristic equation remain unchanged for these generalizations. Compared to [38] we give some additional analysis of the characteristic equation, and obtain an instability result for suitable values of the parameters, in particular, large enough values of the delay functional at zero.

2 Assumptions and typical examples

We adopt the general setting from [53–55]; in particular, we now list a number of hypotheses from these papers with the same numbering as in [53, 55], but in some cases described in slightly different notation. Conditions $(\widetilde{\mathbf{g1}})$ and $(\widetilde{\mathbf{g8}})$ are stronger versions of $(\mathbf{g1})$ and $(\mathbf{g8})$ from [54]; we comment on the assumptions in detail below.

Consider eq. (1.1), and define $U_1 := \left\{\psi \in C^1 \mid (\psi', \psi) \in W\right\}$; this is an open subset of C^1. We shall use the term **'bounding function'** for any nondecreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0 = \lim_{t\to 0} \varphi(t)$. Such bounding functions appear in several assumptions.

(g0) g is continuous (w.r. to $|\ |_{C^0}$ in the first and $|\ |_{C^1}$ in the second argument).

$(\widetilde{\mathbf{g1}})$ **(The delay in the neutral term of (1.1) has a lower bound.)** There exists $\Delta \in (0,h)$ such that for $(\phi_1,\psi), (\phi_2,\psi) \in W \subset C^0 \times C^1$, one has the implication

$$\forall t \in [-h,-\Delta]:\ \phi_1(t) = \phi_2(t) \Longrightarrow g(\phi_1,\psi) = g(\phi_2,\psi). \tag{2.1}$$

(g2) For every $\psi \in U_1 \subset C^1$, there exists $L_2 \geq 0$ and a neighborhood $N \subset W$ of (ψ',ψ) in $C^0 \times C^1$ such that for all $(\phi_1,\psi_1), (\phi_2,\psi_2)$ in N, with ϕ_2 Lipschitz continuous and best possible Lipschitz constant $\mathrm{Lip}(\phi_2)$, we have:

$$|g(\phi_2,\psi_2) - g(\phi_1,\psi_1)| \leq L_2(|\phi_2 - \phi_1|_{C^0} + (\mathrm{Lip}(\phi_2)+1)|\psi_2 - \psi_1|_{C^0}).$$

(g3) The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, and hence also has continuous partial derivatives $D_1 g_1, D_2 g_1 : W_1 \to L_c(C^1,\mathbb{R}^n)$. Every derivative $Dg_1(\phi,\psi) : C^1 \times C^1 \to \mathbb{R}^n$, $(\phi,\psi) \in W_1$, has a continuous linear extension: $D_e g_1(\phi,\psi) \in L_c(C^0 \times C^0,\mathbb{R}^n)$, and the map

$$W_1 \times C^0 \times C^0 \ni (\phi,\psi,\chi,\rho) \mapsto D_e g_1(\phi,\psi)(\chi,\rho) \in \mathbb{R}^n$$

is continuous. The corresponding properties then hold for the partial derivatives and their extensions $D_{1,e}g_1, D_{2,e}g_1 : W_1 \to L_c(C^0,\mathbb{R}^n)$.

(g4) **('Linear' case.)** This condition was used in [53] and essentially requires g to be linear in the first argument; we do not use this assumption.

(g5) (is an additional condition on $D_e g_1(\phi,\psi)$ which we do not use.)

(g6) (Recall that $(0,0) \in W_1$, $g(0,0) = 0$). The map

$$C^1 \times C^1 \supset W_1 \ni (\phi,\psi) \mapsto \|D_{1,e}g_1(\phi,\psi)\|_{L_c(C^0,\mathbb{R}^n)} \in \mathbb{R}$$

(see **(g3)**) is upper semicontinuous at $(0,0)$.

(g7) There exist $c_7 > 0$ and a bounding function ζ_7 so that for every $(\phi,\psi) \in W_1$ with $\max\{|\phi|_{C^0}, |\psi|_{C^0}\} \leq 1$ and for all $\rho \in C^1$, we have

$$|[D_2 g_1(\phi,\psi) - D_2 g_1(0,0)]\rho| \leq \zeta_7(|\phi|_{C^1} + |\psi|_{C^1})|\rho|_{C^0} + c_7 \cdot |\rho|_{C^1}|\psi|_{C^0}.$$

$(\widetilde{\mathbf{g8}})$ **('Nonlinear' case.)** There exist a constant $c_8 > 0$, and a bounding function α such that, with W_1 as in **(g3)** and Δ from **(g1)**, one has for $\phi,\psi \in W_1$ with $\max\{|\phi|_{C^0}, |\psi|_{C^0}\} \leq 1$ and $\chi \in C^1$:

$$|[D_1 g_1(\phi,\psi) - D_1 g_1(0,0)]\chi| \leq c_8|\chi'|_{C^0} \cdot |\psi|_{C^0} + \alpha(|\phi|_{C^0}) \cdot |\chi|_{[-h,-\Delta]}|_{C^0}.$$

(g9) There exist a convex neighborhood $U_2 \subset U_1 \cap C^2$ of 0 in C^2, a constant $c_9 > 0$ and a bounding function ζ_9 such that for $\psi \in U_2$ one has

$$\max_{0 \le s \le 1} |[D_2 g_1(s\psi', s\psi) - D_2 g_1(0,0)]\psi| \le \zeta_9(|\psi|_{C^2})|\psi|_{C^0} + c_9|\psi|_{C^1}|\psi|_{C^0}.$$

Comments on the above hypotheses:

1) Define

$$X_1 := \Big\{ \psi \in U_1 \mid \dot{\psi}(0) = g(\psi', \psi) \Big\}, \text{ and}$$
$$X_{1+} := \Big\{ \psi \in X_1 \mid \psi' \text{ is Lipschitz continuous} \Big\}.$$

Note that the condition defining X_1 is satisfied by any segment $\psi = x_t$ of a solution x of equation (1.1), if $x|_{[t-h,t]}$ is of class C^1. Under assumptions **(g0)**, $\widetilde{\textbf{(g1)}}$, **(g2)**, equation (1.1) defines a (local, in time) semiflow on the set X_{1+} which is continuous with respect to the topology from $\mathbb{R}_0^+ \times C^1$ (see [55], Section 4, in particular, Corollary 4.6). Semiflows on smaller sets, with additional smoothness properties, are restrictions of this one.

2) Condition $\widetilde{\textbf{(g1)}}$ expresses that the values of $g(x_t', x_t)$ do not depend on the 'recent past' of $\dot{x}$, namely, on the values of $\dot{x}$ on $[t - \Delta, t]$. (Our assumption is apparently stronger than the corresponding assumption **(g1)** from [54, 55], since we assume Δ to exist uniformly for W. It was, however, shown in Proposition 2.7 of [55] that Δ can be chosen locally uniformly, so that the difference is actually minimal.)

This condition excludes, in particular, implicit differential equations. This restriction and also the upper bound h on the delay exclude, for example, the famous two-body-problem of electrodynamics, as considered by Driver e.g. in [10, 11], from the framework chosen here.

3) The extension property **(g3)** can be seen as saying that $Dg_1(\phi, \psi)(\chi, \rho)$ does not depend on χ' and ρ'. Such conditions in the context of state-dependent delay equations were employed, e.g., in [36, 49, 53], and seem to go back to Definition 3.2 in [40]. There a corresponding property was called 'almost Fréchet differentiable' and defined as differentiability from a subspace with stronger norm to an ambient space with weaker norm. Extensibility of the derivative to a linear map continuous w.r.t. to the weaker norm was not part of the definition in [40], but was present in the applications there.

4) With X_1 from above, define $\mathcal{M}_2 := X_1 \cap C^2$; this set is called X_2 in [55]. It is shown in Proposition 5.1 of that reference that if g satisfies **(g0), (g1), (g3)**, then $\mathcal{M}_2$ is a C^1-submanifold of C^2 with codimension n; its tangent spaces are given by

$$T_\psi \mathcal{M}_2 = \Big\{ \chi \in C^2 \mid \chi'(0) = Dg_1(\psi', \psi)(\chi', \chi) \Big\}.$$

Note that the condition determining these tangent spaces involves only the first derivative of χ, and using the extension property **(g1)**, one can define the so-called extended tangent spaces

$$T_{e,\psi} \mathcal{M}_2 = \Big\{ \chi \in C^1 \mid \chi'(0) = D_e g_1(\psi', \psi)(\chi', \chi) \Big\}. \tag{2.2}$$

The set $\mathcal{M}_2$ is not invariant under the semiflow on X_1, because the property of being C^2 is not, but the following subset of $\mathcal{M}_2$, which is characterized by a second order compatibility condition, (called X_{2*} in [55]) is invariant:

$$X_2 := \Big\{ \psi \in \mathcal{M}_2 \mid \psi' \in T_{e,\psi} \mathcal{M}_2 \Big\}.$$

Combining the definitions, one gets the following explicit description of X_2:

$$X_2 = \Big\{\psi \in C^2 \mid \text{(i) } \dot{\psi}(0) = g(\psi', \psi); \quad \text{(ii) } \ddot{\psi}(0) = D_e g_1(\psi', \psi)(\psi'', \psi')\Big\} \tag{2.3}$$

Under assumptions **(g0)–(g3)**, the semiflow induced on X_2 is continuous w.r. to the topology from $\mathbb{R}_0^+ \times C^2$, as shown in Section 6 of [55]. It has differentiability properties under the linearity assumption **(g4)** (in brief: differentiation w.r. to initial values is possible w.r. to directions tangent to X_2, is given by a variational equation, and has continuity properties under an additional assumption **(g5)**). Condition **(g5)** is not required for the stability results in the papers [53] and [54], and **(g4)** is assumed in [53], but not in [54]. For the instability result of the present paper we assume neither of these two conditions.

The set $X_2 \subset X_{1,+}$ is invariant under the semiflow, but no smooth submanifold of C^2, and we prove in Section 4 that $\mathcal{M}_4 := X_2 \cap C^4$ is a C^1-submanifold of C^4, under an additional condition on g. We employ that manifold in order to get initial values exhibiting instability.

5) Assumptions **(g6)** and **(g7)** are from the paper [53] on linearized stability – except that here ζ_7 is required to be nondecreasing (clearly, if this would not hold, it could be achieved achieved replacing ζ_7 with $\tilde{\zeta}_7(s) := \sup\Big\{\zeta_7(t) \mid t \in [0,s]\Big\}$), and that the statement here uses the partial derivative. These conditions are used, in particular, to estimate the 'nonlinear part' $r_g(\psi) = g_1(\psi', \psi) - Dg_1(0,0)(\psi', \psi)$ of equation (1.1).

Condition **(g7)** is slightly stronger than **(g9)** from the paper [54](because the arguments ϕ, ψ, ρ are independent in **(g7)**), but we keep this condition in the present paper (see Prop. 2.1 below).

6) Our condition $\widetilde{\textbf{(g8)}}$ is easily seen to imply condition **(g8)** from the 'nonlinear' paper [54], by specialization to the case $\phi := s\psi', \psi := s\psi$, where $s \in [0,1]$, and $\chi := \psi'$. On the other hand, the equations of the primary example class from [54] also satisfy $\widetilde{\textbf{(g8)}}$, as we prove below.

7) Condition **(g9)** above is easily seen to be equivalent with condition **(g9)** from [54]: The $\max\limits_{0 \le s \le 1}$ from [54] disappears in our case since we assume that ζ_9 is a bounding function, and thereby nondecreasing. Therefore we use the same symbol for 'our' condition **(g9)**.

8) One concrete type of 'linear' equation (meaning linear in the delayed derivative) which was shown to satisfy **(g1)–(g7)** in [55] is the scalar equation

$$\dot{x}(t) = a\dot{x}(t - d(x(t))) + f[x(t) - r(x(t))], \tag{2.4}$$

if $a \in \mathbb{R}$ and, for example, $d \in C^2(\mathbb{R}, [\Delta, h])$, $r \in C^2(\mathbb{R}, [0,h])$, and $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$. (Note only that in the notation of the present paper the delays appear with a minus sign.) Correspondingly, the example class from [54] is

$$\dot{x}(t) = A[\dot{x}(t - d(x(t))] + f[x(t - r(x(t)))], \tag{2.5}$$

with a nonlinear C^2 function A and $d \in C^2(\mathbb{R}, [\Delta, h])$, $r \in C^2(\mathbb{R}, [0,h])$, and $f \in C^2(\mathbb{R}, \mathbb{R})$, and $A(0) = f(0) = 0$.

We introduce the additional hypothesis on g, mentioned in point 4) above:

($\mathbf{D^2 g_2}$) The map $g_2 := g_1|_{W_1 \cap (C^2 \times C^2)} : W_1 \cap (C^2 \times C^2) \to \mathbb{R}^n$ induced by g_1 is C^2 on $C^2 \times C^2$, and for $(\psi, \phi) \in W_1 \cap (C^2 \times C^2)$, the continuous bilinear form $D^2 g_2(\psi, \phi) : C^2 \times C^2 \to \mathbb{R}^n$ has a continuous extension $D_e^2 g_2(\psi, \phi)$ to $C^1 \times C^1$.

Proposition 2.1.

a) *Under conditions* **(g0) – (g3)**, **(g6)**, and **(g7)**, $\widetilde{\textbf{(g8)}}$ *(instead of* **(g8)**, **(g9)** *from [54]), the results from [54] remain valid.*

b) *If* $\Delta \in (0,h]$ *and* $A \in C^2(\mathbb{R}^n,\mathbb{R}^n)$, $d \in C^2(\mathbb{R}^n,[-h,-\Delta])$, $r \in C^2(\mathbb{R}^n,[-h,0])$, *and* $f \in C^2(\mathbb{R}^n,\mathbb{R}^n)$, *then equation (2.5) (written in the form (1.1)) satisfies conditions* **(g0)–(g3)**, **(g6)**, **(g7)**, $\widetilde{\textbf{(g8)}}$ *from above, and also condition* $\mathbf{(D^2 g_2)}$.

Proof. a) We show that **(g7)** implies **(g9)** from the present paper, and hence also **(g9)** from [54]: There exists a convex neighborhhod $U_2 \subset U_1$ of 0 in C^2 such for $\psi \in U_2$ and $s \in [0,1]$ one has $(s\psi', s\psi) \in W^1$, and $\max\{|s\psi'|_{C^0}, |s\psi|_{C^0}\} \leq 1$ for $s \in [0,1]$. Then **(g7)** gives

$$\max_{0\leq s\leq 1} |[D_2 g_1(s\psi', s\psi) - D_2 g_1(0,0)]\psi| \leq \max_{0\leq s\leq 1}[\zeta_7(|s\psi'|_{C^1} + |s\psi|_{C^1}) \cdot |\psi|_{C^0} + c_7|\psi|_{C^1}|s\psi|_{C^0}].$$

Using that ζ_7 is nondecreasing, we can estimate the last expression by

$$\zeta_7(2|\psi|_{C^2}) \cdot |\psi|_{C^0} + c_7|\psi|_{C^1}|\psi|_{C^0} = \zeta_9(|\psi|_{C^2}) \cdot |\psi|_{C^0} + c_9|\psi|_{C^1}|\psi|_{C^0},$$

where $c_9 := c_7$ and $\zeta_9(r) := \zeta_7(2r)$. This estimate has the form required in **(g9)**.

It was already remarked that $\widetilde{\textbf{(g8)}}$ implies **(g8)**, so the assertion of a) follows.

Ad b): We can set $W = C^0 \times C^1$ and then have

$$g(\phi,\psi) = A[\phi(-d(\psi(0)))] + f[\psi(-r(\psi(0)))] \text{ for } (\phi,\psi) \in W = C^0 \times C^1. \tag{2.6}$$

The calculation from p. 321 and formula (2.1) on p. 322 from [54] carry over to the n-dimensional case to show that **(g0)** is satisfied, that the restriction g_1 is of class C^1 on $W_1 = C^1 \times C^1$, and that for $(\phi,\psi) \in W_1$, $\chi, \rho \in C^1$ one has

$$\begin{aligned} Dg_1(\phi,\psi)(\chi,\rho) = {} & DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0) + \chi(-d(\psi(0)))] \\ & + Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho(0) + \rho(-r(\psi(0)))]. \end{aligned} \tag{2.7}$$

In particular,

$$D_1 g_1(\phi,\psi)\chi = DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0))), \tag{2.8}$$

$$\begin{aligned} D_2 g_1(\phi,\psi)\rho = {} & DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0)] \\ & + Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho(0) + \rho(-r(\psi(0)))]. \end{aligned} \tag{2.9}$$

Property $\widetilde{\textbf{(g1)}}$ is a direct consequence of formula (2.6) and the assumption that $d(v) \in [\Delta, h]$ for all $v \in \mathbb{R}^n$. The proof of **(g2)** is analogous to the corresponding proof in Proposition 2.1, p. 322 of [54], with one-dimensional balls replaced by n-dimensional balls.

The extension property from **(g3)** and the associated continuity property for $D_e g_1$ are seen from (2.7), mainly since no derivatives of χ and ρ are used. As in [54], p. 323, property **(g6)** is true since, in view of (2.8), for $(\phi,\psi) \in W_1$

$$\begin{aligned} \|D_{1,e} g_1(\phi,\psi)\|_{L_c(C^0,\mathbb{R}^n)} &= \sup_{|\chi|_{C^0}\leq 1} |DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0)))| \\ &\leq \|DA[\phi(-d(\psi(0)))]\|_{L_c(\mathbb{R}^n,\mathbb{R}^n)}, \end{aligned}$$

with equality for appropriately chosen χ, so that the map mentioned in **(g6)** is even continuous.

Proof of **(g7)**: (We use the notation $\|f\|_{\infty,M}$ for $\sup_{x\in M}\|f(x)\|$ for several functions f with values in normed spaces.) For ϕ,ψ and ρ as in **(g7)**, we have from (2.9)

$$
\begin{aligned}
&|[D_2g_1(\phi,\psi)-D_2g_1(0,0)]\rho|\\
&\quad=|DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0)]\\
&\qquad+Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho(0)+\rho(-r(\psi(0)))]\\
&\qquad-DA(0)\cdot 0-Df(0)\rho(-r(0))|\\
&\quad\le\|DA\|_{\infty,B(0,1)}\cdot\|Dd\|_{\infty,B(0,1)}\cdot|\phi|_{C^1}\cdot|\rho|_{C^0}\\
&\qquad+\|Df\|_{\infty,B(0,1)}\cdot\|Dr\|_{\infty,B(0,1)}\cdot|\psi|_{C^1}\cdot|\rho|_{C^0}\\
&\qquad+|Df[\psi(-r(\psi(0)))]\rho(-r(\psi(0)))-Df(0)\rho(-r(0))|.
\end{aligned}
$$

The last term can be estimated by

$$
\begin{aligned}
&|Df[\psi(-r(\psi(0)))-Df(0)]\rho(-r(\psi(0)))|+|Df(0)[\rho(-r(\psi(0)))-\rho(-r(0))]|\\
&\quad\le\sup_{|v|\le|\psi|_{C^0}}|Df(v)-Df(0)|\cdot|\rho|_{C^0}+|Df(0)|\cdot|\rho|_{C^1}\cdot\|Dr\|_{\infty,B(0,1)}\cdot|\psi|_{C^0}\\
&\quad\le\sup_{|v|\le|\psi|_{C^1}}|Df(v)-Df(0)|\cdot|\rho|_{C^0}+|Df(0)|\cdot\|Dr\|_{\infty,B(0,1)}|\rho|_{C^1}\cdot|\psi|_{C^0}.
\end{aligned}
$$

Dropping the index $B(0,1)$ now, we have with $c_7:=|Df(0)|\cdot\|Dr\|_\infty$ and

$$
\zeta_7(u):=\max\{\|DA\|_\infty\cdot\|Dd\|_\infty,\|Df\|_\infty\cdot\|Dr\|_\infty\}\cdot u+\sup_{|v|\le u}|Df(v)-Df(0)|
$$

that $|[D_2g_1(\phi,\psi)-D_2g_1(0,0)]\rho|\le\zeta_7(|\phi|_{C^1}+|\psi|_{C^1})\cdot|\rho|_{C^0}+c_7\cdot|\rho|_{C^1}\cdot|\psi|_{C^0}$.

Proof of **$\widetilde{\text{(g8)}}$**: For ϕ,ψ and χ as in **$\widetilde{\text{(g8)}}$** we obtain from (2.8), using that d has values in $[\Delta,h]$:

$$
\begin{aligned}
&|[D_1g_1(\phi,\psi)-D_1g_1(0,0)]\chi|=|DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0)))-DA(0)\chi(-d(0))|\\
&\quad\le|DA[\phi(-d(\psi(0)))]\,[\chi(-d(\psi(0)))-\chi(-d(0))]|\\
&\qquad+|\{DA[\phi(-d(\psi(0)))]-DA(0)\}\chi(-d(0))|\\
&\quad\le\|DA\|_{\infty,B(0,1)}\cdot|\chi'|_{C^0}\cdot\|Dd\|_{\infty,B(0,1)}|\psi|_{C^0}+\underbrace{\sup_{|v|\le|\phi|_{C^0}}|DA(v)-DA(0)|}_{=:\,\alpha(|\phi|_{C^0})}\cdot|\chi|_{[-h,-\Delta]}|_{C^0}\\
&\quad=c_8\cdot|\chi'|_{C^0}\cdot|\psi|_{C^0}+\alpha(|\phi|_{C^0})\cdot|\chi|_{[-h,-\Delta]}|_{C^0},
\end{aligned}
$$

with $c_8:=\|DA\|_{\infty,B(0,1)}\cdot\|Dd\|_{\infty,B(0,1)}$ and the indicated bounding function α.

Proof of **($\mathbf{D^2g_2}$)**: The evaluation map $\mathrm{ev}:(t,\psi)\mapsto\psi(t)$ is of class C^2 on $[-h,0]\times C^2$. Denoting partial derivatives w.r. to the scalar argument t by ∂_1 and identifying them with vectors, one has for $t\in[-h,0],\psi,\chi\in C^2$

$$
\partial_1^2\mathrm{ev}(t,\psi)=\ddot\psi(t),\ D_2^2\mathrm{ev}(t,\psi)=0,\ \partial_1D_2\mathrm{ev}(t,\psi)\chi=\dot\chi(t).
$$

With the evaluation at zero ev_0 and the canonical projections, g can be represented as on p. 321 of [54]:

$$
g=A\circ\mathrm{ev}\circ((-d\circ\mathrm{ev}_0\circ\mathrm{pr}_2)\times\mathrm{pr}_1)+f\circ\mathrm{ev}\circ((-r\circ\mathrm{ev}_0\circ\mathrm{pr}_2)\times\mathrm{pr}_2),
$$

which shows that under our assumptions the induced map g_2 is C^2 on $W_1\cap(C^2\times C^2)$, as composition of C^2 maps. To prove the extension property, we first compute an expression for

D^2g_2, based on (2.7). (Below, vectors in $\mathbb{R}^n$ are sometimes also multiplied by numbers from the right.) For $(\phi,\psi) \in W_1 \cap (C^2 \times C^2)$ and $(\chi_1,\rho_1),(\chi_2,\rho_2) \in C^2 \times C^2$,

$$\begin{aligned}
D^2g_2(\phi,\psi)[(\chi_1,\rho_1),(\chi_2,\rho_2)] &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[Dg_2(\phi+\varepsilon\chi_2,\psi+\varepsilon\rho_2) - Dg_2(\phi,\psi)](\chi_1,\rho_1)\\
&= D^2A(\phi(-d(\psi(0))))\big[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho_1(0)+\chi_1(-d(\psi(0))),\\
&\qquad -\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho_2(0)+\chi_2(-d(\psi(0)))\big]\\
&\quad + DA(\phi(-d(\psi(0))))\{-\dot\chi_2(-d(\psi(0))Dd(\psi(0))\rho_1(0)\\
&\qquad +\ddot\phi(-d(\psi(0)))[Dd(\psi(0))\rho_2(0)]\cdot[Dd\psi(0)\rho_1(0)]\\
&\qquad -\dot\phi(-d(\psi(0)))D^2d(\psi(0))[\rho_1(0),\rho_2(0)]\\
&\qquad -\dot\chi_1(-d(\psi(0)))Dd(\psi(0))\rho_2(0)\}\\
&\quad + \text{ (a similar expression involving } f \text{ and } r\text{, namely:)}\\
&\quad D^2f(\psi(-r(\psi(0))))\big[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho_1(0)+\rho_1(-r(\psi(0))),\\
&\qquad -\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho_2(0)+\rho_2(-r(\psi(0)))\big]\\
&\quad + Df(\psi(-r(\psi(0))))\{-\dot\rho_2(-r(\psi(0))Dr(\psi(0))\rho_1(0)\\
&\qquad +\ddot\psi(-r(\psi(0)))[Dr(\psi(0))\rho_2(0)]\cdot[Dr(\psi(0))\rho_1(0)]\\
&\qquad -\dot\psi(-r(\psi(0)))D^2r(\psi(0))[\rho_1(0),\rho_2(0)]\\
&\qquad -\dot\rho_1(-r(\psi(0)))Dr(\psi(0))\rho_2(0)\}.
\end{aligned}$$

One sees from the above expressions that $D^2g_2(\phi,\psi)$ has a continuous extension to $C^1 \times C^1$, mainly because no second derivatives of $\chi_1,\rho_1,\chi_2,\rho_2$ appear. □

The formal linearization of equation (1.1) at zero, using the extension property **(g3)**, is $\dot y(t) = D_eg_1(0,0)(y'_t,y_t)$, which can also be written as

$$\dot y(t) = D_{1,e}g_1(0,0)y'_t + D_{2,e}g_1(0,0)y_t, \tag{2.10}$$

Remark 2.2. In the special case of equation (2.5) (but n-dimensional, as in Prop. 2.1 b)), the formal linearization in the sense of equation (2.10) is given by

$$\dot y(t) = DA(0)\dot y(t-d(0)) + Df(0)y(t-r(0)), \tag{2.11}$$

i.e., by the 'frozen delay principle' (linearizing in the same way as if the delays were constant, with the values at equilibrium).

Proof. From the expressions for the partial derivatives in (2.8) and (2.9) we see that for $\chi,\rho \in C^0$ one has $D_{1,e}g_1(0,0)\chi = DA(0)\chi(-d(0))$ and

$$D_{2,e}g_1(0,0)\rho = DA(0)[0] + Df(0)[0+\rho(-r(0)] = Df(0)\rho(-r(0)).$$

Applying this with $\chi = y'_t$ and $\rho = y_t$ shows that in this case (2.10) and (2.11) are equivalent. (See also the remarks at the beginning of Section 3.4, p. 472 in [25], which however refer to the non-neutral case.) □

3 Semigroups, spectra, growth estimates and fundamental matrix

We define $L := D_{1,e}g_1(0,0) \in L_c(C^0,\mathbb{R}^n)$, $R := D_{2,e}g_1(0,0) \in L_c(C^0,\mathbb{R}^n)$, and then rewrite the linearization equation (2.10), shifting the derivative-dependent part to the left-hand side, and writing Y for the phase curve $t \mapsto y_t \in C^0$:

$$\frac{d}{dt}(y - L \circ Y)(t) = Ry_t. \tag{3.1}$$

Complexifications $L_{\mathbb{C}}, R_{\mathbb{C}} \in L_c(C^0_{\mathbb{C}}, \mathbb{C}^n)$ are obtained in the obvious way. It turns out that the latter equation generates a semigroup even on the space C^0, and that this semigroup serves as a kind of linearized approximation of the nonlinear semiflow, but in a sense that must be treated with caution: The domains of the semigroup and of the semiflow are different, and for the error made in approximation by the semigroup to be small in the C^1-norm, the trajectory has to stay in small ball w.r. to the C^2-norm. By a solution of (3.1) we mean a continuous function y on, e.g., $[-h, T)$ such that, with the corresponding phase curve Y, the function $(y - L \circ Y) : [0, T) \to \mathbb{R}^n$ is of class C^1 (meaning the right-hand derivative at $t = 0$), and satisfies the equation. Such a solution of equation (3.1) is in general not necessarily differentiable, only the difference $y - L \circ Y$ is.

Lemma 3.1 (**The semigroup** S^0, see [53, Corollary 6.2, p. 457]). *For every $\chi \in C^0$, there is a uniquely determined continuous solution $y^\chi : [-h, \infty) \to \mathbb{R}^n$ of (3.1) with $y_0^\chi = \chi$. Each linear map $S^0(t) : C^0 \ni \chi \mapsto y_t^\chi \in C^0$, $t \geq 0$ is continuous, and the operators $S^0(t), t \geq 0$, form a strongly continuous semigroup $\{S^0(t)\}_{t \geq 0}$ of operators in $L_c(C^0, C^0)$.*

A large part of this section is concerned with deriving growth estimates for the semigroup S^0 from spectral assumptions on its generator A^0. As always for translation semigroups,. $A^0\varphi = \varphi'$ for $\varphi \in D(A^0)$. Since equation (3.1) is a neutral equation without state-dependent delay, this is not a new topic, but we found that the treatment in the literature does not always provide comfortable reading, and try to improve this in the present paper.

For a complex valued function f, we use the notation $\mathcal{Z}(f)$ for the zero set of f.

Lemma 3.2 (**Spectrum of the generator** A^0 **of** S^0). *The spectrum of its infinitesimal generator A^0 (that is, the spectrum of the complexification $A^0_{\mathbb{C}}$) consists only of isolated eigenvalues of finite multiplicity. These eigenvalues coincide with the solutions of the characteristic equation*

$$\chi(\lambda) := \det(\Delta(\lambda)) = 0$$

where $\Delta(\lambda) \in \mathbb{C}^{n \times n}$ $(\lambda \in \mathbb{C})$ is a so-called characteristic matrix, obtained from the exponential ansatz $y(t) = \exp(\lambda t) \cdot y(0)$ for solutions of equation (3.1). Thus we have

$$\sigma(A^0) = \mathcal{Z}(\chi).$$

Proof. We can write equation (3.1) as $\frac{d}{dt}[(\mathrm{ev}_0 - L)y_t] = Ry_t$, where ev_0 denotes the evaluation at zero. The operator $\mathrm{ev}_0 - L$ corresponds to the operator M in formula (3.1) on p. 510 of [33], and also to the operator M in formula (6.3) on p. 396 of [34]. It satisfies the condition (3.4) on p. 511 from [33] (in the language of [21], p. 6: 'μ uniformly non-atomic at 0'), and the corresponding condition (6.3) from [34]: In our case L and R are given (in the sense of the Riesz representation theorem) by Riemann–Stieltjes integrals of the form

$$L\varphi = \int_{-h}^{0} d\mu(\theta) \cdot \varphi(\theta), \quad R\varphi = \int_{-h}^{0} d\eta(\theta) \cdot \varphi(\theta),$$

with matrix valued functions μ, η having entries of bounded variation, and defining Borel measures on $[-h, 0]$ (see also [46], Theorem 2.14, p. 40). In our situation, μ is constant (in particular, continuous) on $[-\Delta, 0]$ (compare here part b) of Lemma 3.18 below). The assertions now follow from Corollary 3.3 on p. 512 of [33], together with the definition of the characteristic matrix $\Delta(\lambda)$ as introduced in Theorem 3.2 of that reference.

Alternatively, the statements of the present Lemma follow from Theorem 2.1, p. 109 of [26], or from Theorem 1 on p. 17, Section III of [21]. In the latter reference the corresponding condition on the behavior of μ at zero is found in (5) on p. 6, as mentioned above. $\square$

Note that with the representations of L and R as in the above proof, one has

$$\begin{aligned}\Delta(\lambda) &= \lambda I - \lambda \int_{-h}^{0} d\mu(\theta) \exp(\lambda\theta) - \int_{-h}^{0} d\eta(\theta) \exp(\lambda\theta) \\ &= \lambda \cdot \left[I - \int_{-h}^{0} d\mu(\theta) \exp(\lambda\theta) \right] - \int_{-h}^{0} d\eta(\theta) \exp(\lambda\theta). \end{aligned} \tag{3.2}$$

Remark 3.3. In the reference [33], which was employed in the above proof, the resolvent set $\rho(A)$ of a closed operator $A : X \supset D(A) \to X$ with domain $D(A)$ in a complex Banach space X is described in Section I.1.1 on p. 482 as '...the set of complex numbers λ for which the *resolvent* $R(\lambda, A) = (\lambda - A)^{-1}$ exists.' – taken literally, this would include cases where for example the range of $\lambda - A$ is a closed proper subspace $U \subsetneq X$. It is however obvious from the subsequent text on p. 482 of [33] that the existence of the resolvent is understood as an operator in $L_c(X, X)$, i.e., $\lambda \in \rho(A)$ if and only if $\lambda - A : D(A) \to X$ is an isomorphism with continuous inverse. For a closed operator A in a Banach space, this is equivalent to demanding that $\lambda - A$ is bijective onto its image, with a continuous inverse, and that the image is dense in X (it is then automatically all of X).

Lemma 3.4 (The semigroup S^1). *The solutions of equation (3.1) also induce a C^0-semigroup $\{S^1(t)\}_{t\geq 0}$ of linear operators on the space*

$$T^1 = \left\{ \chi \in C^1 \mid \chi'(0) = D_e g_1(0,0)(\chi', \chi) \right\}$$

with the $|\ |_{C^1}$-norm (this is the domain of the generator A^0 of the semigroup $\{S^0(t)\}_{t\geq 0}$ on C^0, with the graph norm). The space T^1 coincides with the extended tangent space $T_{e,0}\mathcal{M}_2$ of $\mathcal{M}_2$ at zero (see (2.2)). The infinitesimal generators A^0 (of the semigroup S^0) and A^1 (of the semigroup S^1) have the same spectra (again, these are the spectra of $A^0_{\mathbb{C}}, A^1_{\mathbb{C}}$). For both operators, these consist only of eigenvalues λ of finite type, which are obtained from the exponential ansatz as in Lemma 3.2. The corresponding (finite dimensional) generalized eigenspaces G_λ of $A^0_{\mathbb{C}}$ and $A^1_{\mathbb{C}}$ coincide.

Proof. From [53], especially the remark on p. 442 preceding condition **(g4)** there, and from [25], Proposition 3.4.1, p. 473, one sees that T^1 coincides with the domain of the 'real' generator A^0, that the restriction of S^0 to T^1 defines a C^0-semigroup with respect to the C^1-topology, and that the spectra/ resolvent sets of the infinitesimal generator A^1 of $\{S^1(t)\}$ and A^0 of $\{S^0(t)\}$ (here we mean the complexified versions) satisfy

$$\sigma(A_1) \subset \sigma(A^0), \text{ and } \rho(A^1) \subset \rho(A^0) \cup \left\{ \lambda \in \mathbb{C} \mid \lambda - A^0 \text{ is injective, not surjective} \right\}. \tag{3.3}$$

It is clear from (2.2) that $T^1 = T_{e,0}\mathcal{M}_2$. Now, from Lemma 3.2, the spectrum of A^0 consists only of eigenvalues of finite type, which are obtained from the exponential ansatz. This shows that the set in brackets in (3.3) above is empty, so $\rho(A_1) \subset \rho(A^0)$. Together with the first inclusion in (3.3) we conclude $\sigma(A_1) = \sigma(A^0)$. The assertion on the eigenspaces follows from part (ii) of Proposition 3.4.1, p. 474 in [25]. $\square$

We shall need results expressing how spectral properties of the generator influence growth properties of the semigroup, in particular, to have a separation between different growth rates on complementary subspaces. The difficulty here lies, in principle, in the nontrivial relation between the so-called spectral bound

$$s(A) := \sup\ \mathrm{Re}(\sigma(A))$$

of the generator A and the growth bound

$$\omega_0(T) := \inf\left\{\beta \in \mathbb{R} \mid \exists M > 0 : \forall t \geq 0 : \|T(t)\| \leq Me^{\beta t}\right\}$$

of a C^0-semigroup $\{T(t)\}_{t\geq 0}$ of operators in $L_c(X, X)$, where X is a Banach space (see, e.g., [13], Section 2 of Chapter IV). In general, one only has $s(A) \leq \omega_0(T)$ instead of equality, and for the spectral radius $r(T(t))$ (which is again defined as $r(T(t)_{\mathbb{C}})$, if necessary):

$$\forall t \geq 0 :\ r(T(t)) = \exp(\omega_0(T)t) \tag{3.4}$$

(see e.g. [13], Chapter IV, Prop. 2.2, p. 251, and the counterexample 2.7 on p. 253, where $s(A) = -1$ and $\omega_0(T) = 0$). A frequently quoted example (in the Hilbert space ℓ^2) is given in the paper by Zabczyk, [57], which mentions the earlier result by Foiaș [14]. See also Lemma 4.2 from Section 74, p. 180 in [20], where the growth bound is called the order of $\{T(t)\}$. Thus, the growth bound for a semigroup is controlled by the spectral radius of one particular $T(t_0)$ with $t_0 > 0$:

Proposition 3.5. *If $\omega \in \mathbb{R}$ satisfies $r(T(t_0)) < \exp(\omega t_0)$ for some $t_0 > 0$ then there exists $M \geq 1$ with*

$$\|T(t)\|_{L_c(X,X)} \leq M \exp(\omega t) \quad \textit{for all } t \geq 0.$$

Proof. Since $r(T(t)) \leq \|T(t)\|_{L_c(X,X)}$ (see e.g. Cor. 1.4, p. 241 of [13]), such an ω must be larger than the growth bound $\omega_0(T)$, in view of (3.4). The estimate then follows from the definition of $\omega_0(T)$. □

We employ the usual notation $\rho(\dots)$ for the resolvent set and $P\sigma(\dots), C\sigma(\dots), R\sigma(\dots)$ for the point spectrum, i.e., the continuous spectrum and the residual spectrum of an operator, compare e.g. [28], Definition 2.16.1, p. 54. The problem of controlling the spectral radius $r(T(t))$ in turn by the spectrum of the generator A stems from the fact that in general one has

$$P\sigma(T(t)) \subset \{0\} \cup \exp[t \cdot P\sigma(A)] \text{ and } R\sigma(T(t)) \subset \{0\} \cup \exp[t \cdot R\sigma(A)]$$

(spectral mapping theorems for the point and residual spectrum, see [13], Theorem 3.7, p. 277), but no corresponding control over possible continuous spectrum $C\sigma(T(t))$. See [28], p. 54 for this subdivision of the spectrum $\sigma(T(t))$, and also Theorems 16.7.2 and 16.7.2 on pages 467 and 469 of [28].

The idea of controlling the spectrum of the semigroup $S^0(t)$ (which is if interest for us) as used in [20], in [26], and also in [17], is to treat $S^0(t)$ as a compact perturbation of a 'simpler' semigroup. For this 'simpler' semigroup an appropriate spectral mapping theorem is known, and then a result for compact perturbations can be used. We start carrying out this approach now.

The operator $L = D_{1,e}g_1(0,0) \in L_c(C^0, \mathbb{R}^n)$ from equation (3.1) has a representation as $L\varphi = \int_{-h}^{0} d\mu(\theta)\varphi(\theta)$ in the sense of the Riesz representation theorem. For simplicity, we introduce assumptions on L which are slightly stronger than needed.

Assumption on L**:** There exist $k \in \mathbb{N}$ and $A_j \in \mathbb{R}^{n\times n}, \tau_j \in (0,h], j = 1,\dots,k$, and an L^1 function $A : [-h,0] \to \mathbb{R}^{n\times n}$ such that L has the form

$$L\varphi = \sum_{j=1}^{k} A_j\varphi(-\tau_j) + \int_{-h}^{0} A(\theta)\varphi(\theta)d\theta \quad (\varphi \in C^0). \tag{3.5}$$

This condition implies the 'non-atomic at zero' and 'no singular part' assumptions made in [21] (Section IV there) and in [26] (assumptions (i) and (ii) on p. 108 there). (In combination with the presently not needed condition $\widetilde{\mathbf{(g1)}}$, Lemma 3.18 below shows that even $A(\theta) = 0$ for almost all θ in $[-\Delta,0]$.) With the difference operator $D_0 : C^0 \to C^0$ defined by

$$D_0\varphi := \varphi(0) - \sum_{j=1}^{k} A_j\varphi(-\tau_j), \tag{3.6}$$

equation (3.1) takes the form $\frac{d}{dt}[D_0 y_t - \int_{-h}^{0} A(\theta)y_t(\theta)d\theta] = Ry_t$, and is therefore (remotely) related to the difference equation $D_0 y_t = 0$, as we see below. In order to use this relation, the next result will be important. It is known, and we include a proof for completeness.

Proposition 3.6. *For fixed* $t \geq 0$*, consider the continuous linear operator*

$$K(t) : C^0 \to C^0([0,t],\mathbb{R}^n),\ \phi \mapsto H(\cdot,\phi),\ \textit{where}$$

$$H(s,\phi) := \int_{-h}^{0} A(\theta)y^{\phi}(s+\theta)d\theta - \int_{-h}^{0} A(\theta)\phi(\theta)d\theta + \int_{0}^{s} R(y^{\phi}_{\sigma})\,d\sigma$$

and $y^{\phi}_s := S^0(s)\phi$*. This operator is compact.*

Proof. Continuity of $K(t)$ is obvious. The middle term in the formula for H just defines a continuous linear functional into $\mathbb{R}^n$, and hence certainly a compact operator. The C^0-semigroup S^0 satisfies an exponential growth estimate of the form $|S^0(t)\phi|_{C^0} \leq c(t)|\phi|_{C^0}$, with c nondecreasing. Using this for the last term in the definition of H shows that this part even produces functions bounded in C^1, if $|\phi|_{C^0} \leq 1$. It follows from the Arzelà–Ascoli theorem that this third part defines a compact operator.

Abbreviating the first term with $U(s,\phi)$, considering $s \in [0,t]$ and $\tau \in [0,h]$ with $s+\tau \in [0,t]$, and extending A to an L^1 function on all of $\mathbb{R}$ (by zero), we obtain for ϕ with $|\phi|_{C^0} \leq 1$:

$$\begin{aligned}
|U(s+\tau,\phi) - U(s,\phi)| &= \left|\int_{-h}^{0} A(\theta)[y^{\phi}(s+\tau+\theta) - y^{\phi}(s+\theta)]d\theta\right| \\
&= \left|\int_{-h+\tau}^{\tau} A(\theta-\tau)y^{\phi}(s+\theta)\,d\theta - \int_{-h}^{0} A(\theta)y^{\phi}(s+\theta)]d\theta\right| \\
&\leq \int_{-h+\tau}^{\tau} |A(\theta-\tau) - A(\theta)|\cdot|y^{\phi}(s+\theta)|\,d\theta \\
&\leq c(t)\cdot|\phi|_{C^0}\int_{-h+\tau}^{\tau} |A(\theta-\tau) - A(\theta)|\,d\theta \leq c(t)\cdot\int_{\mathbb{R}} |A(\theta-\tau) - A(\theta)|\,d\theta.
\end{aligned}$$

Now translation $\mathbb{R} \ni \tau \mapsto A(\cdot+\tau) \in L^1(\mathbb{R},\mathbb{R}^{n\times n})$ is continuous, as follows from approximation by continuous functions with compact support and the Lebesgue convergence theorem. Thus the last term goes to zero as $|\tau| \to 0$. This proves that the functions $U(\cdot,\phi)$, $|\phi|_{C^0} \leq 1$ are a (bounded) and equicontinuous set in $C^0([0,t],\mathbb{R}^n)$, and hence the compactness of also the first part of $K(t)$ follows again from the Arzelà–Ascoli theorem. $\square$

We turn to the analysis of characteristic functions now. For L as in (3.5), the characteristic matrix (compare (3.2)) takes the form

$$\Delta(\lambda) = \lambda \cdot \left[I - \sum_{j=1}^{k} A_j \exp(-\lambda\tau_j) - \int_{-h}^{0} A(\theta)\exp(\lambda\theta)d\theta \right] - \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta). \tag{3.7}$$

In this context, the following functions and sets are important:

Define $\Delta_0(\lambda) := I - \sum_{j=1}^{k} A_j \exp(-\lambda\tau_j)$ and

$$\chi_0(\lambda) := \det \Delta_0(\lambda), \tag{3.8}$$

with the zero set $\mathcal{Z}(\chi_0) = \left\{ \lambda \in \mathbb{C} \mid \det \Delta_0(\lambda) = 0 \right\}$, and

$$Z_0 := \operatorname{Re}(\mathcal{Z}(\chi_0)) = \left\{ \operatorname{Re}(\lambda) \mid \det \Delta_0(\lambda) = 0 \right\}. \tag{3.9}$$

With $M_1(\lambda) := \int_{-h}^{0} A(\theta)\exp(\lambda\theta)d\theta$, $M_2(\lambda) := \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta)$ one has $\Delta(\lambda) = \lambda \cdot [\Delta_0(\lambda) - M_1(\lambda)] - M_2(\lambda)$, so for $\lambda \neq 0$: $\Delta(\lambda) = \lambda \cdot [\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda}M_2(\lambda)]$, and

$$\chi(\lambda) = \det(\Delta(\lambda)) = \lambda^n \cdot \det \left[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda}M_2(\lambda) \right].$$

It follows that the function defined by $\tilde{\chi}(\lambda) := \chi(\lambda)/\lambda^n$ for $\lambda \neq 0$ satisfies

$$\tilde{\chi}(\lambda) = \det \left[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda}M_2(\lambda) \right]. \tag{3.10}$$

For intervals $I \subset \mathbb{R}$, we shall consider $\chi, \tilde{\chi}$ and χ_0 in vertical strips of the form

$$\mathbf{S}_I := \left\{ r + is \mid r \in I,\, s \in \mathbb{R} \right\}.$$

Note that if $I = (\alpha, \beta)$ then the function χ_0 is holomorphic and almost periodic in the strip $\mathbf{S}_I$. (The almost periodicity in the vertical direction corresponds to the definition of H. Bohr in [7], section **104**, p. 86, which includes uniformity w.r. to the real part. Sometimes it is also defined correspondingly for horizontal strips, see e.g. formula (6.09), p. 266 in [39].) The approach in the subsequent results, based on almost periodicity, is essentially contained in [26] and also in sections 12.3 and 12.10 of [20], but we include proofs for completeness.

Lemma 3.7. *Consider a vertical strip $\mathbf{S}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathbb{R}$, and holomorphic functions $f_0, f_1 : \mathbf{S}_{(\alpha,\beta)} \to \mathbb{C}$, with f_0 almost periodic, and such that $f_1(r+is) \to f_0(r+is)$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$.*

a) If $z_0 = r_0 + is_0$ is a zero of f_0 in $\mathbf{S}_{(\alpha,\beta)}$ then there exists a sequence $(z_j) = (r_j + is_j)$ in $\mathcal{Z}(f_0)$ with $r_j \to r_0$ and $s_j \to \infty$ as $j \to \infty$.

b) If z_0 and the sequence $(z_j) \subset \mathcal{Z}(f_0)$ are as in a) then there exists a subsequence $(z_{\psi(j)}) \subset (z_j)$ and a sequence (ζ_j) of zeroes of f_1 such that $|\zeta_j - z_{\psi(j)}| \to 0$ as $j \to \infty$, in particular, $\operatorname{Re}(\zeta_j) \to r_0$.

Proof. Ad a): Assume the opposite; then there exist $\delta > 0$ and $S > 0$ such that the set $\left\{ z = r + is \mid |r - r_0| \le \delta, s \ge S \right\}$ is contained in $\mathbf{S}_{(\alpha,\beta)}$, and disjoint to $\mathcal{Z}(f_0)$. Then all points of the form $r_0 + is$ with $s \ge S + \delta$ would be at least a distance δ away from $\mathcal{Z}(f_0)$. It follows now from the fact that f_0 is both almost periodic and holomorphic that there exists a number $m(\delta) > 0$ such that

$$\forall s \geq S + \delta : \ |f_0(r_0 + is)| \geq m(\delta).$$

[See Lemma 3.1, part (ii) on p. 111 of [26], and Lemma 1 in Section 2 of Chapter VI [39], p. 268. The proof there (stated for horizontal strips) employs the characterization of almost periodic functions (due to Bochner) by the fact that the set of translates of such a function is relatively compact with respect to uniform convergence, see [39], p. 266, and [6], Satz XII on p. 143, where this property is called „Normaleigenschaft".]

Now associated to $\varepsilon := m(\delta)/2$ there exists an ε-almost period $T > 0$ of $f_0(r_0 + i\cdot)$ which satisfies $s_0 + T > S + \delta$, and hence with $s := s_0 + T$ one has

$$\begin{aligned} m(\delta) &\leq |f_0(r_0 + is)| = |f_0(r_0 + i(s_0 + T))| \\ &\leq \underbrace{|f_0(r_0 + is_0)|}_{=0} + \underbrace{|f_0(r_0 + is_0) - f_0(r_0 + i(s_0 + T))|}_{\leq \varepsilon} \leq \varepsilon = m(\delta)/2, \end{aligned}$$

a contradiction.

Ad b): Assume that the sequence $(z_j) = (r_j + is_j)$ is as in a). Since $r_j \to r_0$, we can pick $\delta_0 > 0$ is such that $\forall j \in \mathbb{N} : B(z_j, \delta_0) \subset \mathbf{S}_{(\alpha,\beta)}$. Next, for $j, k \in \mathbb{N}$, the circular rings $R_{j,k} := \Big\{ z \in \mathbb{C} \ \Big| \ \delta_0/(k+1) < |z - z_j| < \delta_0/k \Big\}$ are also contained in $\mathbf{S}_{(\alpha,\beta)}$.
Claim: There exists $k_0 \in \mathbb{N}$ such that $\forall j_0 \in \mathbb{N} \ \exists j \geq j_0 : R_{j,k_0} \cap \mathcal{Z}(f_0) = \emptyset$.
Proof: The opposite of the claim is

$$\forall k_0 \in \mathbb{N} \ \exists j_0 \in \mathbb{N} \ \forall j \geq j_0 : R_{j,k} \cap \mathcal{Z}(f_0) \neq \emptyset. \tag{$*$}$$

Assume that ($*$) holds, and fix $N \in \mathbb{N}$, and take numbers $j_0(1), \ldots, j_0(N)$ corresponding to $k_0 = 1, 2, \ldots, N$ according to ($*$), and set $j^* := \max\{j_0(1), \ldots, j_0(N)\}$. Then one has $R_{j^*,k} \cap \mathcal{Z}(f_0) \neq \emptyset$ for $k = 1, \ldots, N$, i.e., f_0 has at least N zeroes in $B(z_{j*}, \delta_0)$. This argument works for every $N \in \mathbb{N}$, which contradicts the fact that there exists $N^* \in \mathbb{N}$ such that in each rectangle of the form $\Big\{ z = r + is \in \mathbb{C} \ \Big| \ r \in (\alpha, \beta), \, s \in (t, t+1] \Big\}$ (where $t \in \mathbb{R}$), the almost periodic holomorphic function f_0 has at most N^* zeroes. (See [39], Lemma 2, p. 269; the proof again uses Bochner's compactness theorem. See also [26], p. 111, Lemma 3.1, part (i).) The claim is proved.

The above claim allows us to choose a subsequence $(z_{\varphi(j)}) \subset (z_j)$ such that

$$\forall j \in \mathbb{N} : \ R_{\varphi(j),k_0} \cap \mathcal{Z}(f_0) = \emptyset.$$

With $\delta^* := \frac{1}{2}(\delta_0/(k_0+1) + \delta_0/k_0)$, the central circular lines $\partial B(z_{\varphi(j)}, \delta^*)$ of the rings $R_{\varphi(j),k_0}$ all have distance at least $\delta^* - \delta_0/(k_0+1) > 0$ from $\mathcal{Z}(f_0)$. As above, it follows from Lemma 1 on p. 268 of [39] that there exists a number $m > 0$ such that $|f_0| \geq m$ on $\partial B(z_{\varphi(j)}, \delta^*)$ for every $j \in \mathbb{N}$. The assumed uniform convergence of $f_1(r + is)$ to $f_0(r + is)$ as $s \to \infty$ implies that for all large enough j, $|f_1 - f_0| < m$ on $\partial B(z_{\varphi(j)}, \delta^*)$. Then the Rouché theorem implies that f_1 and f_0 have the same number of zeroes in $B(z_{\varphi(j)}, \delta^*)$, in particular, f_1 has a zero in this set.

Together we have proved that, with δ_0 as above, there exists a subsequence $(z_{\varphi(j)}) \subset (z_j)$ and $j_0 \in \mathbb{N}$ such that

$$\forall j \geq j_0 : \mathcal{Z}(f_1) \cap B(z_{\varphi(j)}, \delta_0) \neq \emptyset.$$

In particular, there exist arbitrarily large $j \in \mathbb{N}$ with $\mathcal{Z}(f_1) \cap B(z_{\varphi(j)}, \delta_0) \neq \emptyset$.

Since this argument also works for every smaller positive value of δ_0, we can pick a sequence of numbers $\delta_j > 0$ with $\delta_0 \geq \delta_j \to 0$ and associated indices $\psi(j)$ with $\psi(j+1) > \psi(j)$ (so that $(z_{\psi(j)})$ is a subsequence of (z_j)) such that $\mathcal{Z}(f_1) \cap B(z_{\psi(j)}, \delta_j) \neq \emptyset$ for all $j \in \mathbb{N}$. Choosing ζ_j from the last set for every j, we obtain $(\zeta_j) \subset \mathcal{Z}(f_1)$ and $|\zeta_j - z_{\psi(j)}| < \delta_j \to 0$ as $j \to \infty$. □

We have the following relation between χ_0 and $\tilde{\chi}$, which will allow us to apply the last proposition:

Proposition 3.8. *In a vertical strip* $\mathbf{S}_{(\alpha,\beta)}$, *where* $\alpha, \beta \in \mathbb{R}$, $\tilde{\chi}(r+is) \to \chi_0(r+is)$ *as* $s \to \infty$, *uniformly w.r. to* $r \in (\alpha, \beta)$.

Proof. Recall from (3.8) and (3.10) that

$$\chi_0(\lambda) = \det \Delta_0(\lambda) \quad \text{and} \quad \tilde{\chi}(\lambda) = \det[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda)].$$

We claim that $M_1(r+is) \to 0$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$. The proof is mainly a Riemann–Lebesgue-type argument which we include for completeness. (We choose a matrix norm $\| \ \|$ on $\mathbb{C}^{n\times n}$, and use the corresponding C^0- and L^1-norms.) Set $\mu := \max\{|\alpha|, |\beta|\}$ and let $\varepsilon > 0$ be given. Choose a matrix-valued function $\tilde{A} \in C^1([-h,0], \mathbb{R}^{n\times n})$ with $e^{\mu h} \cdot \|\tilde{A} - A\|_{L^1([-h,0])} \leq \varepsilon/2$. Then one has for $r \in (\alpha, \beta)$

$$M_1(r+is) = \int_{-h}^{0} A(\theta) e^{(r+is)\theta}\, d\theta = \int_{-h}^{0} \tilde{A}(\theta) e^{(r+is)\theta}\, d\theta + \int_{-h}^{0} [A(\theta) - \tilde{A}(\theta)] e^{(r+is)\theta}\, d\theta.$$

The second term can be estimated by $e^{\mu h} \|\tilde{A} - A\|_{L^1([-h,0])} < \varepsilon/2$.

The first term equals, by partial integration,

$$\left[\tilde{A}(\theta) \frac{1}{r+is} e^{(r+is)\theta}\right]_{\theta=-h}^{\theta=0} - \frac{1}{r+is} \int_{-h}^{0} \tilde{A}'(\theta) \cdot e^{(r+is)\theta} d\theta,$$

which for $s > 0$ can be estimated by $\frac{1}{s}[2e^{\mu h} \|\tilde{A}\|_{C^0} + e^{\mu h} \|\tilde{A}'\|_{L^1([-h,0])}]$, and the latter is less than $\varepsilon/2$ for all large enough s. This proves the asserted uniform convergence.

Since $M_2(\lambda)$ is uniformly bounded for $\lambda \in \mathbf{S}_{(\alpha,\beta)}$, we also have $\frac{1}{r+is} M_2(r+is) \to 0$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$.

Note that the matrix $\Delta_0(\lambda)$ is bounded for $\lambda \in \mathbf{S}_{(\alpha,\beta)}$. It follows from the convergence of $M_1(\lambda)$ and $\frac{1}{\lambda} M_2(\lambda)$ to zero as $s \to \infty$ ($\lambda = r + is, r \in (\alpha, \beta)$) that all matrices $\Delta_0(\lambda)$ and $\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda)$ for $s \geq 1$ are contained in a ball in $\mathbb{C}^{n\times n}$, on which the determinant function det is uniformly continuous, so that $|\det(A) - \det(B)| \leq \rho_{\det}(\|A - B\|)$ for A, B in that ball, with a bounding function $\rho_{\det}$. It follows then that for $\lambda = r + is \in \mathbf{S}_{(\alpha,\beta)}$ with $s \geq 1$ one has

$$|\chi_0(r+is) - \tilde{\chi}(r+is)| \leq \rho_{\det}\left(\left\| M_1(r+is) + \frac{1}{r+is} M_2(r+is) \right\|\right) \to 0 \qquad (s \to \infty),$$

uniformly w.r. to $r \in (\alpha, \beta)$. □

Corollary 3.9.

a) If $z_0 = r_0 + is_0$ *is a zero of* χ_0 *then there exist sequences* $(z_j = r_j + is_j)$ *in* $\mathcal{Z}(\chi_0)$ *and* $(\zeta_j = \rho_j + i\sigma_j)$ *in* $\mathcal{Z}(\chi)$ *such that* $r_j \to r_0$, $s_j \to \infty$, *and* $|\zeta_j - z_j| \to 0$ *as* $j \to \infty$.

b) *For the real parts of the zero sets* $Z_0 := \operatorname{Re}(\mathcal{Z}(\chi_0))$ *and* $Z := \operatorname{Re}(\mathcal{Z}(\chi))$ *we have*

$$Z_0 \subset \overline{Z}.$$

Proof. Part a) with $\tilde{\chi}$ in place of χ follows directly from Proposition 3.8 and from Lemma 3.7, since χ_0 is holomorphic and almost periodic in very strip $\mathbf{S}_{(\alpha,\beta)}$ containing z_0. We may assume that all ζ_j of $\tilde{\chi}$ are different from 0, and then they are also zeroes of χ, which proves part a). Part b) is a direct consequence of a), since we also have $\rho_j \to r_0$. $\square$

For a subset $I \subset \mathbb{R}$ we define the corresponding 'circular ring' in $\mathbb{C}$ by

$$\mathbf{R}_I := \Big\{ z \in \mathbb{C} \;\Big|\; |z| \in \exp(I) \Big\},$$

so $\mathbf{R}_I$ is the image of the 'vertical strip' $\mathbf{S}_I = I + \mathbb{R} \cdot i$ under the exponential map. The next auxiliary result has an early precursor in Lemma 5.2 on p. 16 of [19].

Lemma 3.10.

a) *With* D_0 *from* (3.6), *the corresponding difference equation* $D_0 x_t = 0$ *or* $x(t) = \sum_{j=1}^{k} A_j x(t - \tau_j)$ *generates a* C^0*-semigroup* $\{T_{D_0}(t)\}_{t \geq 0}$ *on the kernel of* D_0 *(a subspace of* C^0 *with finite codimension).*

b) *With* $Z_0 = \operatorname{Re}(\mathcal{Z}(\chi_0))$ *from Corollary 3.9 we have for the spectra:*

$$\sigma(T_{D_0}(t)) \subset \{0\} \cup \mathbf{R}_{\overline{Z_0} t} \qquad (t \geq 0).$$

c) *If* $\operatorname{pr}_{D_0} \in L_c(C^0, C^0)$ *is a projection onto* $\ker(D_0)$, *the semigroup* $\{S^0(t)\}$ *generated by equation* (3.1) *can be written as*

$$S^0(t) = T_{D_0}(t) \circ \operatorname{pr}_{D_0} + K(t),$$

with compact operators $K(t) \in L_c(C^0, C^0)$. *(Here, formally,* $T_{D_0}(t) \in L_c(\ker(D_0), \ker(D_0))$ *should be followed by the inclusion map from* $\ker(D_0)$ *to* C^0, *which we omit.)*

Proof. Part a) is stated (not proved) in [26], Section 3, p. 110, and follows from much more general existence results in Chapter 12 of [20]. For D_0 as considered here, and assuming that τ_1 is minimal among the discrete delays τ_j $(j = 1, \ldots, k)$, the forward solution of $D_0 x_t = 0$ (given $x_0 \in \ker(D_0)$) can be directly obtained by stepwise forward definition: $x(t) := \sum_{j=1}^{k} A_j x(t - \tau_j)$ on $[0, \tau_1]$, then by the same formula on $[\tau_1, \tau_1 + 2\tau_1]$, etc.

Part b) is proved in [26], Theorem 3.2, p. 114, based on exponential estimates for $\|T_{D_0}(t)\|$ obtained by Laplace transform methods. The proof in [26] (Lemma 3.4, p. 111) quotes reference [12] of that paper, which apparently was never published. Another proof is given in [17], Theorem 2.1, p. 209. Both proofs use Laplace transform methods and a result due to Cameron and Pitt [8, 43] on exponential expansion $1/h(z)$, if h is almost periodic and holomorphic.

Part c) is proved in [26], Lemma 4.1, p. 116. We sketch the idea: For $\phi \in C^0$ and $\phi_0 := \operatorname{pr}_{D_0}\phi$, $y_t := S^0(t)\phi$ it follows from equations (3.1), (3.5) and (3.6) that

$$D_0 y_t - \int_{-h}^{0} A(\theta) y_t(\theta) d\theta - \left[D_0 \phi - \int_{-h}^{0} A(\theta)\phi(\theta) d\theta \right] = \int_0^t R y_s \, ds,$$

and hence, with H from Proposition 3.6,

$$D_0(y_t - \phi) = \int_{-h}^{0} A(\theta) y_t(\theta) d\theta - \int_{-h}^{0} A(\theta)\phi(\theta) d\theta + \int_0^t R y_s \, ds = H(t, \phi).$$

Thus, setting $\phi_1 := \phi - \phi_0 = (\mathrm{id} - \mathrm{pr}_{D_0})\phi$, $D_0(y_t - \phi) = D_0(y_t - \phi_1)$, and defining $z : [-h, \infty) \to \mathbb{R}^n$ by $z_t := y_t - \phi_1$, we have $z_0 = \phi_0$, so $D_0 z_0 = 0$, and z solves $D_0 z_t = H(t, \phi)$ $(t \geq 0)$, an inhomogeneous version of the equation generating T_{D_0}. The solution theory for this equation implies that, for fixed $t \geq 0$, $z_t = T_{D_0}(t)z_0 + \mathcal{K}(t)H(\cdot, \phi)|_{[0,t]}$, with a continuous and linear operator $\mathcal{K}(t) : C^0([0, t], \mathbb{R}^n) \to C^0$. It follows that

$$S^0(t)\phi = y_t = z_t + \phi_1 = T_{D_0}(t)\mathrm{pr}_{D_0}\phi + \mathcal{K}(t)H(\cdot, \phi)|_{[0,t]},$$

and from Proposition 3.6 we know that the operator $C^0 \ni \phi \mapsto H(\cdot, \phi)|_{[0,t]} \in C^0([0, t], \mathbb{R}^n)$ is compact. The assertion of c) follows. $\square$

We need some functional analytic results of general nature, in particular, a 'compact perturbation' result. The version below suffices for our purposes. (As above, we write $\rho(\dots)$ for the resolvent set and $P\sigma(\dots)$ for the point spectrum, i.e., the eigenvalues of an operator.)

Lemma 3.11. *Assume that X is a real or complex Banach space, and $U, K \in L_c(X, X)$, with K compact.*

(i) *If $G \subset \mathbb{C}$ satisfies*

$$(1) \quad G \subset \rho(U) \quad \text{and} \quad (2) \quad G \cap P\sigma(U + K) = \emptyset, \tag{3.11}$$

then also $G \subset \rho(U + K)$.

(ii) *If $\mu \in \rho(U) \cap \sigma(U + K)$ is an isolated spectral value, then it is an eigenvalue of $U + K$ of finite multiplicity (in the sense that the spectral subspace of $X_{\mathbb{C}}$ associated to μ is finite-dimensional).*

Proof. We can assume that X is a $\mathbb{C}$-Banach space, otherwise we would have to consider the complexifications of spaces and operators, We write $\mathrm{GL}(X, X)$ for the topological linear isomorphisms of X.

Ad (i): For $\lambda \in G$, condition (1) gives that $\lambda - U \in \mathrm{GL}(X, X)$, and one has

$$\lambda - (U + K) = (\lambda - U) \circ \underbrace{[\mathrm{id}_X - (\lambda - U)^{-1}K)]}_{=:F_\lambda} = (\lambda - U) \circ F_\lambda. \tag{3.12}$$

The operators F_λ are of the form $\mathrm{id}_X - K_\lambda$ with compact operators K_λ, and hence Fredholm operators of index zero (see [29], Korollar 25.3., p. 109). This property implies that $F_\lambda \in \mathrm{GL}(X, X)$ if and only if $\ker F_\lambda = \{0\}$. Now assumption (2) shows that the operators $\lambda - (U + K)$ are injective for all $\lambda \in G$, and hence also F_λ is injective, and thereby in $\mathrm{GL}(X, X)$ for $\lambda \in G$. It follows that for these λ also $\lambda - (U + K) \in \mathrm{GL}(X, X)$, so $\lambda \in \rho(U + K)$.

Ad (ii): (The proof here follows the proof of Lemma 5.2, p. 22 in [15].) Assume that μ is as in (ii). Part (i) applied to $G := \rho(U) \setminus P\sigma(U + K)$ shows that we must have $\mu \in P\sigma(U + K)$, so μ is an eigenvalue of $U + K$. For λ close enough to μ, but different from μ, we have $\lambda \in \rho(U + K) \cap \rho(U)$. Thus, for small enough $r > 0$, the spectral projection associated to the spectral set $\{\mu\}$ of $U + K$ is given by $\mathrm{pr}_\mu = \frac{1}{2\pi i} \oint_{|\lambda - \mu| = r} (\lambda - (U + K))^{-1} \, d\lambda$, and from (3.12) we see that $(\lambda - (U + K))^{-1} = F_\lambda^{-1} \circ (\lambda - U)^{-1}$, if $\lambda \in \rho(U + K) \cap \rho(U)$. Switching to resolvent notation we obtain $F_\lambda \circ R(\lambda; U + K) = R(\lambda; U)$ and hence, using the definition of $F_\lambda = \mathrm{id}_X - R(\lambda; U)K$,

$$R(\lambda; U + K) = R(\lambda; U) \circ K \circ R(\lambda; U + K) + R(\lambda; U).$$

Using this in the spectral projection formula we obtain

$$\mathrm{pr}_\mu = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} \{R(\lambda, U) \circ K \circ R(\lambda; U+K) + R(\lambda; U)\}\, d\lambda.$$

The second term under the integral is holomorphic in the neighborhood of μ and hence contributes zero; the first term consists of compact operators, and so we conclude that pr_μ is compact, which (for a projection) means it has finite-dimensional range. (From which it follows again that μ must be an eigenvalue, since $U+K$ induces a finite dimensional endomorphism of image(pr_μ) with spectrum $\{\mu\}$.) $\square$

Remark 3.12. Assume that μ is an isolated eigenvalues with finite-dimensional spectral subspace of the operator $T \in L_c(X, X)$, where X is a complex Banach space (in particular $T := U+K$ and μ as above). Then the space of generalized eigenvectors $\mathcal{G}_\mu := \bigcup_{j=1}^\infty \ker\,(\mu - T)^j$ equals the image of the spectral projection pr_μ, and with its dimension $\nu(\mu)$ one has $\mathcal{G}_{\mu,T} = \ker\,(\mu-T)^{\nu(\mu)}$, and the direct sum decomposition

$$X = \ker(\mu - T)^{\nu(\mu)} \oplus \mathrm{image}(\mu - T)^{\nu(\mu)} = \mathrm{image}(\mathrm{pr}_\mu) \oplus \ker(\mathrm{pr}_\mu), \tag{3.13}$$

with both decompositions coinciding.

Proof. For the first decomposition and the identity $\mathrm{image}(\mathrm{pr}_\mu) = \ker(\mu - T)^{\nu(\mu)}$, see [15], Theorem 2.1, p. 9, and the passage preceding it. Note that these results are independent of the Hilbert space setting of [15], like many results of Chapter I of that reference, see also the first sentence on p. 1 there. The existence of the second decomposition is clear, since pr_μ is a projection, and it remains to prove equality of the spaces to the right of the $\oplus$-signs. If $v \in \mathrm{image}(\mu - T)^{\nu(\mu)}$, there exists $w \in X$ with $v = (\mu - T)^{\nu(\mu)} w$, and then for small $r > 0$

$$\mathrm{pr}_\mu v = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} R(\lambda; T) v\, d\lambda = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} R(\lambda; T)(\mu - T)^{\nu(\mu)} w\, d\lambda = 0,$$

since $\nu(\mu)$ equals the pole order of $R(\cdot; T)$ at μ (see e.g. Theorem 10.1, p. 330 in [47] or formula (2.3) on p. 9 of [15]), and hence the integrand has a holomorphic extension at μ. This shows that $\mathrm{image}(\mu - T)^{\nu(\mu)} \subset \ker(\mathrm{pr}_\mu)$. Since both are direct complements of the same space, equality follows. $\square$

We will need another result of general nature. In the lemma below, the restriction $T|_Y$ of an operator T to an invariant subspace Y of T is meant as simultaneous restriction in the domain and the image space.

Lemma 3.13.

a) *Let X be a complex Banach space and A a closed operator with domain $D(A)$ and range in X, and let Σ be a bounded spectral subset of A. Then the associated spectral subspace X_Σ (with the corresponding projection given by a contour integral with contour enclosing Σ) satisfies $X_\Sigma \subset D(A)$, and $A|_{X_\Sigma}$ is bounded.*

b) *Assume that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup of operators in $L_c(X, X)$, with generator A, and let Σ be a bounded subset of the spectrum $\sigma(A)$, with complement $\Sigma' := \sigma_e(A) \setminus \Sigma$, where $\sigma_e(A)$ denotes the extended spectrum of A (i.e., $\sigma(A) \cup \{\infty\}$ in case A is unbounded). Then the corresponding spectral decomposition $X = X_{\Sigma'} \oplus X_\Sigma$ is invariant under all $S(t)$ $(t \geq 0)$.*

c) The spectral projections $\mathrm{pr}_{M,S(t)}$*, associated to the operator* $S(t)$ *and some spectral subset* M *of* $S(t)$ *for some* $t \geq 0$*, and* $\mathrm{pr}_{\Sigma,A}$*, associated to* A *and* Σ*, commute.*

d) If for some $t > 0$ *one has*

$$\exp(t\Sigma) \cap \sigma(S(t)|_{X_{\Sigma'}}) = \emptyset, \tag{3.14}$$

then the disjoint sets in (3.14) are spectral sets for the operator $S(t)$*. In this case the spectral projection for* $S(t)$ *corresponding to the set* $\exp(t\Sigma)$ *coincides with the spectral projection for* A *corresponding to* Σ*:*

$$\mathrm{pr}_{\exp(t\Sigma),S(t)} = \mathrm{pr}_{\Sigma,A}.$$

Proof. For a), see [47], Theorem 9.2, p. 322.

Ad b): For $t \geq 0$, one has $S(t) \circ A = A \circ S(t)$ on $D(A)$ (see [42], Theorem 2.4 c, p. 5), which implies $(\lambda - A)S(t) = S(t)(\lambda - A)$ on $D(A)$ for all $\lambda \in \mathbb{C}$, and hence for $\lambda \in \rho(A)$: $S(t) = R(\lambda; A)S(t)(\lambda - A)$ and finally $S(t)R(\lambda; A) = R(\lambda; A)S(t)$ on the dense subspace $D(A)$, hence on all of X. Since the spectral projection $\mathrm{pr}_{\Sigma,A}$ associated to A and Σ is given by a contour integral over of $R(\lambda; A)$, it follows that also $\mathrm{pr}_{\Sigma,A}$ and $S(t)$ commute, which proves the invariance.

Ad c): For $z \in \rho(S(t))$ and $w \in \rho(A)$, the fact that $R(w, A)$ and $S(t)$ commute implies

$$R(w; A) = R(w; A)(z - S(t))R(z; S(t)) = (z - S(t))R(w; A)R(z; S(t)),$$

and hence $R(z; S(t)R(w; A) = R(w; A)R(z; S(t))$ (the resolvents commute). Now

$$\mathrm{pr}_{M,S(t)} = \frac{1}{2\pi i}\int_{\Gamma_M} R(z; S(t))\, dz \text{ and } \mathrm{pr}_{\Sigma,A} = \frac{1}{2\pi i}\int_{\Gamma_\Sigma} R(w; A)\, dw$$

with appropriate cycles Γ_M and Γ_Σ, and the fact that both projections commute is obtained from Fubini's theorem and the commuting property of the resolvents.

Ad d): Assume condition (3.14) for some $t > 0$. Since $A|_{X_\Sigma}$ is bounded and obviously the generator of the semigroup $\{S(t)|_{X_\Sigma}\}$, the latter semigroup is uniformly continuous (i.e., continuous with respect to $\|\ \|_{L_c(X,X)}$), see [42], Theorem 1.2, p. 2. The spectral mapping theorem for uniformly continuous semigroups ([13], Lemma 3.1.3, p. 19) shows that $\exp(t\Sigma) = \sigma(S(t)|_{X_\Sigma})$. This set is compact and, in view of (3.14), disjoint to the as well compact set $\sigma(S(t)|_{X_{\Sigma'}})$, but since (in view of b)) $S(t)$ is completely reduced by the subspaces X_Σ and $X_{\Sigma'}$, the union of both sets gives $\sigma(S(t))$ (see [47], Theorem 5.4, p. 289), so both are spectral sets for $S(t)$. Hence the spectral projection $\mathrm{pr}_{\exp(t\Sigma),S(t)}$ is well- defined. We briefly write pr_A for $\mathrm{pr}_{\Sigma,A}$ and pr_S for $\mathrm{pr}_{\exp(t\Sigma),S(t)}$. As above, both are given by appropriate contour integrals over cycles Γ_Σ and $\Gamma_{\exp(t\Sigma)}$ enclosing the respective sets. Note that $\mathrm{pr}_S|_{X_\Sigma} = \frac{1}{2\pi i}\int_{\Gamma_{\exp(t\Sigma)}} R(z; S(t)|_{X_\Sigma})\, dz = \mathrm{id}_{X_\Sigma}$, since $S(t)|_{X_\Sigma}$ has only spectrum in the interior of $\Gamma_{\exp(t\Sigma)}$ (namely, the set $\exp(t\Sigma)$). Hence $X_\Sigma \subset \mathrm{image}(\mathrm{pr}_S)$. Further, $R(\cdot; S(t)|_{X_{\Sigma'}})$ is a holomorphic function in the interior of $\Gamma_{\exp(t\Sigma)}$, due to condition (3.14). It follows that $\mathrm{pr}_S|_{X_{\Sigma'}} = 0$, or $X_{\Sigma'} \subset \ker(\mathrm{pr}_S)$. Now from

$$\begin{aligned} X &= \ker(\mathrm{pr}_A) \oplus \mathrm{image}(\mathrm{pr}_A) = X_{\Sigma'} \oplus X_\Sigma; \\ X &= \ker(\mathrm{pr}_S) \oplus \mathrm{image}(\mathrm{pr}_S) \end{aligned}$$

and the inclusions between the subspaces of both decompositions, equality follows, and hence $\mathrm{pr}_S = \mathrm{pr}_A$. □

We want to obtain an exponential dichotomy (or pseudo-hyperbolicity) result for the semigroup $\{S^0(t)\}_{t\geq 0}$ from assumptions on the spectrum of the generator A_0, i.e., on $\sigma(A_0) = \mathcal{Z}(\chi)$, see Lemma 3.2. The lemma below is a main step.

Lemma 3.14. *Assume L is as in (3.5), and that there exist real numbers α, β with $\alpha < \beta$ such that the spectrum of A^0 decomposes as*

$$\sigma(A^0) = \sigma(A^0_{\mathbb{C}}) = \underbrace{\Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}\Big)}_{=:\Sigma'} \cup \underbrace{\Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big)}_{=:\Sigma},$$

with the set Σ nonempty and finite. Then

(i) *For $t \geq 0$, $\sigma(S^0(t)_{\mathbb{C}}) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup \mathbf{R}_{(t\beta,\infty)}$.*

(ii) *For $t > 0$, $\sigma(S^0(t)_{\mathbb{C}}) \cap \mathbf{R}_{(t\beta,\infty)} = \Big\{\exp(t\lambda) \mid \lambda \in \sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big\}$, and all of these finitely many numbers are eigenvalues of finite multiplicity for $S^0(t)$.*

(iii) *If μ is one of the finitely many spectral values of $S^0(t)_{\mathbb{C}}$ in $\mathbf{R}_{(t\beta,\infty)}$, then the associated spectral space of $S^0(t)$ is given by*

$$\mathcal{G}_{\mu,S^0(t)_{\mathbb{C}}} = \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)},\\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0_{\mathbb{C}}}, \tag{3.15}$$

where $\mathcal{G}_{\lambda,A^0_{\mathbb{C}}}$ denotes the associated finite-dimensional spectral space of $A^0_{\mathbb{C}}$ associated with λ.

(iv) *For every $t > 0$, condition (3.14) from Lemma 3.13 is satisfied with Σ from above.*

Proof. In the proof, we omit the subscript $\mathbb{C}$. From the assumptions, $\sigma(A^0) \cap \mathbf{S}_{(\alpha,\infty)} = \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}$ is finite. Since $\sigma(A^0) = \mathcal{Z}(\chi)$ (Lemma 3.2), we see from Corollary 3.9 that $\mathcal{Z}(\chi_0) \cap \mathbf{S}_{(\alpha,\infty)} = \emptyset$, since any number in this set would imply the existence of infinitely many numbers in $\mathcal{Z}(\chi) \cap \mathbf{S}_{(\alpha,\infty)}$. It follows that, with the notation from Corollary 3.9, $Z_0 \subset (-\infty, \alpha]$ and hence also $\overline{Z_0} \subset (-\infty, \alpha]$. We obtain for $t \geq 0$ that $t \cdot \overline{Z_0} \subset (-\infty, t\alpha]$, and Lemma 3.10 b) shows that $\sigma(T_{D_0}(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$. Since $\sigma[T_{D_0}(t) \circ \mathrm{pr}_{D_0}] = \{0\} \cup \sigma(T_{D_0}(t))$, we conclude that

$$\forall t \geq 0 : \ \sigma[T_{D_0}(t) \circ \mathrm{pr}_{D_0}] \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}. \tag{3.16}$$

For $t \geq 0$, Lemma 3.10 c) allows us to apply Lemma 3.11 with $U := T_{D_0}(t) \circ \mathrm{pr}_{D_0}$ and $K := K(t)$ (so that $U + K = S^0(t)$), and with $G := \mathbf{R}_{(t\alpha,\infty)} \setminus P\sigma(U + K)$, which obviously satisfies the second condition in (3.11). We see from (3.16) that the first condition in (3.11) also holds, and so we can conclude that $G \subset \rho(S^0(t))$. It follows that

$$\sigma(S^0(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup P\sigma(S^0(t)). \tag{3.17}$$

Now the spectral mapping theorem for the point spectrum (see Theorem 16.7.2 on page 467 of [28], or Theorem 3.7 on p. 277 of [13]) gives $P\sigma(S^0(t)) \setminus \{0\} = \exp(t \cdot P\sigma(A^0))$. Since $P\sigma(A^0) = \sigma(A^0)$ and A^0 has no spectrum in $\mathbf{S}_{(\alpha,\beta]}$, we conclude that $\sigma(S^0(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup \mathbf{R}_{(t\beta,\infty)}$, and assertion (i) is proved.

Ad (ii): Assume $t > 0$. We also see from (3.17) and the spectral mapping theorem for the point spectrum that

$$\begin{aligned}\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} &= P\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} \\ &= \exp(t \cdot P\sigma(A^0)) \cap \mathbf{R}_{(t\beta,\infty)} = \exp[t \cdot (P\sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)})].\end{aligned}$$

(Note that $\exp(t\lambda) \in \mathbf{R}_{(t\beta,\infty)}$ if and only if $\operatorname{Re}(t\lambda) > t\beta$, which means $\operatorname{Re}(\lambda) > \beta$.) Thus we obtain that $\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} = \left\{\exp(t\lambda) \mid \lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}\right\}$, and that these numbers are all eigenvalues of $S^0(t)$. Finite multiplicity can be seen as follows: Consider a spectral value $\mu = \exp(\lambda t)$ of $S^0(t)$, where $\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}$. Then, with U and K defined as above, μ is obviously an isolated spectral value of $U + K$ and (3.16) shows that $\mu \in \rho(U)$. Finite multiplicity now follows from part (ii) of Lemma 3.11.

Ad (iii): We briefly write $\mathcal{G}$ for $\mathcal{G}_{\mu,S^0(t)}$. Remark 3.12 shows that $\mathcal{G} = \ker\,(\mu - S^0(t))^{\nu(\mu)}$, and this space is invariant under all $S^0(s)$, since the null space of an operator T_1 is invariant under a second operator T_2 that commutes with T_1. Hence $\{S^0(s)|_{\mathcal{G}}\}_{s\geq 0}$ is a semigroup on the finite dimensional space $\mathcal{G}$. Its generator is defined on all of $\mathcal{G}$ and coincides with $A^0|_{\mathcal{G}}$. Since the spectrum is natural with respect to restriction to spectral subspaces (see [47], Theorem 9.2 and Corollary 9.3, pp. 322-323), we have $\sigma(S^0(t)|_{\mathcal{G}}) = \{\mu\}$, and the finite-dimensional spectral mapping theorem for $\exp(t\cdot)$ (which follows easily from the Jordan canonical form theorem, but also e.g. from Lemma 3.13 on p. 19 of [13]) gives

$$\sigma(S^0(t)|_{\mathcal{G}}) = \exp[t \cdot \sigma(A^0|_{\mathcal{G}})].$$

It follows that $\sigma(A^0|_{\mathcal{G}}) \subset \left\{\lambda \in \mathbb{C} \mid \exp(t\lambda) = \mu\right\}$, and since $\mu \in \mathbf{R}_{(t\beta,\infty)}$, we conclude $\sigma(A^0|_{\mathcal{G}}) \subset \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)} \cap \left\{\lambda \in \mathbb{C} \mid \exp(t\lambda) = \mu\right\}$. It follows now from the Jordan canonical form theorem, with the obvious notation for the generalized eigenspaces of $A^0|_{\mathcal{G}}$, that

$$\mathcal{G} = \bigoplus_{\lambda \in \sigma(A^0|_{\mathcal{G}})} \mathcal{G}_{\lambda,A^0|_{\mathcal{G}}} = \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}, \\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0|_{\mathcal{G}}} \subset \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}, \\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0},$$

where the last inclusion is obvious. This proves the inclusion '$\subset$' in (3.15). To prove the inclusion '$\supset$', it suffices to prove that for λ as on the right hand side one has $\mathcal{G}_{\lambda,A^0} \subset \mathcal{G}_{\mu,S^0(t)}$. For such λ, the space $\mathcal{G}_{\lambda,A^0}$ is finite dimensional, contained in the domain of A^0 and invariant under A^0, and $A_0|_{\mathcal{G}_{\lambda,A^0}} = \lambda + N_\lambda$, with a nilpotent operator N_λ. Further,

$$\begin{aligned} S^0(t)|_{\mathcal{G}_{\lambda,A^0}} &= \exp[tA_0|_{\mathcal{G}_{\lambda,A^0}}] = \exp[t(\lambda + N_\lambda)] = \exp(t\lambda) \circ \exp(tN_\lambda) = \mu \circ [\mathrm{id}_{\mathcal{G}_{\lambda,A^0}} + \tilde{N}] \\ &= \mu + \hat{N}, \end{aligned}$$

where $\hat{N}$ is also a nilpotent endomorphism of $\mathcal{G}_{\lambda,A^0}$. It follows that $(\mu - S^0(t))^k|_{\mathcal{G}_{\lambda,A^0}} = 0$ for some $k \in \mathbb{N}$ (certainly for $k = \dim \mathcal{G}_{\lambda,A^0}$), and hence $\mathcal{G}_{\lambda,A^0} \subset \bigcup_{j=1}^{\infty} \ker\,(\mu - S^0(t))^j = \mathcal{G}_{\mu,S^0(t)}$, see Remark 3.12.

Ad (iv): Consider $\mu \in \exp(t\Sigma)$. To the isolated eigenvalue μ of $S^0(t)$ corresponds a spectral projection pr_μ, and from Lemma 3.13 c) we see that it commutes with pr_{Σ,A^0}. From (3.15) we see that $\mathcal{G}_{\mu,S^0(t)} \subset C^0_\Sigma$ (where the last symbol denotes the spectral subspace of A^0 corresponding to Σ), which implies that

$$\mathrm{pr}_{\Sigma,A^0} \circ \mathrm{pr}_\mu = \mathrm{pr}_\mu. \tag{3.18}$$

The operator $\mu - S^0(t)$ induces an isomorphism on $\ker(\mathrm{pr}_\mu)$, since $\mu \notin \sigma(S^0(t)|_{\ker(\mathrm{pr}_\mu)})$. We show that $C^0_{\Sigma'} \subset \ker(\mathrm{pr}_\mu)$: Since $C^0_{\Sigma'} = \ker(\mathrm{pr}_{\Sigma,A^0})$, we obtain using the commuting property and (3.18):

$$\mathrm{pr}_\mu|_{C^0_{\Sigma'}} = \mathrm{pr}_\mu(\mathrm{id} - \mathrm{pr}_{\Sigma,A^0})|_{C^0_{\Sigma'}} = (\mathrm{pr}_\mu - \mathrm{pr}_{\Sigma,A^0}\mathrm{pr}_\mu)|_{C^0_{\Sigma'}} = 0.$$

Thus, $\mu - S^0(t)$ also induces an isomorphism on the space $C^0_{\Sigma'}$ (which, as we know from Lemma 3.13b), is invariant under $S^0(t)$), and we conclude $\mu \notin \sigma(S^0(t)|_{C^0_{\Sigma'}})$, so condition (3.14) holds. ☐

Putting together the above results on spectra and characteristic functions, we arrive at the theorem below.

Theorem 3.15 (Exponential separation for S^0). *Assume L is as in (3.5), and that there exist real numbers α, β with $\alpha < \beta$ such that the spectrum $\sigma(A^0)$ of the generator of S^0 can be split as in Lemma 3.14:*

$$\sigma(A^0_{\mathbb{C}}) = \Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big) \cup \Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}\Big), \tag{3.19}$$

and $\big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\big)$ is a nonempty finite set. Then the following hold:

a) *For $t > 0$, the decomposition $C^0_{\mathbb{C}} = E^+_{\mathbb{C}} \oplus E^-_{\mathbb{C}}$ into spectral subspaces of $A^0_{\mathbb{C}}$ according to (3.19) is invariant under $S^0(t)_{\mathbb{C}}$. These spaces coincide with the spectral subspaces of $S^0(t)_{\mathbb{C}}$ coming from the spectral sets $\sigma(S^0(t)_{\mathbb{C}}) \cap \mathbf{R}_{(t\beta,\infty)}$ and $\sigma(S^0(t)_{\mathbb{C}}) \cap \big(\{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}\big)$ (see Lemma 3.14 (i)).*

b) *Analogous to a), setting $E^\pm := \operatorname{Re}(E^\pm_{\mathbb{C}})$, the 'real' decomposition $C^0 = E^+ \oplus E^-$ is invariant under the 'real' operator family $\{S^0(t)\}_{t\geq 0}$.*

c) *There exists a constant $K > 0$ such that for all $t \geq 0$*

$$|S^0(t)\varphi|_{C^0} \geq K^{-1}\exp(\beta t)|\varphi|_{C^0} \quad \textit{for } \varphi \in E^+. \tag{3.20}$$

d) *If $\tilde{\alpha} \in (\alpha, \beta)$, then there exists $\tilde{K} > 0$ such that for all $t \geq 0$*

$$|S^0(t)\varphi|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha} t)|\varphi|_{C^0} \quad \textit{for } \varphi \in E^-. \tag{3.21}$$

Proof. Ad a): Fix $t > 0$. Lemma 3.13 b) applied to the semigroup $S^0(\cdot)_{\mathbb{C}}$ and with $\Sigma := \sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}$ and Σ' as in Lemma 3.14 gives the invariance of the spaces $E^\pm_{\mathbb{C}}$ under $S^0(t)_{\mathbb{C}}$. Further, we see from Lemma 3.14(ii) that $\exp(\Sigma t) = \sigma(S^0(t)_{\mathbb{C}} \cap \mathbf{R}_{(t\beta,\infty)}$ and from part (iv) of the same lemma that condition (3.14) holds. Hence part d) of Lemma 3.13 gives that the spectral projections $\mathrm{pr}_{\exp(t\Sigma),S^0(t)_{\mathbb{C}}}$ and $\mathrm{pr}_{\Sigma,A^0_{\mathbb{C}}}$ onto $E^+_{\mathbb{C}}$ coincide, and hence the complementary spectral subspaces (corresponding to the set $\{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$ for $S^0(t)_{\mathbb{C}}$ and to Σ' for $A^0_{\mathbb{C}}$) also coincide, namely, with $E^-_{\mathbb{C}}$.

b) Follows from a) taking real parts of the involved spaces; noting that $S^0(t)$ is the restriction of $S^0(t)_{\mathbb{C}}$ to C^0, and the analogous property for the spaces $E^\pm_{\mathbb{C}}$ and $E^\pm$.

Ad c): E^+ is finite-dimensional, with

$$\min\Big\{ \operatorname{Re}(\lambda) \;\Big|\; \lambda \text{ is eigenvalue of } A^0_{\mathbb{C}}|_{E^+_{\mathbb{C}}} \Big\} > \beta.$$

Hence estimate (3.20) for $S^0(t)|_{E^+}$ with respect to the C^0-norm is obtained in a standard way, as for ordinary differential equations (even in the case of possible multiple eigenvalues, since their minimal real part is larger than β). Alternatively, one can also use that $\sigma(S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}) \subset \mathbf{R}_{(t\beta,\infty)}$ for $t > 0$ implies $\sigma([S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}]^{-1}) \subset \Big\{z \in \mathbb{C} \;\Big|\; |z| < \exp(-\beta t)\Big\}$ and then use Proposition 3.5 for the semigroup $t \mapsto \sigma([S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}]^{-1})$ of inverse operators. The analogous 'real' estimate is then obtained by restriction.

Ad d): From a), we have $\sigma(S^0(t)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$, which for $\tilde{\alpha} \in (\alpha,\beta)$ implies that the spectral radius satisfies $r(S^0(t)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}) < \exp(\tilde{\alpha}t)$. Proposition 3.5 applied to the semigroup $\{S^0(s)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}\}_{s\geq 0}$ gives estimate (3.21) first for the complexification, and the real version follows. □

We can now easily obtain a result corresponding to the above theorem for the semigroup S^1 on the space $T^1 = D(A^0)$ from Lemma 3.4, which is what we actually need later.

Corollary 3.16 (Exponential separation for S^1). *Under the assumptions and with the notation of Theorem 3.15, one has $E^+ \subset T^1$ and the S^1-invariant decomposition*

$$T^1 = E^+ \oplus (T^1 \cap E^-) \tag{3.22}$$

With respect to the C^1-norm, the semigroup S^1 satisfies estimates analogous to (3.20) and (3.21) on these spaces.

Proof. 1. From Lemma 3.13, applied with Σ as in Lemma 3.14, we see that $E^+_{\mathbb{C}} \subset D(A^0_{\mathbb{C}})$, which implies $E^+ = \mathrm{Re}(E^+_{\mathbb{C}}) \subset \mathrm{Re}(D(A^0_{\mathbb{C}})) = D(A^0) = T^1$. The complex spectral projection $\mathrm{pr}_{\Sigma,A^0_{\mathbb{C}}} \in L_C(C^0_{\mathbb{C}}, C^0_{\mathbb{C}})$ onto $E^+_{\mathbb{C}}$ induces a projection $\mathrm{pr}_{\Sigma,A^0} \in L_C(C^0, C^0)$ onto E^+ which corresponds to the decomposition in Theorem 3.15b). For $\varphi \in T^1$ one has also $\varphi - \mathrm{pr}_{\Sigma,A^0}\varphi \in T^1$ (and certainly $\varphi - \mathrm{pr}_{\Sigma,A^0}\varphi \in E^-$), so we have the decomposition $T^1 = E^+ \oplus (T^1 \cap E^-)$. It is invariant under all $S^1(t)$ $(t \geq 0)$, since the spaces $E^\pm$ are invariant under $S^0(t)$, of which $S^1(t)$ is a restriction.

2. On the finite-dimensional space E^+ all norms are equivalent, hence it is clear that an estimate analogous to (3.20) also holds w.r. to the C^1-norm, and hence for $S^1(t)$ restricted to this space.

3. Since $T^1 = D(A^0)$ and since $S^0(t)$ and A^0 commute on $D(A^0)$ ([42], Theorem 2.4 c, p. 5), we have for $\varphi \in T^1$ and $t \geq 0$ in view of (3.21):

$$|(S^1(t)\varphi)'|_{C^0} = |(S^0(t)\varphi)'|_{C^0} = |A^0 S^0(t)\varphi|_{C^0} = |S^0(t)A^0\varphi|_{C^0} = |S^0(t)\varphi'|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha}t)|\varphi'|_{C^0}.$$

In combination with estimate (3.21) for the C^0-norm, it is now obvious that we obtain an analogous estimate for the C^1-norm:

$$|S^1(t)\varphi|_{C^1} = |S^1(t)\varphi|_{C^0} + |(S^1(t)\varphi)'|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha}t)|\varphi|_{C^0} + \tilde{K}\exp(\tilde{\alpha}t)|\varphi'|_{C^0} = \tilde{K}\exp(\tilde{\alpha}t)|\varphi|_{C^1}.$$

□

The remark below may explain why we decided to give proofs for Theorem 3.15 and its prerequisites, although a number of related references exist.

Remark 3.17 (on related literature). a) Recall the sets Z_0 and Z from Corollary 3.9 (in [26], Theorem 4.1, p. 117, the set Z_0 is named Z). In Theorem 4.2 of [26], which essentially describes consequences of a splitting of the spectrum at real part = $\alpha \in \mathbb{R}$, there is no assumption like $\alpha \notin \overline{Z}$ (in the notation from that paper), which one would expect, in view of the preparations leading to that theorem. The proof of Theorem 4.2 in [26] uses Theorem 4.1 of the same reference, and that does have assumptions on Z, so their absence from the hypotheses of Theorem 4.2 is surprising. This can be explained using ideas sketched in the first remark on p. 18 of [26], but such an explanation is not given in [26]. We tried to carry this out in the proof of Lemma 3.14 above.

b) Some results of [26] take reference to the paper of the same author titled 'Adjoint theory and boundary value problems for neutral linear FDEs', which apparently was never published.

c) Contrary to Theorem 4.2 in [26], the somewhat analogous Theorem 6.1 of [34] contains assumptions on both the zeroes of $\lambda \mapsto \det \Delta(\lambda)$ and $\lambda \mapsto \det \Delta_0(\lambda)$ – this is apparently due to the more general form of the operator L considered in [34]; compare the remark after condition (J) on p. 397 of [34].

d) Theorem 6.4 from the section with application to neutral delay equations from [34] would allow to transfer the hyperbolic splitting from Theorem 6.1 of the same paper to a splitting by some growth rate $\exp(\alpha t)$ for nonzero α, but the proof contains an unclear point: It uses a rescaling argument familiar in semigroup theory (see e.g. Section 2 of Chapter II in [13]). But the rescaled semigroup and its generator are not necessarily obtained from a neutral delay equation as the original ones.

e) Above we proved and used the 'compact perturbation' result Lemma 3.11. It is a simpler form of Lemma 5.2 from p. 22 of [15], which however is stated in a Hilbert space context, and also a simpler form of Lemma 4.2 from p. 117 of [26], where it is claimed that the proof can be obtained by modification of the proof from [15]. In the corresponding passage of [17] (Lemma 2.4 on p. 211), a reference from the well-known book of Kato [35] is quoted with a misleading number, and the result of Theorem 5.26 from Chapter IV of that book (which was possibly meant) does not seem to fit well. In the book [20], the reader is referred to Section 12.12 of [20] for references concerning the 'compact perturbation' result, (Lemma 3.4 of Section 12.3, p. 285), but Section 12.12 does not seem to contain such references.

The last part of this section prepares the treatment of nonlinear equations in Section 3. It will be important later that for particular solutions y of equation 3.1 and short time intervals, on which the phase curve Y satisfies $L \circ Y = 0$, y will still be C^1 on a short interval to the right of zero.

For $\nu \in (0, h)$ we define the space

$$\mathcal{N}_\nu := \left\{ \varphi \in C^0 \mid \varphi = 0 \text{ on } [-h, -\nu] \right\}.$$

Recall the set W_1 from condition **(g3)**. Condition $\widetilde{\textbf{(g1)}}$ implies the following property:

Lemma 3.18. *Assume $(\psi, \phi) \in W_1$, further that $\chi_1, \chi_2, \in C^0$, that $\hat{\chi} \in \mathcal{N}_\Delta$ and $\hat{\psi} \in C^1 \cap \mathcal{N}_\Delta$, and that also $(\psi + \hat{\psi}, \phi) \in W_1$.*

a) Then $D_e g_1(\psi + \hat{\psi}, \phi)(\chi_1 + \hat{\chi}, \chi_2) = D_e g_1(\psi, \phi)(\chi_1, \chi_2)$.

b) In particular, $D_{1,e} g_1(0,0) = 0$ on $\mathcal{N}_\Delta$, and hence also $L = 0$ on $\mathcal{N}_\Delta$.

Proof. Property $\widetilde{\textbf{(g1)}}$ implies

$$g(\tilde{\psi} + \hat{\psi}, \tilde{\phi}) = g(\tilde{\psi}, \tilde{\phi})$$

for $(\tilde{\psi}, \tilde{\phi})$ in a neighborhood of (ψ, ϕ) in $C^1 \times C^1$. Hence,

$$Dg_1(\psi + \hat{\psi}, \phi) = Dg_1(\psi, \phi), \text{ and consequently } D_e g_1(\psi + \hat{\psi}, \phi) = D_e g_1(\psi, \phi).$$

(Note that the extensions to $C^0 \times C^0$ are unique, due to density of C^1 in $(C^0, |\ |_{C^0})$.)

It follows that

$$\begin{aligned} D_e g_1(\psi+\hat{\psi},\phi)(\chi_1+\hat{\chi},\chi_2) &= D_e g_1(\psi,\phi)(\chi_1+\hat{\chi},\chi_2) \\ &= D_e g_1(\psi,\phi)(\chi_1,\chi_2) + D_e g_1(\psi,\phi)(\hat{\chi},0). \end{aligned} \tag{3.23}$$

In case $\hat{\chi} \in C^1 \cap \mathcal{N}_\Delta$, the last term equals $\lim_{s\to 0} \frac{1}{s}[g_1(\psi + s\hat{\chi},\phi) - g_1(\psi,\phi)] = 0$, since $g_1(\psi + s\hat{\chi},\phi) - g_1(\psi,\phi) = 0$ for s sufficiently small. By density, it follows that $D_e g_1(\psi,\phi)(\hat{\chi},0) = 0$ for $\hat{\chi} \in \mathcal{N}_\Delta$. Assertion a) then follows from (3.23).

Proof of b): Assertion a), specialized to the case $\psi = \hat{\psi} = \phi = \chi_1 = \chi_2 = 0$, gives $D_{1,e}g_1(0,0)\hat{\chi} = D_e g_1(0,0)(\hat{\chi},0) = D_e g_1(0,0)(0,0) = 0$ if $\hat{\chi} \in \mathcal{N}_\Delta$, which shows b). □

Associated to equation (3.1), there are not only the semigroups $\{S^0(t)\}_{t\geq 0}$ and $\{S^1(t)\}_{t\geq 0}$, but also the so-called fundamental solution $\mathbf{X} : [-h,\infty) \to \mathbb{C}^{n\times n}$; the column functions $t \mapsto \mathbf{X}_j(t)$ $(j = 1,\dots,n)$ are zero on $[-h,0)$, equal to the j-th unit vector e_j at $t = 0$, continuous on $[0,\infty)$, and solve equation (3.1) on $[0,\infty)$ in the 'integral' sense of formula (3.24) explained below (see Section 6 of [53]): The description of the operators L and R in $L_c(C^0,\mathbb{R}^n)$ by integrals can be naturally extended from continuous functions to bounded Borel-measurable functions, leading to extended operators $\hat{L}$ and $\hat{R}$. The X_j then satisfy

$$\mathbf{X}_j(t) - \hat{L}\mathbf{X}_{j,t} = e_j + \int_0^t \hat{R}\mathbf{X}_{j,s}\,ds \quad (t \geq 0), \tag{3.24}$$

where $\mathbf{X}_{j,t} = \mathbf{X}_j(t+\cdot)|_{[-h,0]}$ denotes the segment of $\mathbf{X}_j$ at time t, and the integral is a Lebesgue integral. Compare Prop. 6.7, p. 459 in [53]. In this sense the fundamental solution can be seen as an extension of the solution operators to discontinuous initial segments (which are zero on $[-h,0)$), and it is helpful for the description of solutions to inhomogeneous equations.

Lemma 3.19 ([53, Corollary 6.8])**.** *Let $c \geq 1$, $\omega \in \mathbb{R}$ be given with*

$$|S^0(t)\chi|_{C^0} \leq ce^{\omega t}|\chi|_{C^0}, \quad \text{for all } \; t \geq 0, \chi \in C^0. \tag{3.25}$$

Then the columns $\mathbf{X}_j$ of the fundamental matrix satisfy

$$|\mathbf{X}_j(t)| \leq ce^{\omega t} \quad \text{for all } j \in \{1,\dots,n\} \text{ and } t \geq 0. \tag{3.26}$$

It will be important how the semigroup S^0 acts on functions in the space $\mathcal{N}_{\Delta/2}$, because such functions span an n-dimensional complement of $T^1 = T_{e,0}\mathcal{M}_2$ in C^2. Although we cannot expect a general solution of the linear equation (3.1) to be of class C^1, it will be important that such solutions for special initial functions (namely, in the space $\mathcal{N}_{\Delta/2}$) are C^1 when restricted to the time interval $[0,\Delta/2]$. We shall see that a similar property holds for the additional term present in solutions of the nonlinear equation 1.1, which term involves the fundamental matrix.

Lemma 3.20. *Assume $\psi \in \mathcal{N}_{\Delta/2}$.*

a) The restriction of the solution y of equation (3.1) with $y_0 = \psi$ (i.e., $y_t = S^0(t)\psi$) to $[0,\Delta/2]$ is of class C^1 (at $t = 0$, this refers to the right hand derivative), and satisfies $\dot{y}(0+) = R\psi$, and there exists a constant $M_1 \geq 1$ such that for all such ψ and $t \in [0,\Delta/2]$, one has

$$\max\{|\dot{y}(t)|,|y(t)|\} \leq M_1 \cdot |\psi|_{C^0}.$$

b) *The fundamental matrix* $\mathbf{X}$ *is absolutely continuous on* $[0, \Delta/2]$ *(right-continuous at* 0*), hence differentiable Lebesgue-almost everywhere on* $[0, \Delta/2]$*, and satisfies* $|\mathbf{X}(t)| \le \tilde{M}_1$ *for* $t \in [0, \Delta/2]$*, with an appropriate* $\tilde{M}_1 \ge 1$.

Proof. Solutions y of equation (3.1) with $y_0 = \psi \in \mathcal{N}_{\Delta/2}$ actually follow the non-neutral retarded equation $\dot{y}(t) = Ry_t$ on $[0, \Delta/2]$, since the segments y_t satisfy $y_t \in \mathcal{N}_\Delta \subset \ker(L)$ for $t \in [0, \Delta/2]$. For the semigroup S^0, there exist constants $M \ge 1$ and $\Omega > 0$ such that for all $t \ge 0$ one has $\|S^0(t)\|_{L_c(C^0, C^0)} \le M \cdot \exp(\Omega t)$ (see [42], Theorem 2.2, p. 4, or Proposition 3.5 of the present paper). Writing $\|R\|$ for $\|R\|_{L_c(C^0, \mathbb{R}^n)}$, it follows that for such y and $t \in [0, \Delta/2]$ one has

$$|\dot{y}(t)| = |Ry_t| \le \|R\| \cdot |y_t|_{C^0} \le \|R\| \cdot M \cdot \exp(\Omega t) \cdot |\psi|_{C^0} \le \|R\| \cdot M \cdot \exp(\Omega\Delta/2) \cdot |\psi|_{C^0},$$

and clearly $|y(t)| \le |y_t|_{C^0} \le M \cdot \exp(\Omega\Delta/2)|\psi|_{C^0}$. Set $M_1 := \max\{\|R\|, 1\} M \cdot \exp(\Omega\Delta/2)$.

Ad b): Continuity on $[0, \infty)$ was already remarked above, and the estimate with $\tilde{M}_1 := M \cdot \exp(\Omega\Delta/2)$ follows from Lemma 3.19. Further, the segments $\mathbf{X}_{j,t}$ are zero on $[-h, -\Delta]$ (even on $[-h, -\Delta/2)$) for $t \in [0, \Delta/2]$, and the operator $\hat{L}$ is zero on such segments, as extension of L which is zero on $\mathcal{N}_\Delta$ (see also [56], Prop. 5.3). In view of equation (3.24), we see that the $\mathbf{X}_j$ actually satisfy

$$\mathbf{X}_j(t) = e_j + \int_0^t \hat{R}\mathbf{X}_{j,s}\, ds \qquad (t \in [0, T])$$

(compare also formula (6.2) in [56]). The integrand here is of class L^1, and it follows that $\mathbf{X}_j$ is absolutely continuous, with derivative $\hat{R}\mathbf{X}_{j,t}$ for Lebesgue-almost every $t \in [0, \Delta/2]$ (see [27], Satz 131.2, p. 113, and [16], Theorem 29, Chap. X, p. 208). ☐

We turn to inhomogeneous equations now.

Lemma 3.21 (**Variation of constants,** Corollary 6.12, p. 460 in [53]). *For every* $\phi \in C^0([-h, 0], \mathbb{C}^n)$ *and every continuous function* $f : [0, \infty) \to \mathbb{C}^n$ *there is a unique continuous solution of the inhomogeneous equation*

$$\frac{d}{dt}(y - L_{\mathbb{C}} \circ Y)(t) = R_{\mathbb{C}} y_t + f(t), \qquad t > 0, \tag{3.27}$$

with $y_0 = \phi$. *(The notion of a solution here is analogous to the case* $f = 0$.*) For all* $t \ge 0$*, one has with* $v_t := S^0(t)_{\mathbb{C}}\phi$ *(i.e.,* v *is the solution of the homogeneous equation* (3.1) *with* $v_0 = \phi$*) the representation*

$$y(t) = v(t) + \int_0^t \mathbf{X}(t - s) f(s)\, ds.$$

4 Solutions of the nonlinear equation

Recall the set X_2 from the introduction, described in (2.3).

Theorem 4.1 (**Semiflow on** $\mathbf{X_2}$). *Assume* **(g0)–(g3)**. For each $\varphi \in X_2$, the corresponding solution x^φ of (1.1) is twice continuously differentiable, and for all t in the maximal existence interval $[0, t_\varphi)$, one has $x_t^\varphi \in X_2$. These solutions define a semiflow Φ on X_2 by setting $\Phi(t, \varphi) := x_t^\varphi$ (a restriction of the semiflow on X_{1+}), which is continuous w.r.t. to the obvious topology on $[0, \infty) \times C^2$ induced by $|\ |_{C^2}$ on C^2.

Proof. See Propositions 6.1 and 6.2, and the passage before condition **(g4)** in [55]. ☐

Recall the set W_1 from condition **(g3)**. As indicated in point 5) of the comments on the hypotheses, we define the map

$$r_g : \{\psi \in C^2 : (\psi', \psi) \in W_1\} \ni \psi \mapsto g_1(\psi', \psi) - Dg_1(0,0)(\psi', \psi) \in \mathbb{R}^n. \tag{4.1}$$

This map is continuously differentiable w.r.t. to $| \; |_{C^2}$ on its domain (and the ordinary topology on $\mathbb{R}^n$).

Lemma 4.2 ([53, Proposition 3.3]). *The twice continuously differentiable solutions $y : [-h, t_\varphi) \to \mathbb{R}^n$ of (1.1) with $y_0 = \varphi \in X_2$ as in Theorem 4.1 are also solutions of the inhomogeneous equation*

$$\frac{d}{dt}(y - L \circ Y)(t) = Ry_t + r_g(y_t). \tag{4.2}$$

Corollary 4.3. *If $t_0 > 0$ and $x : [-h, t_0] \to \mathbb{R}^n$ is a C^2 solution of (1.1) as in Lemma 4.2, then for $t \in [0, t_0]$ one has*

$$x_t = S^0(t)x_0 + N_t, \tag{4.3}$$

where $N(t) = 0$ for $t \in [-h, 0]$ and $N(t) = \int_0^t \mathbf{X}(t-s) r_g(x_s)\, ds$ for $t \in [0, t_0]$.

Proof. The proof follows from Lemma 4.2 and Lemma 3.21. □

For the term N in formula (4.3), we have a result similar to part a) of Lemma 3.20.

Lemma 4.4. *With x and N as in Corollary 4.3, and $T \in (0, \Delta] \cap (0, t_0]$, the function N restricted to $[0, T]$ is of class C^1, satisfies $\dot{N}(t) = RN_t + r_g(x_t)$ for $t \in [0, T]$, and in particular $\dot{N}(0+) = r_g(x_0)$.*

Proof. N is continuous, and one sees from formula (6.2) in Prop. 6.2 of [56] that N satisfies the integral equation

$$N(t) - LN_t = \int_0^t RN_s\, ds + \int_0^t r_g(x_s)\, ds.$$

Now for $t \in [0, T]$, the segments N_t are in $\mathcal{N}_\Delta$, so that $LN_t = 0$, and N actually satisfies

$$N(t) = \int_0^t RN_s\, ds + \int_0^t r_g(x_s)\, ds \qquad (t \in [0, T]).$$

Both integrands here are continuous, since $s \mapsto N_s$ is continuous, and since the map $s \mapsto (x_s', x_s) \in C^1 \times C^1$ is continuous (compare formula (4.1)). Observe here that x is C^2, which (using locally uniform continuity of $\ddot{x}$) implies that, in particular, $s \mapsto (x_s)'' \in C^0$ is continuous, and hence $s \mapsto (x_s)' \in C^1$ is continuous. It follows from the fundamental theorem of calculus that N is C^1 on $[0, T]$, with $\dot{N}(t) = RN_t + r_g(x_t)$, in particular, $\dot{N}(0+) = 0 + r_g(x_0)$. □

We turn to an estimate for the nonlinear term r_g in equation (4.2) now.

Lemma 4.5. *Assume* **$\widetilde{\text{(g1)}}$–(g3), (g6), (g7)** *and* **$\widetilde{\text{(g8)}}$**. *Then there exists a neighborhood U_2 in C^2 of 0 and a bounding function $\tilde{\zeta}$ such that for $\psi \in U_2$ the following estimate holds:*

$$|r_g(\psi)| \leq \tilde{\zeta}(|\psi|_{C^2}) \cdot |\psi|_{C^0} + \alpha(|\psi|_{C^1}) |\psi'|_{[-h,-\Delta]}|_{C^0}.$$

Proof. (The proof follows the proof of part (v) of Proposition 3.1, p. 324 in [54].) With W_1 from assumption **(g3)**, there exists a neighborhood U_2 in C^2 of 0 such that for $\psi \in U_2$ one has $(\psi', \psi) \in W_1$. For such ψ one has

$$\begin{aligned} r_g(\psi) &= \int_0^1 [Dg_1(s\psi', s\psi) - Dg_1(0,0)](\psi', \psi))\, ds = \\ &= \int_0^1 [D_1g_1(s\psi', s\psi) - D_1g_1(0,0)]\psi'\, ds + \int_0^1 [D_2g_1(s\psi', s\psi) - D_2g_1(0,0)]\psi\, ds. \end{aligned}$$

Using $\widetilde{\textbf{(g8)}}$ and that α is nondecreasing, the first term can be estimated by

$$\max_{0\le s\le 1} c_8|\psi''|_{C^0}|s\psi|_{C^0} + \alpha(|s\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0} \le c_8|\psi''|_{C^0}|\psi|_{C^0} + \alpha(|\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0}.$$

In a similar way, using **(g7)**, the second term is estimated by

$$\zeta_7(|\psi'|_{C^1} + |\psi|_{C^1}) \cdot |\psi|_{C^0} + c_7|\psi|_{C^1}|\psi|_{C^0}.$$

Adding both estimates gives

$$\begin{aligned} |r_g(\psi)| &\le \left[c_8|\psi''|_{C^0} + \zeta_7(|\psi'|_{C^1} + |\psi|_{C^1}) + c_7|\psi|_{C^1}\right] \cdot |\psi|_{C^0} + \alpha(|\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0} \\ &\le [c_8|\psi|_{C^2} + \zeta_7(2|\psi|_{C^2}) + c_7|\psi|_{C^2}] \cdot |\psi|_{C^0} + \alpha(|\psi|_{C^1}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0}. \end{aligned}$$

The assertion follows by defining $\tilde{\zeta}(s) := (c_8 + c_7)s + \zeta_7(2s)$ for $s \in [0, \infty)$. □

The following coarser estimate for r_g will be convenient to use.

Corollary 4.6. *Under the assumptions of Lemma 4.5, there exists a bounding function ρ_g such that*

$$\forall \psi \in U_2 : \ |r_g(\psi)| \le \rho_g(|\psi|_{C^2}) \cdot |\psi|_{C^1}.$$

(The proof is obvious, setting $\rho_g(|\psi|_{C^2}) := \tilde{\zeta}(|\psi|_{C^2}) + \alpha(|\psi|_{C^2})$.)

Under more restrictive assumptions than in the present paper (in particular, the linearity condition **(g4)**), it is possible to show that the nonlinearity r_g satisfies an estimate of the form $|r_g(\psi)| \le \text{const} \cdot |\psi|_{C^2} \cdot |\psi|_{C^0}$, see Proposition 3.2, p. 448 in [53]. In the present work (in view of the estimate in Corollary 4.6) we have to work with the C^1-norm. For this purpose we will replace the decomposition in formula (4.3) by a more suitable one, using that $X_2 \subset \mathcal{M}_2$, and the local graph representation of $\mathcal{M}_2$ at zero.

As another consequence of condition $\widetilde{\textbf{(g1)}}$ (together with continuity properties of Dg_1) we shall next obtain an estimate of $|x_t|_{C^1}$ in terms of $|x_0|_{C^1}$ for solutions of (1.1), if $t \in [0, \Delta]$. It follows from assumptions **(g6)** and **(g7)** that there exists a ball B_2 around zero in C^2 such that $B_2 \subset U_2$, and

$$\|D_{1,e}g_1\|_{\infty,B_2} := \sup\left\{\|D_{1,e}g_1(\psi)\|_{L_c(C^0,\mathbb{R}^n)} \mid \psi \in B_2\right\} < \infty,$$

$$\exists \tilde{D}_2 > 0\ \forall \psi \in B_2\ \forall s \in [0,1] : |D_2g_1(s\psi', s\psi)\psi| < \tilde{D}_2|\psi|_{C^0}, \text{ and hence}$$

$$\bar{D} := \max\{\|D_{1,e}g_1\|_{\infty,B_2}, \tilde{D}_2, 1\} < \infty.$$

(Note that the property concerning $\|D_{1,e}g_1\|_{\infty,B_2}$ would even hold on a ball in C^1 around zero.)

Lemma 4.7. *Assume that $B_2 \subset C^2$ and $\bar{D}$ are as described above, and that $x : [-h, t_1] \to \mathbb{R}^n$ is a solution of* (1.1) *with segments $x_t \in X_2 \cap B_2$, and with $t_1 \in (0, \Delta]$.*

Then, setting $C_1 := \bar{D}[1 + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta)]$, one has

$$\forall t \in [0, t_1] : \ |x_t|_{C^1} \leq C_1 |x_0|_{C^1}.$$

Proof. Set $\varphi := x_0 \in X_2$, and (as in [55], Proposition 2.3) define the affine-linear C^1 extension $\varphi^d : [-h, \Delta] \to \mathbb{R}^n$ by $\varphi^d(t) := \varphi(0) + t\dot{\varphi}(0) \ (t \in [0, \Delta])$. For $t \in [0, t_1]$ we have $\dot{x}(t + \theta) = \dot{\varphi}(t + \theta) = \dot{\varphi}^d(t + \theta)$ if $\theta \in [-h, -\Delta]$ and hence

$$x_t' - (\varphi^d)_t' \in \mathcal{N}_\Delta.$$

Using Lemma 3.18 a) we obtain for these t

$$\begin{aligned} \dot{x}(t) = g(x_t', x_t) &= \int_0^1 [Dg_1(sx_t', sx_t)(x_t', x_t)\, ds = \int_0^1 [D_e g_1(sx_t', sx_t)(x_t', x_t)\, ds \\ &= \int_0^1 D_e g_1(sx_t', sx_t)((\varphi^d)_t', x_t)\, ds = \int_0^1 [D_{1,e} g_1(sx_t', sx_t)(\varphi^d)_t' + D_2 g_1(sx_t', sx_t)x_t]\, ds, \end{aligned}$$

and hence

$$\begin{aligned} |\dot{x}(t)| &\leq \|D_{1,e}g_1\|_{\infty, B_2} |(\varphi^d)_t'|_{C^0} + \tilde{D}_2 |x_t|_{C^0} \\ &\leq \|D_{1,e}g_1\|_{\infty, B_2} |\varphi'|_{C^0} + \tilde{D}_2 |x_t|_{C^0} \leq \bar{D}(|\varphi|_{C^1} + |x_t|_{C^0}). \end{aligned} \tag{4.4}$$

It follows that for $t \in (0, t_1] \subset [0, \Delta]$

$$|x(t)| \leq |\varphi(0)| \ + \int_0^t |\dot{x}(s)|\, ds \leq |\varphi(0)| + \Delta \cdot \bar{D}|\varphi|_{C^1} + \bar{D} \int_0^t |x_s|_{C^0}\, ds.$$

Setting $\mu(t) := \max_{s \in [-h,t]} |x(s)|$ for $t \in [0, \Delta]$, we see that

$$\mu(t) \leq (1 + \Delta \cdot \bar{D})|\varphi|_{C^1} + \bar{D} \int_0^t \mu(s)\, ds,$$

and Gronwall's lemma gives

$$\mu(t) \leq (1 + \Delta \cdot \bar{D})|\varphi|_{C^1} \exp[\bar{D}t] \leq [1 + \Delta \cdot \bar{D}] \cdot \exp[\bar{D}\Delta] \cdot |\varphi|_{C^1}.$$

Since $\bar{D} \geq 1$, the last estimate and the definition of C_1 imply

$$|x_t|_{C^0} \leq [1 + \Delta \cdot \bar{D}] \cdot \exp[\bar{D}\Delta] \cdot |\varphi|_{C^1} \leq C_1 |\varphi|_{C^1} \text{ for } t \in [0, t_1]. \tag{4.5}$$

Combining the first inequality in (4.5) with (4.4) we conclude

$$\begin{aligned} |\dot{x}(t)| \leq \bar{D}[|\varphi|_{C^1} + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta) \cdot |\varphi|_{C^1}] &= \bar{D}[1 + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta)] \cdot |\varphi|_{C^1} \\ = C_1 |\varphi|_{C^1}. \end{aligned}$$

This estimate together with the second inequality in (4.5) gives the result. □

5 Manifolds, graph representation, and decomposition of solutions.

Recall the set U_1 from the beginning of Section 2, and consider the map $F_2 : U_1 \cap C^2 \to \mathbb{R}^n$, $F_2(\psi) := \dot{\psi}(0) - g_1(\psi', \psi)$. This map is of class C^1 (when considered with $|\ |_{C^2}$ on its domain). It is shown in [55, Proposition 5.1] that, if g satisfies $\widetilde{\textbf{(g1)}}$ and **(g3)**, then $\mathcal{M}_2 := F_2^{-1}(0)$ (called X_2 in the mentioned reference) is a submanifold of class C^1 of the space C^2. The proof in [55] is based on the fact that the differential $DF_2(\psi)$ at every point of $\mathcal{M}_2$ is surjective. In particular, $DF_2(0)$ is surjective, and given by

$$DF_2(0)\chi = \dot{\chi}(0) - Dg_1(0,0)(\chi', \chi) = \dot{\chi}(0) - L\chi' - R\chi.$$

It is also shown in part 2 of the proof of [55, Proposition 5.1] that even $DF_2(0)|_{\mathcal{N}_\Delta}$ is surjective, and hence there exist functions $\psi_1, \dots, \psi_n \in \mathcal{N}_\Delta \cap C^2$ with

$$DF_2(0)\psi_j = e_j \text{ (the } j\text{-th unit vector)}, \quad j = 1, \dots, n. \tag{5.1}$$

It is clear that one can also choose the ψ_j such that $\psi_j \in \mathcal{N}_{\Delta/2}$, which we assume from now on. Since $L = 0$ on $\mathcal{N}_\Delta$, we have $\dot{\psi}_j(0) - R\psi_j = e_j$, $j = 1, \dots, n$. The tangent space $T_0\mathcal{M}_2$ satisfies $T_0\mathcal{M}_2 = \ker DF_2(0) = \left\{\chi \in C^2 \mid \chi'(0) = Dg_1(0,0)(\chi', \chi)\right\}$, and the so-called extended tangent space to $\mathcal{M}_2$ at zero is

$$T_{e,0}\mathcal{M}_2 = \left\{\chi \in C^1 \mid \chi'(0) = D_e g_1(0,0)(\chi', \chi)\right\} = T^1$$

(see Lemma 3.4). We have the decompositions

$$C^2 = \ker DF_2(0) \oplus \bigoplus_{j=1}^{n} \mathbb{R} \cdot \psi_j, \tag{5.2}$$

and correspondingly also

$$C^1 = T^1 \oplus \bigoplus_{j=1}^{n} \mathbb{R} \cdot \psi_j, \tag{5.3}$$

since T^1 is the kernel of the continuous linear functional $C^1 \ni \chi \mapsto \chi'(0) - D_e g_1(0,0)(\chi', \chi)$, and the ψ_j also span its n-dimensional complement in C^1. Recall also that T^1 is the domain of the generator of the semigroup S^0, which induces the C^0-semigroup S^1 on $(T^1, |\ |_{C^1})$ (see Lemma 3.4).

Segments $\varphi \in \mathcal{M}_2$ close to zero w.r. to $|\ |_{C^2}$, say, with $|\varphi|_{C^2} < \delta_2$, have a graph representation $\varphi = \bar{\varphi} + \sum_{j=1}^n m_j(\bar{\varphi}) \cdot \psi_j$, with the projection $\bar{\varphi} \in T_0\mathcal{M}_2 \subset T_{e,0}\mathcal{M}_2 = T^1$ of φ to $T_0\mathcal{M}_2$ according to the decomposition (5.2), and with real-valued functions m_j defined on a neighborhood of zero in $T_0\mathcal{M}_2$ and of class C^1 w.r. to the C^2-topology, satisfying $m_j(0) = 0, Dm_j(0) = 0$. Clearly we can choose $\delta_2 > 0$ such that $B_{|\ |_{C^2}}(0, \delta_2) \subset B_2 \subset U_2$, with B_2 as in Lemma 4.7.

For segments in the state space X_2 of the semiflow from Theorem 4.1(recall $X_2 \subset \mathcal{M}_2$) we have the following close relation between the functions m_j and the components r_j of the nonlinear term r_g in equation (4.2):

Lemma 5.1. *For $\varphi \in X_2 \subset \mathcal{M}_2$ with $|\varphi|_{C^2} < \delta_2$ one has*

$$\varphi = \underbrace{\bar{\varphi}}_{\in T^1} + \sum_{j=1}^{n} m_j(\bar{\varphi}) \cdot \psi_j = \bar{\varphi} + \underbrace{\sum_{j=1}^{n} r_j(\varphi) \cdot \psi_j}_{=:\varphi_*} = \bar{\varphi} + \varphi_*, \tag{5.4}$$

and $|\varphi_*|_{C^1} \le \rho_*(|\varphi|_{C^2}) \cdot |\varphi|_{C^1}$, with a bounding function ρ_*.

Proof. For φ as in the statement of the lemma, the first equality is clear from the above remarks. Further,

$$\begin{aligned}0 = F_2(\varphi) &= \dot\varphi(0) - g_1(\varphi', \varphi) = \dot\varphi(0) - Dg_1(0,0)(\varphi', \varphi) - r_g(\varphi)\\ &= DF_2(0)\varphi - r_g(\varphi) = DF_2(0)[\bar\varphi + \sum_{j=1}^{n} m_j(\bar\varphi)\cdot\psi_j] - r_g(\varphi)\\ &= \underbrace{DF_2(0)\bar\varphi}_{=0} + \sum_{j=1}^{n} m_j(\bar\varphi)\cdot \underbrace{DF_2(0)\psi_j}_{=e_j} - r_g(\varphi)\\ &= \sum_{j=1}^{n}[m_j(\bar\varphi) - r_j(\varphi)]\cdot e_j,\end{aligned}$$

so $r_j(\varphi) = m_j(\bar\varphi)), j = 1, \dots, n$, which shows the second equality in (5.4). Finally (recall that we use the 1-norm on $\mathbb{R}^n$), from Corollary 4.6 we get

$$|\varphi_*|_{C^1} \le \sum_{j=1}^{n} |r_j(\varphi)|\cdot|\psi_j|_{C^1} \le |r_g(\varphi)|\cdot\max_j|\psi_j|_{C^1} \le \underbrace{\rho_g(|\varphi|_{C^2})\cdot\max_j|\psi_j|_{C^1}}_{=:\rho_*(|\varphi|_{C^2})}\cdot|\varphi|_{C^1},$$

so the stated estimate follows with the indicated definition of ρ_*. □

The decomposition of solutions from Corollary 4.3 is now replaced by the subsequent one (recall the semiflow Φ from Theorem 4.1).

Corollary 5.2. *For φ and $\bar\varphi$ as in Lemma 5.1 and $t \ge 0$ such that $\Phi(t, \varphi)$ is defined, one has for the corresponding solution x^φ of eq.* (1.1)

$$x_t^\varphi = \Phi(t, \varphi) = S^1(t)\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j + N_t. \tag{5.5}$$

Proof. Using Corollary 4.3 and Lemma 5.1 one gets

$$x_t^\varphi = S^0(t)[\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] + N_t = S^1(t)\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j + N_t.$$ □

Remark 5.3. Note that in this decomposition, since x_t^φ is of class C^1 (even C^2), and $S^1(t)\bar\varphi$ is of class C^1, the remaining sum of two terms $\sum_{j=1}^n r_j(\varphi)\cdot S^0(t)\psi_j + N_t$ is also of class C^1. This is not true for the parts $\sum_{j=1}^n r_j(\varphi)\cdot S^0(t)\psi_j$ and N_t as defined in Corollary 4.3, but as long as $t < \Delta/2$, we have the 'partial smoothness' results from Lemmas 3.20 and 4.4, since the ψ_j are in $\mathcal{N}_{\Delta/2}$. It is also instructive to see how the jump discontinuities at 0 in the derivatives of the middle and the last term in (5.5) cancel (as it must be, since the sum of both terms is C^1 on $[-h, t]$): We have $\dot N(0+) = r_g(\varphi)$, $\dot N(0-) = 0$, hence $\dot N(0+) - \dot N(0-) = r_g(\varphi)$. For the middle term one has, setting $\mu(t) := [\sum_{j=1}^n r_j(\varphi)\cdot S^0(t)\psi_j](0) \in \mathbb{R}^n$,

$$\dot\mu(0-) = \sum_{j=1}^{n} r_j(\varphi)\dot\psi_j(0) = \sum_{j=1}^{n} r_j(\varphi)(R\psi_j + e_j) = [\sum_{j=1}^{n} r_j(\varphi)\cdot R\psi_j] + r_g(\varphi), \text{ while}$$

$$\dot\mu(0+) = (L+R)[\sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] = R[\sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] = \sum_{j=1}^{n} r_j(\varphi)\cdot R\psi_j,$$

so that $\dot\mu(0+) - \dot\mu(0-) = -r_g(\varphi)$, which just cancels the jump of $\dot N$ at 0.

Lemma 5.4. *There exists a bounding function ρ such that for every $\delta \in (0, \delta_2]$ and every solution $x = x^\varphi$ with segments $x_t \in X_2 \cap B_{|\,|_{C^2}}(0;\delta)$ for $t \in [0, \Delta/2]$ and $\bar{\varphi}$ as in Lemma 5.1 , one has*

$$\forall t \in [0, \Delta/2]:\ |x_t^\varphi - S^1(t)\bar{\varphi}|_{C^1} \le \rho(\delta) \cdot |\varphi|_{C^1}.$$

Proof. For $\delta \in (0, \delta_2]$ (with δ_2 as in Lemma 5.1), the conclusion of Lemma 4.7 holds with an appropriate number C_1 for solutions with segments in $X_2 \cap B_{|\,|_{C^2}}(0, \delta) \subset X_2 \cap B_2$. Corollary 4.6 and the choice of δ_2 show that

$$\forall \varphi \in B_{|\,|_{C^2}}(0, \delta_2):\ |r_g(\varphi)| \le \rho_g(|\varphi|_{C^2}) \cdot |\varphi|_{C^1}. \tag{5.6}$$

Set $M^* := \max_{j=1,\ldots,n} |\psi_j|_{C^1}$, and with $M_1, \tilde{M}_1$ as in Lemma 3.20, and set

$$C := \max\left\{M_1 M^*, C_1\tilde{M}_1 \cdot (\Delta/2), \|R\|_{L_c(C^0,\mathbb{R}^n)} \cdot C_1\tilde{M}_1 \cdot (\Delta/2) + C_1\right\}.$$

Consider now a solution x as in the assertion, which then has a decomposition according to formula (5.5). From Lemma 3.20 a) and Lemma 4.4 we see that the functions y_j defined by $(y_j)_t := [S^0(t)\psi_j]$, $j = 1, \ldots, n$, as well as the function N, are C^1 when restricted to $[0, \Delta/2]$ and to $[-h, 0]$, with a jump discontinuity of the first derivative at $t = 0$. With $z(t) := \sum_{j=1}^n r_j(\varphi) \cdot y_j(t)$ for $t \in [-h, \Delta/2]$ we have from formula (5.5)

$$x^\varphi(t) - (S^1(t)\bar{\varphi})(0) = z(t) + N(t) \qquad (t \in [0, \Delta/2]),$$

and $x_0^\varphi - \bar{\varphi} = \varphi - \bar{\varphi} = \sum_{j=1}^n r_j(\varphi) \cdot \psi_j = z_0$.

For $t \in [0, \Delta/2]$ we see from Lemma 3.20 that $\max\{|\dot{y}_j(t)|, |y_j(t)|\} \le M_1 \cdot |\psi_j|_{C^0}$, $j = 1, \ldots, n$. Since we use the 1-norm on $\mathbb{R}^n$, it follows from (5.6) and the definition of M^* that for these t

$$\begin{aligned}\max\{|\dot{z}(t)|, |z(t)|\} &\le \left(\sum_{j=1}^n |r_j(\varphi)|\right) \cdot M_1 \cdot \max_j |\psi_j|_{C^0} = |r_g(\varphi)| M_1 \max_j |\psi_j|_{C^0} \\ &\le \rho_g(|\varphi|_{C^2}) \cdot M_1 M^* |\varphi|_{C^1} \le C \cdot \rho_g(\delta)|\varphi|_{C^1}.\end{aligned}$$

For $t \in [-h, 0]$ we have $\max\{|\dot{z}(t)|, |z(t)|\} \le |r_g(\varphi)| M^* \le \rho_g(|\varphi|_{C^2}) M^* |\varphi|_{C^1}$, so that the last estimate holds also for these t, since $M_1 \ge 1$.

Now $N = 0$ on $[-h, 0]$, and for $t \in [0, \Delta/2]$ we obtain, using Lemma 3.20 b), again estimate (5.6), and Lemma 4.7:

$$\begin{aligned}|N(t)| = \left|\int_0^t \mathbf{X}(t-s) r_g(x_s)\, ds\right| &\le \tilde{M}_1 \int_0^t \rho_g(|x_s|_{C^2}) |x_s|_{C^1}\, ds \\ &\le \tilde{M}_1 \rho_g(\delta) \int_0^t C_1 |\varphi|_{C^1}\, ds \le C_1 \tilde{M}_1 \cdot (\Delta/2) \rho_g(\delta) |\varphi|_{C^1} \\ &\le C \rho_g(\delta) |\varphi|_{C^1}.\end{aligned}$$

Further, for $t \in [0, \Delta/2]$, Lemma 4.4, the second last inequality in the last estimate, and Lemma 4.7 (again) give

$$\begin{aligned}|\dot{N}(t)| &\le \|R\|_{L_c(C^0,\mathbb{R}^n)} |N_t|_{C^0} + |r_g(x_t)| \le \|R\|_{L_c(C^0,\mathbb{R}^n)} C_1 \tilde{M}_1 (\Delta/2) \rho_g(\delta) |\varphi|_{C^1} + \rho_g(\delta) C_1 |\varphi|_{C^1} \\ &= \left[\|R\|_{L_c(C^0,\mathbb{R}^n)} C_1 \tilde{M}_1 (\Delta/2) + C_1\right] \cdot \rho_g(\delta) |\varphi|_{C^1} \le C \rho_g(\delta) |\varphi|_{C^1}.\end{aligned}$$

Combining the above estimates for $z, \dot{z}, N$, and $\dot{N}$ we see that for $t \in [0, \Delta/2]$

$$|x_t^\varphi - S^1(t)\bar{\varphi}|_{C^1} \le \max_{t\in[-h,0]\cup[0,\Delta/2]} \left\{|z(t)| + |N(t)| + |\dot{z}(t)| + |\dot{N}(t)|\right\} \le 4C \cdot \rho_g(\delta)|\varphi|_{C^1},$$

which proves the assertion with $\rho(\delta) := 4C \cdot \rho_g(\delta)$. □

We now use condition $(\mathbf{D^2g_2})$ from Section 2 to obtain a manifold containing initial values with unstable behavior.

The proof of the lemma below is methodically similar to the proof that $\mathcal{M}_2$ (called X_2 in [55]) is a submanifold of C^2 in Proposition 5.1 of [55].

Lemma 5.5. *Under the additional assumption* $(\mathbf{D^2g_2})$*, the set* $\mathcal{M}_4 := X_2 \cap C^4$ *is a* C^1*-submanifold of* C^4*. The tangent space at* $0 \in C^4$ *to* $\mathcal{M}_4$ *satisfies*

$$T_0\mathcal{M}_4 = \Big\{\chi \in C^4 \;\big|\; \dot{\chi}(0) = Dg_1(0,0)(\chi', \chi),\\ \ddot{\chi}(0) = Dg_1(0,0)(\chi'', \chi')\Big\}.$$

Proof. From (2.3), we have

$$\mathcal{M}_4 = \Big\{\psi \in U_1 \cap C^4 \;\big|\; \text{(i) } \dot{\psi}(0) = g(\psi', \psi);\\ \text{(ii) } \ddot{\psi}(0) = D_e g_1(\psi', \psi)(\psi'', \psi')\Big\}$$

Note that for $\psi \in C^4$, we can write g_1 instead of g and Dg_2 instead of $D_e g_1$ in the definition of X_2 and the description of $\mathcal{M}_4$. Thus, with $F_4 : U_1 \cap C^4 \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$F_4(\psi) := [\dot{\psi}(0) - g_1(\psi', \psi), \ddot{\psi}(0) - Dg_2(\psi', \psi)(\psi'', \psi')],$$

we have $\mathcal{M}_4 = F_4^{-1}\{(0,0)\}$. F_4 is of class C^1, because the maps

$$C^4 \ni \psi \mapsto (\psi'', \psi') \in C^2 \times C^3 \subset C^2 \times C^2$$

and $C^4 \ni \psi \mapsto (\psi', \psi) \in C^3 \times C^4 \subset C^2 \times C^2$ are linear and continuous, the latter maps $U_1 \cap C^4$ into W_1, and Dg_2 is C^1 on $W_1 \cap (C^2 \times C^2)$, due to assumption $(\mathbf{D^2g_2})$.

Further, for $\psi \in U_1 \cap C^4$ and $\chi \in C^4$, we calculate

$$\begin{aligned} DF_4(\psi)\chi = &[\dot{\chi}(0) - Dg_1(\psi', \psi)(\chi', \chi), \ddot{\chi}(0) \\ &- Dg_2(\psi', \psi)(\chi'', \chi') - D^2g_2(\psi', \psi)[(\psi'', \psi'), (\chi', \chi)]]. \end{aligned} \tag{5.7}$$

(Note that in the term with Dg_2 of the last formula, it makes no difference if we use Dg_1 or Dg_2.) If $\chi \in C^4 \cap \mathcal{N}_\Delta$ (then also $\chi' \in \mathcal{N}_\Delta$ and $\chi'' \in \mathcal{N}_\Delta$), Lemma 3.18 implies

$$Dg_2(\psi', \psi)(\chi', \chi) = Dg_1(\psi', \psi)(\chi', \chi) = Dg_1(\psi', \psi)(0, \chi) = D_e g_1(\psi', \psi)(0, \chi),$$

and

$$Dg_1(\psi', \psi)(\chi'', \chi') = Dg_1(\psi', \psi)(0, \chi') = D_e g_1(\psi', \psi)(0, \chi').$$

Further, using $\chi' \in \mathcal{N}_\Delta$ and Lemma 3.18 again, one obtains

$$\begin{aligned}
D^2 g_2(\psi',\psi)[(\psi'',\psi'),(\chi',\chi)] &= \lim_{s\to 0}\frac{1}{s}Dg_2(\psi'+s\chi',\psi+s\chi)(\psi'',\psi') \\
&= \frac{1}{s}Dg_1(\psi'+s\chi',\psi+s\chi)(\psi'',\psi') \\
&= \lim_{s\to 0}\frac{1}{s}Dg_1(\psi',\psi+s\chi)(\psi'',\psi') \\
&= D^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)] \\
&= D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)].
\end{aligned}$$

Thus, for $\chi \in C^4 \cap \mathcal{N}_\Delta$, we obtain

$$\begin{aligned}
DF_4(\psi)\chi = &\big[\dot\chi(0) - D_e g_1(\psi',\psi)(0,\chi), \ddot\chi(0) - D_e g_1(\psi',\psi)(0,\chi') \\
&- D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)]\big].
\end{aligned} \tag{5.8}$$

Take now $j \in \{1,\dots,n\}$. Slightly modifying the argument from the proof of [55, Proposition 5.1] to C^4-smoothness, one can find a sequence $(\chi_m^{(j)})_{m\in\mathbb{N}} \subset C^4 \cap \mathcal{N}_\Delta$ with $\dot\chi_m^{(j)}(0) = e_j$ (the j-th unit vector in $\mathbb{R}^n$) and

$$\dot\chi_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,\chi_m^{(j)}) \to e_j \quad (m\to\infty).$$

(Here $|\chi_m^{(j)}|_{C^0} \to 0$ as $m\to\infty$, which together with the continuity property of $D_e g_1$ implies $D_e g_1(\psi',\psi)(0,\chi_m^{(j)}) \to 0$. This is why we use the notation $D_e g_1(\psi',\psi)$ here, although $\chi_m^{(j)} \in C^3 \subset C^1$.)

$(\chi_m^{(j)})$ can be also chosen such that the sequences $(\chi_m^{(j)}) \subset C^1$ and $(\ddot\chi_m^{(j)}(0)) \subset \mathbb{R}^n$ are bounded, so that the sequence

$$\ddot\chi_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,(\chi_m^{(j)})') - D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi_m^{(j)})]$$

is bounded in $\mathbb{R}^n$. Hence we can assume that this sequence converges to a vector $f_j \in \mathbb{R}^n$.

Together we obtain

$$DF_4(\psi)\chi_m^{(j)} \to (e_j, f_j) \;\text{ as }\; m\to\infty. \tag{5.9}$$

Next, we find a sequence $(\zeta_m^{(j)}) \subset C^4 \cap \mathcal{N}_\Delta$ such that $\dot\zeta_m^{(j)}(0) = 0$, $|\zeta_m^{(j)}|_{C^1} \to 0$, $\ddot\zeta_m^{(j)}(0) = e_j$. With the 'minimal delay' Δ, it suffices to define $\zeta_m^{(j)}$ for m with $1/m < \Delta$. Take for example

$$\zeta_m^{(j)}(t) = \begin{cases} 0, & -h \le t \le -\dfrac{1}{m} \\ \dfrac{m^5}{2}\cdot t^2\left(t+\dfrac{1}{m}\right)^5\cdot e_j, & -\dfrac{1}{m} \le t \le 0. \end{cases}$$

Then

$$\begin{aligned}
|\zeta_m^{(j)}|_{C^0} &\le \frac{m^5}{2}\cdot\frac{1}{m^2}\cdot\frac{1}{m^5}|e_j| \to 0 \;(m\to\infty), \\
|(\zeta_m^{(j)})'|_{C^0} &\le \frac{m^5}{2}\cdot\left[\frac{2}{m}\left(\frac{1}{m}\right)^5 + \frac{1}{m^2}\cdot\frac{5}{m^4}\right]\cdot|e_j| \to 0 \qquad (m\to\infty), \\
\ddot\zeta_m^{(j)}(0) &= \frac{m^5}{2}\cdot 2\cdot\frac{1}{m^5}\cdot e_j = e_j.
\end{aligned}$$

Therefore, we have for the first part in expression (5.8) for $DF_4(\psi)\zeta_m^{(j)}$:

$$\dot{\zeta}_m^{(j)}(0) - D_e g_1(\psi', \psi)(0, \zeta_m^{(j)}) \to 0 \quad \text{as } m \to \infty,$$

and for the second part

$$\begin{aligned}\ddot{\zeta}_m^{(j)}(0) &- D_e g_1(\psi', \psi)(0, (\zeta_m^{(j)})') - D_e^2 g_2(\psi', \psi)[(\psi'', \psi'), (0, \zeta_m^{(j)})] \\ &= e_j - D_e g_1(\psi', \psi)(0, (\zeta_m^{(j)})') - D_e^2 g_2(\psi', \psi)[(\psi'', \psi'), (0, \zeta_m^{(j)})] \to e_j \quad \text{as } m \to \infty,\end{aligned}$$

since $(\zeta_m^{(j)})' \to 0$ in C^0 and $|\zeta_m^{(j)}|_{C^1} \to 0$ $(m \to \infty)$), in view of the continuity properties of $D_e g_1(\psi', \psi)$ and $D_e^2 g_2(\psi', \psi)$. We see now that

$$DF_4(\psi)\zeta_m^{(j)} \to (0, e_j) \quad \text{as } m \to \infty. \tag{5.10}$$

From (5.9) and (5.10), one sees that the $2n$ vectors (e_j, f_j) and $(0, e_j)$, $j = 1, \ldots, n$, which are a basis of $\mathbb{R}^{2n}$, are in the closure of the image of $DF_4(\psi)$. It follows that $DF_4(\psi) : C^4 \to \mathbb{R}^n \times \mathbb{R}^n \approx \mathbb{R}^{2n}$ is surjective. As in the proof of [55, Proposition 5.1], this is sufficient to show that $\mathcal{M}_4$ is a C^1–submanifold of C^4, with codimension $2n$.

We prove the statement about $T_0\mathcal{M}_4$ now: For $v \in T_0\mathcal{M}_4$ there exists $\varepsilon > 0$ and a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}_4$ differentiable at 0, with $\gamma(0) = 0, \dot{\gamma}(0) = v$. It follows that

$$0 = \frac{d}{dt}|_{t=0} F_4(\gamma(t)) = DF_4(0)\dot{\gamma}(0) = DF_4(0)v.$$

Hence $T_0\mathcal{M}_4 \subset \ker DF_4(0)$. Since both spaces have codimension $2n$ in C^3, they are equal. Now since $D^2 g_2(0,0)[(0,0), (\chi'', \chi')] = 0$ for all $\chi \in C^4$, we have (see (5.7))

$$\begin{aligned}T_0\mathcal{M}_4 &= \ker DF_4(0) \\ &= \left\{\chi \in C^4 \mid \dot{\chi}(0) = Dg_1(0,0)(\dot{\chi}, \chi),\ \ddot{\chi}(0) = Dg_1(0,0)(\chi'', \chi')\right\}.\end{aligned}$$

□

Remark 5.6.

1) The continuous extension property for $D^2 g_2$ that is actually used in the proof of (5.10) above is weaker than assumption **($\mathbf{D^2 g_2}$)**, because the proof uses only that $D_e^2 g_2(\psi', \psi)[(\psi'', \psi'), (0, \delta_m)] \to 0$ as $|\delta_m|_{C^1} \to 0$.

2) Convenient application of the chain rule to obtain C^1 smoothness of the map F_4 above was the main reason for constructing the manifold $\mathcal{M}_4$ as a subset of C^4.

3) With the last lemma, we have

$$X_1 \supset X_{1,+} \supset X_1 \cap C^2 = \mathcal{M}_2 \supset X_2 \supset X_2 \cap C^4 = \mathcal{M}_4,$$

where the X_j are invariant, but not smooth submanifolds of C^j, and the $\mathcal{M}_j$ are C^1-submanifolds of C^j, but not invariant under the semiflow on $X_{1,+}$. It seems that, based on increasingly higher smoothness assumptions on g, one could continue this construction to obtain a decreasing sequence of invariant subsets X_j containing (non-invariant) submanifolds $\mathcal{M}_j$, such that the semiflow restricted to X_j has higher order smoothness properties with increasing j. We do not pursue this here.

Recall the semigroup $\{S^0(t)\}_{t\geq 0}$ defined by the solutions of (3.1). Of course, a $\mathbb{C}^n$-valued solution $a+ib$ of (3.1) is to be understood in the sense that its real part a and its imaginary part b are $\mathbb{R}^n$-valued solutions of (3.1).

Proposition 5.7. *Assume* $(\mathbf{D^2g_2})$. *If $y : \mathbb{R} \to \mathbb{C}^n$ is a C^4 solution of equation* (3.1) *then $\zeta :=$ Re(y) solves also $\dot{\zeta}(t) = Dg_1(0,0)[(\dot{\zeta})_t, \zeta_t]$ $(t \in \mathbb{R})$, and all segments ζ_t are contained in $T_0\mathcal{M}_4$.*

Proof. Set $\zeta :=$ Re(y). The maps $p_0 : t \mapsto \zeta_t \in C^1$ and $p_1 : t \mapsto \dot{\zeta}_t \in C^1$ are of class C^1, with $\dot{p}_0(t) = (\dot{\zeta})_t$, $\dot{p}_1(t) = (\ddot{\zeta})_t$, and in the equation $\frac{d}{dt}(\zeta(t) - L\zeta_t) = R\zeta_t$, the terms in the bracket are both individually differentiable w.r.t. to t. Thus we have

$$\begin{aligned}\dot{\zeta}(t) &= L(\dot{\zeta})_t + R\zeta_t = D_{1,e}g_1(0,0)(\dot{\zeta})_t + D_{2,e}g_1(0,0)(\zeta)_t \\ &= D_e g_1(0,0)[(\dot{\zeta})_t, \zeta_t] = Dg_1(0,0)[(\dot{\zeta})_t, \zeta_t],\end{aligned} \tag{5.11}$$

where the last equality holds because $(\dot{\zeta})_t$ and ζ_t are in C^1 (even in C^3). Since ζ is of class C^4, differentiation of (5.11) gives for $t \in \mathbb{R}$: $\ddot{\zeta}(t) = Dg_1(0,0)[(\ddot{\zeta})_t, (\dot{\zeta})_t]$. It follows that for $t \in \mathbb{R}$ the segment $\chi := \zeta_t$ satisfies

$$\dot{\chi}(0) = Dg_1(0,0)(\chi', \chi) \quad \text{and} \quad \ddot{\chi}(0) = Dg_1(0,0)(\chi'', \chi').$$

We also have $\chi \in C^4$, and from Lemma 5.5 we see that $\chi \in T_0\mathcal{M}_4$. □

Corollary 5.8. *Under assumption* $(\mathbf{D^2g_2})$, *the following hold:*

a) If $\lambda \in \mathbb{C}$ is an eigenvalue of the infinitesimal generator $A^0_{\mathbb{C}}$ of the (complexified) semigroup $\{S^0(t)\}_{t\geq 0}$ then the corresponding finite-dimensional generalized eigenspace $\mathcal{G}_{\lambda, A^0_{\mathbb{C}}}$ (see Lemma 3.2 and Lemma 3.14) satisfies

$$\text{Re}\,(\mathcal{G}_{\lambda, A^0_{\mathbb{C}}}) \subset T_0\mathcal{M}_4.$$

b) In the situation of Corollary 3.16, one has $E^+ \subset T_0\mathcal{M}_4$.

Proof. Ad a): (We omit the subscript $\mathbb{C}$ in the proof.) For $\varphi \in \mathcal{G}_{\lambda, A^0}$ there exists a solution $y : \mathbb{R} \to \mathbb{C}^n$ of equation 3.1 of the form $y(t) = e^{\lambda t} \cdot p(t)$, where p is a polynomial with coefficients in $\mathbb{C}^n$, with $y_0 = \varphi$. (A solution of the finite-dimensional ODE generated by $A^0|_{\mathcal{G}_{\lambda,A^0}}$ on $\mathcal{G}_{\lambda,A^0}$.) Application of Proposition 5.7 to y at $t = 0$ gives Re$(\varphi) =$ Re$(y_0) \in T_0\mathcal{M}_4$.

Assertion b) follows from a) since $E^+ = \text{Re}\big[\bigoplus_{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}} \mathcal{G}_{\lambda,A^0}\big]$. □

Under assumption $(\mathbf{D^2g_2})$, the submanifold $\mathcal{M}_4$ of C^4 is locally a graph over its tangent space. Hence there exist neighborhoods W_4 of zero in C^4, U_4 of zero in $T_0\mathcal{M}_4$, and a C^1 function (w.r.t. to $|\ |_{C^4}$) $m_4 : U_4 \to C^4$ with $m_4(0) = 0, Dm_4(0) = 0$ such that

$$\mathcal{M}_4 \cap W^4 = \Big\{\psi + m_4(\psi) \mid \psi \in U_4\Big\}.$$

If we add the assumptions of Corollary 3.16, so the space E^+ is defined, then $E^+ \subset T_0\mathcal{M}_4$ (Corollary 5.8 b)), and the set $U_4 \cap E^+$ is a neighborhood of zero in E^+. We can then set $m_4^+ := m_4|_{U_4 \cap E^+}$ and define

$$\mathcal{M}_4^+ := \Big\{\psi + m_4^+(\psi) \mid \psi \in U_4 \cap E^+\Big\},$$

which is a submanifold of $\mathcal{M}_4$ tangent to E^+ at zero in the C^4-topology. Clearly $\mathcal{M}_4 \subset C^1$, and in view of the decompositions (5.3) and (3.22), we have

$$C^1 = E^+ \oplus (T^1 \cap E^-) \oplus C^1_*, \tag{5.12}$$

where $C^1_* := \bigoplus_{j=1}^n \mathbb{R} \cdot \psi_j$. Hence we can assume that m_4^+ is a map $m_4^+ : U_4 \cap E^+ \to (T^1 \cap E^-) \oplus C^1_*$.

Corollary 5.9. *Assume* $(\mathbf{D^2 g_2})$ *and the conditions of Corollary 3.16. There exists a bounding function ρ_4 such that all $\varphi \in \mathcal{M}_4^+$ have a representation (in the sense of (5.12))*

$$\varphi = \varphi^+ + \varphi^- + \varphi_*, \quad \textit{with } \max\{|\varphi^-|_{C^1}, |\varphi_*|_{C^1}\} \leq \rho_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}.$$

Proof. First, the properties $m_4^+(0) = 0$ and $Dm_4^+(0) = 0$ imply that for $\varphi \in \mathcal{M}_4^+, \varphi = \varphi^+ + \varphi^- + \varphi_*$, where $\varphi^- + \varphi_* = m_4^+(\varphi^+)$, one has $|\varphi^- + \varphi_*|_{C^4} = \tilde{\rho}_4(|\varphi^+|_{C^4}) \cdot |\varphi^+|_{C^4}$, with a bounding function $\tilde{\rho}_4$. Equivalence of the C^4 and the C^1 norms on the finite dimensional space E^+ gives a related bounding function $\hat{\rho}_4$ such that

$$|\varphi^- + \varphi_*|_{C^1} \leq |\varphi^- + \varphi_*|_{C^4} \leq \hat{\rho}_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}.$$

Since the spaces in (5.12) are closed subspaces w.r. to $|\ |_{C^1}$, the corresponding projections are continuous w.r. to this norm, and the C^1-norm on $(T^1 \cap E^-) \oplus C^1_*$ is equivalent to the norm defined by $\psi^- + \psi_* \mapsto \max\{|\psi_-|_{C^1}, |\varphi_*|_{C^1}\}$ on this space. The asserted estimate with a third bounding function ρ_4 follows. $\square$

6 The linearized instability theorem

Before using the preparations from the previous sections to prove our main theorem, we found it worth while to state an 'abstract' version of the main arguments in the lemma below, which reveals the essential structures. It is an adapted version of Lemma 3.3 from [30], p. 5389 and, like the latter, inspired by [3].

Lemma 6.1. *Let $(E, |\ |)$ be a Banach space. We make the subsequent assumptions:*

(i) *E has a decomposition $E = E_u \oplus E_s \oplus E_*$ into subspaces closed w.r. to $|\ |$, so the corresponding projections π_u, π_s, π_* are continuous as maps from $(E, |\ |)$ into itself. (We use the notation $x = x_u + x_s + x_*$ in obvious meaning.)*

(ii) *$X \subset E$ is a subset and $P : X \to E$ is a map which takes the form*

$$P(x) = P_L(\pi_u x + \pi_s x) + P_N(x), \textit{ with a map } P_L : E_u \oplus E_s \to E_u \oplus E_s$$

satisfying the subsequent properties.

(iii) *There exist a norm $\|\ \|$ on $E_u \oplus E_s$ equivalent to $|\ ||_{E_u \oplus E_s}$ and numbers a, b with $a < b$ and $b > 1$ such that for $x_u \in E_u$, $x_s \in E_s$ one has $\|x_u + x_s\| = \max\{\|x_u\|, \|x_s\|\}$, and*

$$\|\pi_u P_L(x_u + x_s)\| \geq b\|x_u\|, \ \|\pi_s P_L(x_u + x_s)\| \leq a\|x_s\|.$$

Under these assumptions there exists $c > 0$ such that for $x_s \in E_s$, $x_u \in E_u$ and $x \in E$ one has

$$\|x_s\| \leq c|x_s|,\ \|x_u\| \leq c|x_u|,\ |x_u + x_s| \leq c\|x_u + x_s\|, \text{ and } |\pi_u x| \leq c|x|,\ |\pi_s x| \leq c|x|.$$

With such a number c, define now $\kappa := \min\{1/2, \frac{b-a}{4c^3}, \frac{b-1}{4c^3}\}$.

If then $x = x_s + x_u + x_* \in X$ satisfies

$$\|x_u\| \geq \|x_s\| \quad \text{and} \quad \max\{|\pi_* x|, |P_N(x)|\} \leq \kappa \cdot |x|, \tag{6.1}$$

and $y = P(x) = y_u + y_s + y_*$, then also $\|y_u\| \geq \|y_s\|$ (cone invariance), and with $q := \frac{b+1}{2}$ one has $\|y_u\| \geq q\|x_u\|$ (expansion).

Proof. The existence of $c > 0$ as above is clear from equivalence of the norms and continuity of the projections. For x as in the assertion one has

$$|x| = |x_s + x_u + \pi_* x| \leq |x_s + x_u| + |\pi_* x| \leq |x_s + x_u| + \kappa|x|,$$

so $\kappa \leq 1/2$ and (6.1) imply

$$|x| \leq \frac{1}{1-\kappa}|x_s + x_u| \leq 2|x_s + x_u| \leq 2c\|x_s + x_u\| = 2c\max\{\|x_u\|, \|x_s\|\} = 2c\|x_u\|. \tag{6.2}$$

Further,

$$\begin{aligned} \|y_u\| = \|\pi_u P(x)\| &= \|\pi_u P_L(x_u + x_s) + \pi_u P_N(x)\| \geq b\|x_u\| - \|\pi_u P_N(x)\| \\ &\geq b\|x_u\| - c|\pi_u P_N(x)| \geq b\|x_u\| - c^2|P_N(x)| \geq b\|x_u\| - c^2\kappa|x| \\ &\geq (b - 2\kappa c^3)\|x_u\|, \end{aligned}$$

where we used (6.2) in the last estimate. Similarly,

$$\|y_s\| \leq a\|x_s\| + 2\kappa c^3\|x_s + x_u\| = a\|x_s\| + 2\kappa c^3\|x_u\| \leq (a + 2\kappa c^3)\|x_u\|.$$

The choice of κ implies that $b - 2\kappa c^3 \geq a + 2\kappa c^3$ and also $b - 2\kappa c^3 \geq b - (b-1)/2 = (b+1)/2 = q$, from which the assertions follow. $\square$

For simplicity, we chose the cone defined by $\|x_u\| \geq \gamma\|x_s\|$ with $\gamma = 1$ above; similar arguments are possible with different cones. The theorem below is the main result of the present work:

Theorem 6.2 (Linearized Instability Principle)**.** *Consider equation (1.1), and assume that g satisfies conditions* **(g0)**, $\widetilde{\textbf{(g1)}}$, **(g2)**, **(g3)**,**(g6)**, **(g7)**, $\widetilde{\textbf{(g8)}}$, *and* $\mathbf{(D^2g_2)}$.

Further, assume that the operator L is as in (3.5), and that with appropriate numbers $\alpha < \beta$, the spectrum of the generator $A^0_{\mathbb{C}}$ (given by the zeroes of the characteristic function χ) splits into $\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}$ and $\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}$ as in Theorem 3.15. With $\mathcal{M}^+_4$ as in Corollary 5.9, the zero equilibrium is then unstable for the semiflow Φ on X_2 in the following sense:

There exists a ball $\mathcal{B}_2$ around zero in C^2 such that for all nonzero $\varphi \in \mathcal{B}_2 \cap \mathcal{M}^+_4 \subset \mathcal{B}_2 \cap X_2$, there exists a time $t(\varphi) > 0$ with $\Phi(t(\varphi), \varphi) \notin \mathcal{B}_2$.

Proof. 1. in view of the spectral splitting assumption, Corollary 3.16 gives the decomposition $T^1 = E^+ \oplus (T^1 \cap E^-)$, and choosing $\tilde{\alpha} \in (\alpha, \beta)$ we have estimates analogous to (3.20) and (3.21) for the semigroup S^1 and $|\ |_{C^1}$. There exists a norm $\|\ \|$ equivalent to $|\ |_{C^1}|_{T^1}$ such that the estimates hold with K and $\tilde{K}$ replaced by 1 w.r. to this norm, and that $\|\varphi^+ + \varphi^-\| =$

$\max\{\|\varphi^+\|, \|\varphi^-\|\}$ for $\varphi^+ \in E^+$, $\varphi^- \in E^- \cap T^1$. (Extend S^1 to a group on E^+, define $\|\varphi^+\| := \sup_{t\le 0} \exp(\beta|t|)|S^1(t)\varphi^+|_{C^1}$ for $\varphi^+ \in E^+$, and $\|\varphi^-\| := \sup_{t\ge 0} \exp(-\tilde{\alpha}t)|S^1(t)\varphi^+|_{C^1}$ for $\varphi^+ \in T^1 \cap E^-$. Compare e.g. [3], Lemma 2.1, p. 10.) In particular, for $t := \Delta/2$ we obtain for the linear map $P_L := S^1(\Delta/2)$, which has E^+ and $T^1 \cap E^-$ as invariant subspaces, and with $b := \exp(\beta\Delta/2) > 1$, $a := \exp(\tilde{\alpha}\Delta/2) < b$ that

$$\|P_L\varphi^+\| \ge b\|\varphi^+\| \text{ for } \varphi^+ \in E^+, \ \|P_L\varphi^-\| \le a\|\varphi^-\| \text{ for } \varphi^- \in T^1 \cap E^-. \tag{6.3}$$

2. We want to apply Lemma 6.1 with $(C^1, |\,|_{C^1})$ in place of $(E, |\,|)$, and with the decomposition (5.12) in place of $E = E_u \oplus E_s \oplus E_*$. We see from (6.3) that the new norm, P_L, a and b are as required in Lemma 6.1. Thus we obtain numbers $c, \kappa > 0$ as in that lemma. Consider now δ_2 and the bounding function ρ from Lemma 5.4, and the bounding function ρ_* from Lemma 5.1. Choose $\delta_2^* \in (0, \delta_2]$ such that

$$\rho(\delta_2^*) \le \kappa \quad \text{and} \quad \rho_*(\delta_2^*) \le \kappa.$$

Next choose $\hat{\delta}_2 \in (0, \delta_2^*]$ such that with $B := B_{|\,|_{C^2}}(0, \hat{\delta}_2)$, for every $\varphi \in B \cap X_2$, the corresponding solution x^φ (with segments $x_t^\varphi \in X_2$) is defined at least on $[-h, \Delta/2]$, and satisfies

$$|x_t^\varphi|_{C^2} < \delta_2^* \qquad (t \in [0, \Delta/2]). \tag{6.4}$$

This is possible since the semiflow is continuous w.r.t. to $|\,|_{C^2}$ (Theorem 4.1); see also e.g. [1], Lemma (10.5), p. 125 and the obvious modification for semiflows, for the lower semicontinuity of the existence time. Then the map

$$P : B \cap X_2 \to X_2 \subset C^1, \qquad \varphi \mapsto x^\varphi_{\Delta/2}$$

is well-defined. For $\varphi \in B \cap X_2$ we have, in view of Lemma 5.1, $\varphi = \bar{\varphi} + \varphi_* \in T^1 \oplus C^1_*$, with

$$|\varphi_*|_{C^1} \le \rho_*(|\varphi|_{C^2}) \cdot |\varphi|_{C^1} \le \rho_*(\delta_2^*) \cdot |\varphi|_{C^1} \le \kappa|\varphi|_{C^1}.$$

Further, for such $\varphi = \bar{\varphi} + \varphi_*$, property (6.4) and Lemma 5.4 show that

$$P(\varphi) = x^\varphi_{\Delta/2} = S^1(\Delta/2)\bar{\varphi} + P_N(\varphi) = P_L(\bar{\varphi}) + P_N(\varphi),$$

with $|P_N(\varphi)|_{C^1} \le \rho(\delta_2^*) \cdot |\varphi|_{C^1} \le \kappa|\varphi|_{C^1}$.

3. We have proved in step 2 that for all $\varphi \in B \cap X_2$ the second condition in (6.1) is satisfied. In order to find initial functions $\varphi = \varphi^+ + \varphi^- + \varphi_* \in B$ which also satisfy the first condition in (6.1), that is, $\|\varphi^+\| \ge \|\varphi^-\|$, we employ Corollary 5.9, which first shows that for $\varphi \in B \cap \mathcal{M}_4^+$ one has $|\varphi^-|_{C^1} \le \rho_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}$. The equivalence of the norms $\|\,\|$ and $|\,|_{C^1}$ on T^1 implies that, with a related bounding function $\tilde{\rho}_4$, one also has $\|\varphi^-\| \le \tilde{\rho}_4(|\varphi^+|_{C^1}) \cdot \|\varphi^+\|$ for these φ. Now we can choose a ball $\mathcal{B}_2 \subset B$ w.r. to $|\,|_{C^2}$ such that for $\varphi \in \mathcal{B}_2 \cap \mathcal{M}_4^+$ one has $\tilde{\rho}_4(|\varphi|_{C^1}) \le 1$. For these φ then $\|\varphi^-\| \le \|\varphi^+\|$, i.e., the first condition in (6.1) also holds.

4. We prove now that the subset $\mathcal{B}_2 \cap \mathcal{M}_4^+$ of X_2 has the asserted property: For $\varphi \ne 0$ in this set, $P(\varphi)$ is defined, invariance of X_2 under the semiflow gives that also $\psi := P(\varphi) \in X_2$, and Lemma 6.1 shows that $\psi = \psi^+ + \psi^- + \psi_*$ again satisfies the first condition in (6.1), and $\|\psi^+\| \ge q\|\varphi^+\|$. In case that still $\psi \in \mathcal{B}_2$, also the second condition from (6.1) holds for ψ, and we can apply Lemma 6.1 again to obtain $P(\psi) = P^2(\varphi)$ with $\|P(\psi)^+\| \ge q\|\psi^+\| \ge q^2\|\varphi^+\|$, and $P(\psi)$ again allows application of that lemma, in case $P(\psi) \in \mathcal{B}_2$. As long as this iteration is possible, we obtain a sequence $P^j(\varphi), j = 1, 2, \ldots$ with exponentially growing E^+- component. Thus there must exist a $j \in \mathbb{N}$ such that $P^j(\varphi)$ is defined, but not in $\mathcal{B}_2$, which implies the assertion. $\square$

Remark 6.3.

1. In the above proof the manifold $\mathcal{M}_4^+$ was needed only to satisfy the first condition in (6.1) in the beginning – it is then preserved under iteration. The proof also shows that for nonzero $\varphi \in \mathcal{B}_2 \cap \mathcal{M}_4^+$, the corresponding trajectory has to leave the ball B (not only $\mathcal{B}_2$).

2. It would be interesting to know if in the situation of Theorem 6.2 solutions can stay in small C^0-neighborhood of zero, with only the C^2-norm growing such that the ball B_2 from above is left; for example, solutions with segments x_t even going to zero in the C^0-norm, but the C^2-norm growing (which would require rapid oscillations). We do at present not have an example.

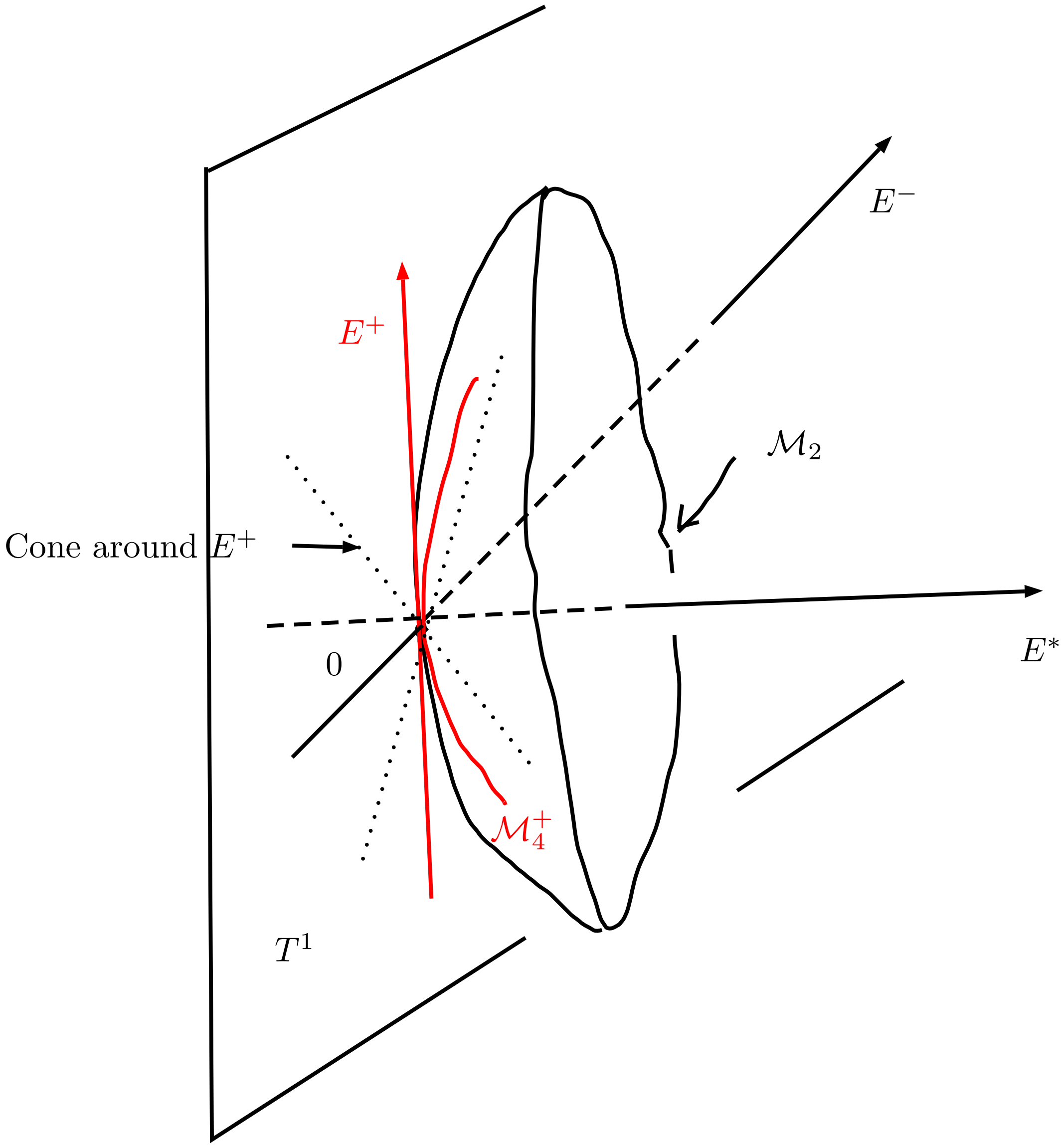

Figure 6.1: Symbolic illustration of some of the geometric objects

7 Application to examples

We show that generalizations of the mechanical example from [38] fit into our framework. In [38], the system below was considered as a model for hybrid experimental testing of a mechanical system, built by suspending a pendulum on a mass-spring-damper (MSD) system. The 'hybrid' testing consists of replacing the MSD by a computer simulation plus an actuator, which exerts the calculated force upon the pendulum, and is the source of delay in the system. $y(t)$ describes the (calculated, vertical) motion of the MSD system, while $\theta(t)$ describes the (angular) motion of the pendulum, see Fig. 1 in [38].

The equations used were (in the absence of external forcing)

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau) + m\ell[\ddot{\theta}(t-\tau)\sin(\theta(t-\tau)) + \dot{\theta}^2(t-\tau)\cos(\theta(t-\tau))] = 0,$$
$$m\ell^2\ddot{\theta}(t-\tau) + \kappa\dot{\theta}(t-\tau) + mg\ell\sin(\theta(t-\tau)) + m\ell\ddot{y}(t-\tau)\sin(\theta(t-\tau)) = 0.$$

This corresponds to eq. (2.2) on p. 1274 in [38] with $k = 0$, with positive constants M, C, K, m, ℓ, κ. Here C and κ are friction coefficients, M and m are the masses of the MSD system and the pendulum, and ℓ is the pendulum length. The terms with a factor m in the first equation represent the inertial reaction force, the force from the angular acceleration, and from the radial acceleration of the pendulum mass, in this order. In the second equation, the first three terms correspond to the pendulum with fixed point of suspension, and the last term represents the force coming from the (in this case, simulated) MSD system. Obviously, $t - \tau$ may be replaced by t in all terms of the second equation.

So far, the delay τ is a fixed number, but one can imagine situations where it is state-dependent and of the form $\tau = \tau(y(t), \theta(t), \dot{y}(t), \dot{\theta}(t))$, with a maximal value $h > 0$ and a minimal value $\Delta \in (0, h]$. In addition, the coupling terms with delayed derivatives may involve nonlinearities, for example present in the devices providing measurements to the simulating computer. Then, rewriting the above system as a four-dimensional system of first order, one could for example obtain

$$\begin{cases} \dot{y}(t) = v(t) \\ \dot{\theta}(t) = \omega(t) \\ M\dot{v}(t) = -Cv(t) - Ky(t) - mf_1(\dot{v}(t-\tau)) \\ \qquad\qquad - m\ell f_2[\dot{\omega}(t-\tau)\sin(\theta(t-\tau)), \omega^2(t-\tau)\cos(\theta(t-\tau))], \\ m\ell^2\dot{\omega}(t) = -\kappa\omega(t) - mg\ell\sin(\theta(t)) - m\ell\dot{v}(t)\sin(\theta(t)), \end{cases} \tag{7.1}$$

with suitably smooth functions $\tau = \tau[y(t), \theta(t), v(t), \omega(t)]$ and $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R}^2 \to \mathbb{R}$, with $f_1(0) = f_2(0,0) = 0$. The function f_2 would be irrelevant for the linear approximation, and we can assume $f_1'(0) = 1$. (The terms $-Cv(t) - Ky(t)$ could also be replaced by corresponding nonlinear terms with derivative $-C$ and $-K$ at zero; we do not pursue this obvious generalization.)

System (7.1) is then an equation of the class as described in equation (2.5), with dimension $n = 4$. Formal linearization of that system at the zero solution $y(t) = \theta(t) = v(t) = \omega(t) = 0$ in the sense of Remark 2.2 (using the 'frozen delay principle') gives a linear system, with y and v decoupled from θ, ω, since all terms in (2.5) coupling these variables are of second order. The y-equation of that system, written again as second order equation, is the neutral equation with constant delay

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau_0) = 0, \tag{7.2}$$

where $\tau_0 = \tau(0,0,0,0)$ (the value of the state-dependent delay at $(0,0,0,0) \in \mathbb{R}^4$). The second equation of the linearized system is just the equation of a harmonic oscillator with friction, hence contributes only to the stable part of the spectrum and will not be considered.

The first-order version of eq. (7.2) is

$$\begin{cases} \dot{y}(t) = v(t) \\ \dot{v}(t) = \dfrac{1}{M}[-Cv(t) - Ky(t) - m \cdot \dot{v}(t-\tau_0)] \end{cases} \tag{7.3}$$

and generates a semigroup S^0 on $C^0([-h,0],\mathbb{R}^2)$. In analogy to eq. (3.1) the last equation can be rewritten as

$$\frac{d}{dt}\left[\begin{pmatrix} y \\ v \end{pmatrix}(t) + \begin{pmatrix} 0 & 0 \\ 0 & m/M \end{pmatrix} \cdot \begin{pmatrix} y \\ v \end{pmatrix}(t-\tau_0)\right] = \begin{pmatrix} 0 & 1 \\ -K/M & -C/M \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}(t). \tag{7.4}$$

We see from Lemma 3.2 that the spectrum of its infinitesimal generator A consists only of isolated eigenvalues of finite multiplicity, and that these eigenvalues coincide with the solutions of the characteristic equation obtained from the exponential ansatz $y(t) = \exp(\lambda t) \cdot y(0)$ for solutions of equation (7.2). As in [38], we introduce the positive parameters

$$\omega_2 := \sqrt{K/M}, \qquad p := m/M, \qquad \zeta := \frac{C}{2\sqrt{MK}}, \qquad \hat{\tau} := \omega_2 \tau_0 \tag{7.5}$$

and set $z(t) := y(t/\omega_2)$. Then equation (7.2) is equivalent to the equation

$$\ddot{z}(t) + 2\zeta\dot{z}(t) + z(t) + p\ddot{z}(t-\hat{\tau}) = 0, \tag{7.6}$$

and the characteristic equation associated to the latter is

$$\chi(\lambda) := \lambda^2 + 2\zeta\lambda + 1 + p\lambda^2 \exp(-\lambda\hat{\tau}) = 0 \quad \text{(eq. (3.3) in [38]).} \tag{7.7}$$

This equation is analyzed in detail in [38], with a number of precursors, e.g. [4], [5] and [9]. We repeat some results from [38], adding additional pieces of information. For a nonzero complex number w we denote by $\arg(w)$ the unique angle $\varphi \in [0, 2\pi)$ with $w = |w|\exp(i\varphi)$.

From now on we make the following assumptions on the parameters:

$$p < 1, \quad \zeta < 1/\sqrt{2} \quad \text{and} \quad 1 - p^2 < (1-2\zeta^2)^2, \tag{7.8}$$

so that with the abbreviations $z := 1 - 2\zeta^2$, $q := 1 - p^2$ we have

$$z > 0, \qquad q < z^2. \tag{7.9}$$

First we show that for fixed parameters C, M, m, K, and hence for fixed ω_2, p and ζ, the following is true: For all small enough $\tau_0 > 0$, all zeroes of χ have negative real part. This is natural because equation (7.6) for $\tau_0 = 0$ is a harmonic oscillator with friction (see the corresponding remark after formula (3.3) on p. 1275 of [38]). However, the perturbation from delay zero to positive delay is not completely harmless, so we include a proof here.

Lemma 7.1. *For $\tau_0 > 0$ close enough to zero, all zeroes of χ have negative real part.*

Proof. Choose $R_1 > 0$ such that
$R_1(R_1(1-p) - 2\zeta) - 1 > 0$. If $\operatorname{Re}(\lambda) \geq 0$ and $|\lambda| > R_1$ then for all $\tau \geq 0$ one has

$$|\chi(\lambda)| \geq |\lambda|^2(1-p) - 2\zeta|\lambda| - 1 \geq R_1(R_1(1-p) - 2\zeta) - 1 > 0.$$

Choose $r_1 \in (0, R_1)$ such that $1 - [r_1^2(1+p) + 2\zeta r_1] > 0$. If $\operatorname{Re}(\lambda) \geq 0$ and $|\lambda| < r_1$ then for all $\tau \geq 0$

$$|\chi(\lambda)| \geq 1 - [r_1^2(1+p) + 2\zeta r_1] > 0.$$

On the compact set $K_1 := \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}(\lambda) \geq 0,\ r_1 \leq |\lambda| \leq R_1\right\}$ the function χ converges uniformly to the function χ^* given by $\chi^*(\lambda) = \lambda^2(1+p) + 2\zeta\lambda + 1$ as $\hat{\tau} \to 0$, and hence also as $\tau_0 \to 0$. The zeroes of χ^* have negative real parts (depending on p and ζ), so it follows that for all sufficiently small τ_0 the characteristic function χ also has no zeroes in K_1, and hence no zeroes in the closed right half plane. □

Lemma 7.2. *Assume the inequalities* (7.8)*. Then*

(i) *$\lambda = i\omega$ is a purely imaginary zero of the characteristic function χ with $\omega > 0$ if and only if $\omega\hat{\tau} = \arg(1 - \omega^2 - 2i\zeta\omega) + 2\pi n$ for some $n \in \mathbb{N}_0$, and $(1-p^2)\omega^4 + (4\zeta^2 - 2)\omega^2 + 1 = 0$.*

(ii) *In the situation of* (i) *one has $\chi'(i\omega) \neq 0$, so that the eigenvalue $\lambda = i\omega$ can be locally continued as a C^1 function of the parameters, in particular, of $\hat{\tau}$.*

(iii) *The second equation in* (i) *has exactly two positive solutions $\omega_+ > \omega_- > 0$ (depending on p and ζ, but not on $\hat{\tau}$), and $\operatorname{Re}(\lambda(\cdot))$ has a positive derivative with respect to $\hat{\tau}$ at $\hat{\tau}$, if $\lambda(\hat{\tau}) = i\omega_+$, and a negative derivative if $\lambda(\hat{\tau}) = i\omega_-$.*

(iv) *In the situation of* (iii)*, the angles*

$$\varphi_\pm := \arg(1 - \omega_\pm^2 - 2i\zeta\omega_\pm) \quad \textit{are equal to} \quad 2\pi - \arccos\left(\frac{1-\omega_\pm^2}{p\omega_\pm^2}\right)$$

and both contained in $(\pi, 2\pi)$, with $\varphi_- > \varphi_+$. The corresponding $\hat{\tau}$-values obtained from the first equation in (i) *are*

$$\tau_\pm(n) := \frac{\varphi_\pm + 2\pi n}{\omega_\pm}, \qquad n = 0, 1, 2, \ldots$$

Proof. Ad (i): For $\omega > 0$, $\chi(i\omega) = 0$ is equivalent to $-\omega^2 + 2\zeta i\omega + 1 - p\omega^2 \exp(-i\omega\hat{\tau}) = 0$, and hence to

$$\omega\hat{\tau} = \underbrace{\arg(1 - \omega^2 - 2i\zeta\omega)}_{\in(\pi,2\pi)} + 2\pi n, \qquad n \in \{0, 1, 2, 3, \ldots\}, \quad \text{and} \tag{7.10}$$

$$p^2\omega^4 = (1-\omega^2)^2 + 4\zeta^2\omega^2, \quad \text{or} \quad (1-p^2)\omega^4 + (4\zeta^2 - 2)\omega^2 + 1 = 0. \tag{7.11}$$

Ad (ii): We have for $\lambda \in \mathbb{C}$

$$\begin{aligned}\chi'(\lambda) &= 2\lambda + 2\zeta + 2\lambda p \exp(-\lambda\hat{\tau}) - \hat{\tau} p \lambda^2 \exp(-\lambda\hat{\tau}) \\ &= 2(\lambda + \zeta) + p\lambda(2 - \lambda\hat{\tau}) \exp(-\lambda\hat{\tau}).\end{aligned} \tag{7.12}$$

If $\chi(\lambda) = 0$ then $\lambda \neq 0$ and $p\lambda^2 \exp(-\lambda\hat{\tau}) = -(\lambda^2 + 2\zeta\lambda + 1)$, and hence $p\lambda \exp(-\lambda\hat{\tau}) = -(\lambda^2 + 2\zeta\lambda + 1)/\lambda$ and $\chi'(\lambda) = 2(\lambda + \zeta) - \frac{(\lambda^2+2\zeta\lambda+1)(2-\lambda\hat{\tau})}{\lambda}$, so if also $\chi'(\lambda) = 0$ then

$$2(\lambda^2 + \lambda\zeta) = (\lambda^2 + 2\zeta\lambda + 1)(2 - \lambda\hat{\tau}), \quad \text{and hence}$$

$$0 = 2\lambda\zeta + 2 - \lambda\hat{\tau}(\lambda^2 + 2\zeta\lambda + 1).$$

In particular, if this would occur for $\lambda = i\omega$ then (from the real part) $0 = 2 + 2\zeta\omega^2\hat{\tau}$, which is impossible. Thus $\chi'(i\omega) \neq 0$ if $\chi(i\omega) = 0$; the remaining statement follows from the implicit function theorem.

Ad (iii): Writing u for ω^2, equation (7.11) gives $(1 - p^2)u^2 + (4\zeta^2 - 2)u + 1 = 0$. With the notation from (7.9), we obtain the solutions

$$u_\pm = \frac{1 - 2\zeta^2 \pm \sqrt{(1 - 2\zeta^2)^2 - (1 - p^2)}}{1 - p^2} = \frac{z \pm \sqrt{z^2 - q}}{q}, \tag{7.13}$$

and thus the corresponding two solutions $\omega_\pm = \sqrt{u_\pm}$ with

$$0 < \omega_- < \omega_+.$$

In view of (ii), if $\chi(i\omega_*) = 0$ (where $* = +$ or $* = -$) for some values of the parameters $\hat{\tau}, \zeta$ and p, then, in particular, this eigenvalue can be locally viewed as a C^1 function of $\hat{\tau}$, so we can consider $\frac{d}{d\hat{\tau}} \operatorname{Re}(\lambda(\hat{\tau}))$. The assertion that the sign of this expression coincides with $*$ is contained in [38] (proof of Lemma 3.2, p. 1277 there), with the details of the calculation omitted, and with a misprint ($e^{-\lambda\tau}$ instead of $e^{\lambda\tau}$) in the formula for $\left(\frac{d\lambda}{d\tau}\right)^{-1}$. Therefore we show the main steps, and for this purpose we omit the hat in the symbol $\hat{\tau}$. Whenever an eigenvalue $\lambda(\neq 0)$ satisfies $\chi'(\lambda) \neq 0$ and is hence locally a unique C^1 function of τ, differentiation of the characteristic equation gives

$$0 = \chi'(\lambda)\frac{d\lambda}{d\tau} + p\lambda^2(-\lambda)e^{-\lambda\tau}, \quad \text{so} \quad \frac{d\lambda}{d\tau} = \frac{p\lambda^3 e^{-\lambda\tau}}{\chi'(\lambda)}.$$

Then, using (7.12), one gets

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2(\lambda + \zeta) + p\lambda(2 - \tau\lambda)e^{-\lambda\tau}}{p\lambda^3 e^{-\lambda\tau}} = 2\frac{(\lambda + \zeta)e^{\lambda\tau} + p\lambda}{p\lambda^3} - \frac{\tau}{\lambda} \\ &= 2\left[(\lambda + \zeta)\frac{e^{\lambda\tau}}{p\lambda^3} + \frac{1}{\lambda^2}\right] - \frac{\tau}{\lambda} \quad \text{(compare also formula (3.9), p. 75 in [37]).}\end{aligned}$$

Since for a complex number $w \neq 0$ one has $\operatorname{sign}(\operatorname{Re}(w)) = \operatorname{sign}(\operatorname{Re}(w^{-1}))$, and since for $\lambda = i\omega$ the term τ/λ is imaginary, we get (omitting the factor 2)

$$\operatorname{sign}\ \operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\lambda = i\omega} = \operatorname{sign}\ \operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\lambda = i\omega} = \operatorname{sign}\ \operatorname{Re}\left((\lambda + \zeta)\frac{e^{\lambda\tau}}{p\lambda^3} + \frac{1}{\lambda^2}\right)|_{\lambda = i\omega}.$$

Substituting $e^{\lambda\tau}$ by $\frac{-p\lambda^2}{\lambda^2+2\zeta\lambda+1}$ (from the characteristic equation), inserting $\lambda = i\omega$, and multiplying by $\omega > 0$, the last expression is transformed to

$$\operatorname{sign}\left\{\frac{-\omega(1 - \omega^2) + 2\zeta^2\omega}{(1 - \omega^2)^2 + 4\zeta^2\omega^2} - \frac{1}{\omega}\right\}.$$

In view of eq. (7.11), the denominator of the first fraction equals $p^2\omega^4$ (if $\chi(i\omega) = 0$), and so multiplying with this factor one obtains

$$\operatorname{sign}\{-\omega(1-\omega^2) + 2\zeta^2\omega - p^2\omega^3\} = \operatorname{sign}\{2\zeta^2 - 1 + \omega^2(1-p^2)\}.$$

According to whether $\omega = \omega_+$ or $\omega = \omega_-$, the last expression equals in the notation of (7.13) $\operatorname{sign}\{-z + q \cdot u_\pm\} = \operatorname{sign}\{\pm\sqrt{z^2 - q}\}$. It is now obvious that this sign is positive for $\omega = \omega_+$ and negative for $\omega = \omega_-$.

Ad (iv): Since the imaginary parts are negative, the angles $\varphi_\pm = \arg(1 - \omega_\pm^2 - 2i\zeta\omega_\pm)$ are given by (consider the antipodal complex numbers)

$$\varphi_\pm = \arccos\left(\frac{\omega_\pm^2 - 1}{\sqrt{(\omega_\pm^2 - 1)^2 + 4\zeta^2\omega_\pm^2}}\right) + \pi \in (\pi, 2\pi),$$

which in view of equation (7.11) coincides with

$$\varphi_\pm = \arccos\left(\frac{\omega_\pm^2 - 1}{p\omega_\pm^2}\right) + \pi = 2\pi - \arccos\left(\frac{1 - \omega_\pm^2}{p\omega_\pm^2}\right).$$

(Compare [38], the passage after formula (3.7) on p. 1276 there.) Now since arccos is strictly decreasing and $\frac{d}{du}\left[\frac{u-1}{pu}\right] = \frac{1}{pu^2} > 0$, we see that $\omega_- < \omega_+$ implies $\varphi_+ < \varphi_-$. The assertion on the corresponding $\hat{\tau}$-values is clear. □

Corollary 7.3. *Assume in addition to (7.8) the followimg condition (which is more restrictive than the second inequality of (7.9)):*

$$\frac{q}{z^2} < \frac{17}{81}, \quad \text{i.e.,} \quad \frac{1-p^2}{(1-2\zeta^2)^2} < \frac{17}{81}. \tag{7.14}$$

Then the numbers $\tau_\pm(n)$ from Lemma 7.2 satisfy

$$\tau_+(n) < \tau_+(n+1) < \tau_-(n) < \tau_-(n+1) \qquad (n \in \mathbb{N}_0),$$

where the first and last inequality are true independently of (7.14).

Proof. $\frac{q}{z^2} < \frac{17}{81}$ implies $\sqrt{1 - \frac{q}{z^2}} > \sqrt{\frac{64}{81}} = 8/9$, and hence

$$\frac{u_+}{u_-} = \frac{z + \sqrt{z^2 - q}}{z - \sqrt{z^2 - q}} = \frac{1 + \sqrt{1 - q/z^2}}{1 - \sqrt{1 - q/z^2}} > \frac{1 + 8/9}{1 - 8/9} = 17,$$

so with $\varphi_\pm \in (\pi, 2\pi)$ one sees that

$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{u_+}{u_-}} > 4 = \frac{4\pi}{\pi} > \frac{\varphi_+ + 2\pi}{\varphi_-},$$

or $\omega_-(\varphi_+ + 2\pi) < \omega_+\varphi_-$, which in view of $0 < \omega_- < \omega_+$ implies for $n \in \mathbb{N}_0$

$$\omega_-\varphi_+ + \omega_- 2\pi + \omega_- 2\pi n < \omega_+\varphi_- + \omega_+ 2\pi n.$$

Thus we obtain

$$\omega_-(\varphi_+ + 2\pi(n+1)) < \omega_+(\varphi_- + 2\pi n), \quad \text{or}$$

$$\tau_+(n+1) = \frac{\varphi_+ + 2\pi(n+1)}{\omega_+} < \frac{\varphi_- + 2\pi n}{\omega_-} = \tau_-(n).$$

The remaining two inequalities are obvious since $\tau_\pm(n) = \frac{\varphi_\pm + 2\pi n}{\omega_\pm}$. □

Corollary 7.4 (Instability)**.** *Under the conditions of Corollary 7.3, the total number $N_+(\hat\tau)$ of zeroes of χ in the right half plane (counted with multiplicity) is even and satisfies*

$$N_+(\hat\tau) \geq 2 \text{ if } \hat\tau > \tau_+(0) = \frac{\varphi_+}{\omega_+}.$$

Proof. We know from Lemma 7.1 that for $\tau_0 > 0$ close to zero, and correspondingly, $\hat\tau > 0$ close to zero, all zeroes lie in the left half plane. If we keep p and ζ satisfying the conditions of Lemma 7.2 fixed and increase $\hat\tau$ from zero to positive values, we obtain the following from Lemma 7.2: At every value $\tau_+(n)$ $(n \in \mathbb{N}_0)$ a simple eigenvalue (and its conjugate) cross the imaginary axis from left to right at $\pm i\omega_+$, and at every value $\tau_-(n)$ $(n \in \mathbb{N}_0)$ a zero and its conjugate cross the imaginary axis from right to left at $\pm\omega_-$, and these are the only $\hat\tau$-values where such crossings happen. Corollary 7.3 shows that, in particular, $\tau_+(0) < \tau_-(0)$. If $\hat\tau \in (\tau_+(0), \tau_-(0))$ then $N_+(\hat\tau) \geq 2$. For $\hat\tau \geq \tau_-(0)$ the set $\left\{n \in \mathbb{N}_0 \mid \tau_-(n) \leq \hat\tau\right\}$ is not empty, and Corollary 7.3 shows that it is contained in the set $\left\{n \in \mathbb{N}_0 \mid \tau_+(n) \leq \hat\tau\right\}$. We can thus define $c_-(\hat\tau) := \max\left\{n \in \mathbb{N}_0 \mid \tau_-(n) \leq \hat\tau\right\}$. Then one sees from Corollary 7.3 that the number $c_+(\hat\tau) := \max\left\{n \in \mathbb{N}_0 \mid \tau_+(n) \leq \hat\tau\right\}$ satisfies $c_+(\hat\tau) \geq c_-(\hat\tau) + 1$, and hence we have

$$N_+(\hat\tau) = 2(c_+(\hat\tau) - c_-(\hat\tau)) \geq 2.$$

□

Remark 7.5. The situation described in Corollary 7.4 corresponds to p-values larger than p_1 in Figure 2 on p. 1276 of [38], and to $\hat\tau$-values larger than $\tau_+(0)$ (calculated for p and ζ with (7.8) and (7.14)), so that the point $(\hat\tau, p)$ lies in the non-shaded region of Figure 2 of [38]. The lower estimate for p corresponding to condition (7.14) is explicit, but will be larger than p_1 from [38].

For the statement of the theorem below we recollect the assumptions on the parameters, expressing them in a fashion slightly closer to the original parameters. Recall that $\hat\tau = \omega_2\tau_0 = \sqrt{K/M}\tau_0$. In this notation, the assumptions made above read as follows:

$$m < M, \quad \text{and that} \quad q := 1 - \left(\frac{m}{M}\right)^2 \quad \text{and} \quad z := 1 - \frac{C^2}{2MK} \quad \text{satisfy} \quad z > 0 \quad \text{and} \quad \frac{q}{z^2} < \frac{17}{81}.$$

Further, setting

$$u_+ := \frac{z + \sqrt{z^2 - q}}{q} \quad \text{and} \quad \varphi_+ := \arccos\left[\frac{M(u_+ - 1)}{mu_+}\right] + \pi$$

we assume that with the delay function τ one has

$$\tau_0 = \tau(0,0,0,0) > \frac{\varphi_+}{\sqrt{u_+}\sqrt{K/M}}.$$

Theorem 7.6. *Consider system (7.1) with f_1, f_2 and the delay function τ of class C^2, and with $f_1(0) = f_2(0) = 0, f_1'(0) = 1$. Also assume the above conditions on the parameter values. Then the spectrum of the infinitesimal generator of the semigroup generated by eq. (7.3) splits as required in Theorem 6.2, with an even number of eigenvalues in the right half plane. Hence the zero solution of system (7.1) is then unstable as described in that theorem.*

Proof. We see from Corollary 7.4 that under the given assumptions we can split the zeroes of χ in the ones with real part less or equal zero, and the even nonzero number of zeroes with real part larger than, e.g., $\tilde{\beta} := \frac{1}{2} \min \left\{ \operatorname{Re}(\lambda) \mid \chi(\lambda) = 0, \operatorname{Re}(\lambda) > 0 \right\}$. Due to the time rescaling in going from eq. (7.3) to eq. (7.6), the eigenvalues of the generator of the semigroup S^0 differ from the zeros of χ only by the factor $\omega_2 = \sqrt{K/M}$, and hence allow a splitting as required in Theorem 6.2 with $\alpha := 0$ and $\beta := \omega_2 \tilde{\beta}$. Next, by restricting to a suitable neighborhood, we can assume that τ takes values only in an interval of the form $[\Delta, h]$, where $0 < \Delta < \tau_0 < h$. Also, in view of Prop 2.1 b), system (7.1) fits in the framework of Theorem 6.2, from which the result follows. □

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