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Multiple positive solutions for a fractional Kirchhoff type equation with logarithmic and singular nonlinearities

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Abstract. In this paper, we study the following fractional Kirchhoff type equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = |u|^{q-2}u\ln|u|^2+\frac{\lambda}{u^\gamma}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0<s<1<p$, $0<\gamma<1$, $a>0, b\geq 0$, $N>ps$, $2p<q<q+2<p_s^*$, $p_s^*=\frac{Np}{N-ps}$ is the fractional critical exponent, $\lambda>0$ is a real parameter. By using the critical point theory for nonsmooth functionals and analytic techniques, the existence and multiplicity of positive solutions are obtained.

Keywords: Kirchhoff type equation, singular nonlinearity, positive solution.

2020 Mathematics Subject Classification: 35J20, 35B33, 58E05.

1 Introduction and main result

We consider the following fractional Kirchhoff type equation involving singular nonlinearity

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = |u|^{q-2}u\ln|u|^2+\frac{\lambda}{u^\gamma}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0<s<1<p$, $0<\gamma<1$, $a>0, b\geq 0$, $N>ps$, $2p<q<q+2<p_s^*$, $p_s^*=\frac{Np}{N-ps}$ is the fractional critical exponent, $\lambda>0$ is a real parameter.

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Problem (1.1) was proposed by Kirchhoff in [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, which have the following stationary analogue of the Kirchhoff equation

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u = f(x,u).$$

Recently a great attention has been focused on studying the fractional problems, which are derived from the study of optimization, finance, phase transitions, stratified materials, anomalous diffusion, ultra-relativistic limits of quantum mechanics, water waves and so on, we can see [19] for more details. Many authors are interested in the existence of solutions for the fractional Kirchhoff type equation with logarithmic or singular terms. In [6], the authors dealt with the fractional p-Laplacian Choquard logarithmic equation involving critical and subcritical nonlinearities, they proved the existence and multiplicity of nontrivial solutions by using genus theory and the mountain pass lemma. Fan et al. in [7, 8] studied the fractional critical Schrödinger equation with logarithmic nonlinearity, by applying the Nehari manifold and the variational methods, the existence of positive ground state solutions and ground state sign-changing solutions were showed. Truong studied the fractional p-Laplacian equation with logarithmic nonlinearity on whole space, by the Nehari manifold method, the author obtained the existence of nontrivial solutions in [23].

In particular, the authors considered the following fractional Kirchhoff equation with logarithmic and critical nonlinearities

$$\begin{cases} M([u]^p_{s,p})(-\Delta)^s_p u = \lambda h(x)|u|^{q-2}u\ln|u|^2 + |u|^{p^*_s-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N > ps$ with $s \in (0,1)$, $p > 1$ and

$$[u]^p_{s,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy.$$

When $M([u]^p_{s,p}) = a + b[u]^p_{s,p}$ and $h(x) = 1$, by using constraint variational methods, Liang and Rădulescu in [15] dealt with the existence and least energy sign-changing solutions of (1.2). Under some assumptions on M and $p \geq 2$, the authors [14] obtained the existence of solutions in the case of high perturbations of (1.2) for λ sufficiently large. When $h(x) > 0$, Lv and Zheng in [17] showed the existence of a nontrivial ground state solution for λ sufficiently small. When $M([u]^p_{s,p}) = [u]^{(\theta-1)p}_{s,p}$ with $\theta \geq 1$, the authors [24] established the least energy solutions for (1.2) with $\theta p < q < p^*_s$ and $h(x) > 0$ and two local least energy solutions with $1 < q < \theta p$ and $h(x)$ is a sign-changing function by the Nehari manifold approach.

In [9], Fiscella and Mishra studied the following fractional Kirchhoff type equation with singular and critical growths

$$\begin{cases} M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda f(x)u^{-\gamma} + g(x)|u|^{2^*_s-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.3}$$

where $N > 2s$ with $s \in (0,1)$, $0 < \gamma < 1$, $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, by the Nehari manifold method, they proved that (1.3) has at least two positive solutions for λ sufficiently small. In [21], by the variational methods and truncation arguments, the authors obtained the existence of multiple positive solutions for (1.3) with singular and Choquard critical nonlinearities. In addition, the existence of positive solutions for the fractional problems involving singular nonlinearity has been paid much attention by many authors, we can see [1,3,10,11,22,25,26] and so on.

Recently, Lei et al. in [13] investigated the following logarithmic elliptic equation with singular nonlinearity

$$\begin{cases} -\Delta u = u\log|u|^2 + \frac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $\gamma \in (0,1)$, by using the variational methods and the critical point theory for a nonsmooth functional, they obtained the existence of two positive solutions. In [20], the authors proved the existence of positive solutions for a logarithmic Schrödinger–Poisson system with singular nonlinearity.

Define the fractional Sobolev space $W^{s,p}(\Omega)$ is given by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\},$$

with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

Let $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$, we define

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\}.$$

The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

where the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. The space X_0 is defined as $X_0 = \{u \in X : u = 0 \text{ on } \mathcal{C}\Omega\}$, for all $p > 1$, it is a uniformly convex Banach space endowed with the norm

$$\|u\| := \|u\|_{X_0} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}. \tag{1.4}$$

The dual space of X_0 will be denoted by X_0^*. Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the integral in (1.4) can be extended to $\mathbb{R}^N \times \mathbb{R}^N$. We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e. $S_\rho = \{u \in X_0 : \|u\| = \rho\}$, $B_\rho = \{u \in X_0 : \|u\| \leq \rho\}$.

Let S be the best fractional Sobolev constant

$$S = \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy}{\left(\displaystyle\int_\Omega |u|^{p_s^*} dx \right)^{p/p_s^*}}.$$

The energy functional associated with (1.1) has the form

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln|u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx.$$

Since the energy functional fails to be finite and loses C^1 smoothness on its natural Sobolev spaces, the classical critical point theory can not be applied directly, we overcome this hurdle by the critical point theory for nonsmooth functionals. Moreover, logarithmic nonlinearity is sign-changing, it becomes much more difficult than usual to obtain estimates of the energy functional. Our difficulties are as follows: (i) The singular term leads to the non-differentiability of the energy functional I_λ corresponding to (1.1) in [13]; (ii) The appearance of logarithmic and singular nonlinearities makes it more difficult for us to prove the convergence of the (PS) sequence; (iii) The fractional p-Laplacian operators also cause great difficulties for the existence of positive solutions.

Now we state our main result.

Theorem 1.1. *Assume that $0 < \gamma < 1$ and $2p < q < q + 2 < p_s^*$ hold, there exists $\Lambda_0 > 0$ such that for all $\lambda \in (0, \Lambda_0)$, equation (1.1) has at least two positive solutions.*

2 Preliminaries

In this section, we first recall some concepts adapted from critical point theory for nonsmooth functionals in [4,16].

Definition 2.1. Let (Y, d) be a complete metric space, $f : Y \to \mathbb{R}$ be a continuous functional in Y. Denote by $|Df|(u)$ the supremum of κ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\sigma : B_\delta(u) \times [0, \delta] \to Y$ satisfying

$$\begin{cases} f(\sigma(z,t)) \le f(z) - \kappa t, & (z,t) \in B_\delta(u) \times [0,\delta], \\ d(\sigma(z,t), z) \le t, & (z,t) \in B_\delta(u) \times [0,\delta]. \end{cases} \tag{2.1}$$

The extended real number $|Df|(u)$ is called the weak slope of f at u.

Definition 2.2. A sequence $\{u_n\}$ of Y is called (PS) sequence of the functional f, if $|Df|(u_n) \to 0$ as $n \to \infty$ and $f(u_n)$ is bounded. We say that $u \in Y$ is a critical point of f if $|Df|(u) = 0$. Since $u \to |Df|(u)$ is lower semicontinuous, any accumulation point of a (PS) sequence is clearly a critical point of f.

Since we are looking for positive solutions of (1.1), we consider the functional I_λ as defined on the closed positive cone P of X_0

$$P = \{u \mid u \in X_0,\ u(x) \ge 0,\ \text{a.e. } x \in \Omega\}.$$

P is a complete metric space and I_λ is a continuous functional on P. Then we have the following lemma.

Lemma 2.3. *Suppose that $|DI_\lambda|(u) < +\infty$ holds, then for all $v \in P$ such that*

$$\begin{aligned} (a + b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[(v-u)(x)-(v-u)(y)]}{|x-y|^{N+ps}} dxdy \\ - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx + |DI_\lambda|(u)\|v-u\| \ge \lambda\int_\Omega \frac{(v-u)}{u^\gamma} dx. \end{aligned} \tag{2.2}$$

Proof. Let $|DI_\lambda|(u) < \mu$, $\delta < \frac{1}{2}\|v-u\|$, $v \in P$ and $v \neq u$. Define the mapping $\sigma : B_\delta(u) \times [0,\delta] \to P$ by

$$\sigma(z,t) = z + t\frac{v-z}{\|v-z\|}.$$

Thus, we have $\|\sigma(z,t) - z\| = t$, by (2.1), there exists a pair $(z,t) \in B_\delta(u) \times [0,\delta]$ such that

$$I_\lambda(\sigma(z,t)) > I_\lambda(z) - \mu t.$$

Consequently, we assume that there exist sequences $\{u_n\} \subset P$ and $\{t_n\} \subset [0,\infty)$, such that $u_n \to u$, $t_n \to 0^+$, and

$$I_\lambda\left(u_n + t_n\frac{v-u_n}{\|v-u_n\|}\right) \geq I_\lambda(u_n) - \mu t_n,$$

that is

$$I_\lambda(u_n + s_n(v-u_n)) \geq I_\lambda(u_n) - \mu s_n\|v-u_n\|, \tag{2.3}$$

where $s_n = \frac{t_n}{\|v-u_n\|} \to 0^+$ as $n \to \infty$. Divided by s_n in (2.3), we have

$$\begin{aligned}
&\frac{a}{p}\frac{\|u_n + s_n(v-u_n)\|^p - \|u_n\|^p}{s_n} + \frac{b}{2p}\frac{\|u_n + s_n(v-u_n)\|^{2p} - \|u_n\|^{2p}}{s_n} \\
&\quad + \int_\Omega \frac{f(u_n + s_n(v-u_n)) - f(u_n)}{s_n}dx + \mu\|v-u_n\| \\
&\geq \frac{\lambda}{1-\gamma}\int_\Omega \frac{|u_n + s_n(v-u_n)|^{1-\gamma} - |u_n|^{1-\gamma}}{s_n}dx,
\end{aligned}$$

where

$$f(u_n) = \frac{2}{q^2}\int_\Omega |u_n|^q dx - \frac{1}{q}\int_\Omega |u_n|^q \ln|u_n|^2 dx.$$

Notice that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{f(u_n + s_n(v-u_n)) - f(u_n)}{s_n}dx \\
&= \lim_{n\to\infty}\frac{2}{q^2}\int_\Omega \frac{|u_n + s_n(v-u_n)|^q - |u_n|^q}{s_n}dx \\
&\quad - \lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{(|u_n + s_n(v-u_n)|^q - |u_n|^q)\ln|u_n + s_n(v-u_n)|^2}{s_n}dx \\
&\quad - \lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{|u_n|^q(\ln|u_n + s_n(v-u_n)|^2 - \ln|u_n|^2)}{s_n}dx \\
&= \frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx - \frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx \\
&= -\int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx.
\end{aligned}$$

In fact, from [15], for all $r \in (q, p_s^*)$ and $2p < q < p_s^*$, we have that

$$\lim_{t\to 0}\frac{|t|^{q-1}\ln|t|^2}{|t|^{p-1}} = 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{|t|^{q-1}\ln|t|^2}{|t|^{r-1}} = 0.$$

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{q-1}\ln|t|^2 \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1}. \tag{2.4}$$

It follows from $u_n(x) \to u(x)$ a.e in Ω and $u_n \to |u_n|^q \ln |u_n|^2$ is continuous that

$$|u_n(x)|^q \ln |u_n(x)|^2 \to |u(x)|^q \ln |u(x)|^2, \quad \text{a.e. in } \Omega.$$

Thus, by the Lebesgue dominated convergence theorem and (2.4), we get

$$\int_\Omega |u_n|^q \ln |u_n|^2 dx \to \int_\Omega |u|^q \ln |u|^2 dx, \quad \text{as } n \to \infty.$$

Set

$$I_{1,n} = \int_\Omega \frac{|u_n + s_n(v - u_n)|^{1-\gamma} - |(1 - s_n)u_n|^{1-\gamma}}{s_n(1-\gamma)} dx,$$

and

$$I_{2,n} = \frac{(1-s_n)^{1-\gamma} - 1}{s_n(1-\gamma)} \int_\Omega |u_n|^{1-\gamma} dx.$$

Notice that

$$I_{1,n} = \int_\Omega \frac{\xi_n^{-\gamma} s_n v}{s_n} dx = \int_\Omega \xi_n^{-\gamma} v dx,$$

where $\xi_n \in (u_n - s_n u_n, u_n + s_n(v - u_n))$, which implies that $\xi_n \to u$ $(u_n \to u)$ as $s_n \to 0^+$. Since $I_{1,n} \geq 0$ for all n, by the Fatou lemma, we obtain that

$$\liminf_{n\to\infty} I_{1,n} \geq \int_\Omega \frac{v}{u^\gamma} dx,$$

for all $v \in P$. For $I_{2,n}$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} I_{2,n} = -\int_\Omega u^{1-\gamma} dx.$$

From the above information, we get

$$\begin{aligned}
&(a + b\|u\|^p) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[(v-u)(x) - (v-u)(y)]}{|x-y|^{N+ps}} dxdy \\
&\quad - \int_\Omega |u|^{q-2} u(v - u) \ln |u|^2 dx + \mu\|v - u\| \\
&\quad \geq \liminf_{n\to\infty} (I_{1,n} + I_{2,n}) \geq \lambda \int_\Omega \frac{(v-u)}{u^\gamma} dx,
\end{aligned}$$

for every $v \in P$. Since $|DI_\lambda|(u) < \mu$ is arbitrary. The proof is complete. □

Lemma 2.4. *Let $2p < q < q + 2 < p_s^*$, there exist constants $\alpha, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$. Then the functional I_λ satisfies the following conditions:*

(i) $I_\lambda|_{u\in S_\rho} \geq \alpha > 0$; $\inf_{u\in B_\rho} I_\lambda(u) < 0$;

(ii) *There exists $e \in X_0$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) Since $\ln|u|^2 \le |u|^2$, by the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln|u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\
&\ge \frac{a}{p}\|u\|^p - \frac{1}{q}\int_\Omega |u|^{q+2}dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\
&\ge \frac{a}{p}\|u\|^p - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}\left(\int_\Omega |u|^{p_s^*}dx\right)^{\frac{q+2}{p_s^*}} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}\left(\int_\Omega |u|^{p_s^*}dx\right)^{\frac{1-\gamma}{p_s^*}} \\
&\ge \frac{a}{p}\|u\|^p - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}S^{-\frac{q+2}{p}}\|u\|^{q+2} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}S^{-\frac{1-\gamma}{p}}\|u\|^{1-\gamma} \\
&= \|u\|^{1-\gamma}\left(\frac{a}{p}\|u\|^{p-1+\gamma} - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}S^{-\frac{q+2}{p}}\|u\|^{q+1+\gamma} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}S^{-\frac{1-\gamma}{p}}\right).
\end{aligned}$$

Set

$$h(t) = \frac{a}{p}t^{p-1+\gamma} - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}S^{-\frac{q+2}{p}}t^{q+1+\gamma}$$

for $t > 0$, thus, there exists a constant

$$\rho = \left[\frac{aq(p-1+\gamma)S^{\frac{q+2}{p}}}{p(q+1+\gamma)|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}}\right]^{\frac{1}{q+2-p}} > 0,$$

such that $\max_{t>0} h(t) = h(\rho) > 0$. Let

$$\Lambda_0 = \frac{h(\rho)(1-\gamma)S^{\frac{1-\gamma}{p}}}{|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}},$$

thus, $I_\lambda|_{u\in S_\rho} \ge \alpha > 0$ for all $\lambda \in (0,\Lambda_0)$. Moreover, for $u \in X_0\backslash\{0\}$, we get

$$\lim_{t\to 0^+}\frac{I_\lambda(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx < 0.$$

Therefore, we obtain that $I_\lambda(tu) < 0$ for t small enough. Consequently, for $\|u\|$ small enough, we have

$$d \triangleq \inf_{u\in B_\rho} I_\lambda(u) < 0. \tag{2.5}$$

(ii) For all $u \in X_0\backslash\{0\}$ and $t > 0$, we have

$$\begin{aligned}
I_\lambda(tu) &= \frac{at^p}{p}\|u\|^p + \frac{bt^{2p}}{2p}\|u\|^{2p} + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{t^q}{q}\int_\Omega |u|^q\ln|tu|^2 dx \\
&\quad - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\
&= \frac{at^p}{p}\|u\|^p + \frac{bt^{2p}}{2p}\|u\|^{2p} + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{2t^q}{q}\int_\Omega |u|^q\ln|u| dx \\
&\quad - \frac{2t^q}{q}\int_\Omega |u|^q \ln t dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \to -\infty
\end{aligned}$$

as $t \to +\infty$, which implies that $I_\lambda(tu) < 0$ for $t > 0$ large enough. Thus, we can find $e \in X_0$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Lemma 2.5. *Suppose that $2p < q < p^*$ and $0 < \gamma < 1$ hold, the functional I_λ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset P$ be a (PS) sequence for I_λ at the level c, that is

$$I_\lambda(u_n) \to c, \quad \text{and} \quad |DI_\lambda|(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

It follows from (2.2) and (2.6) that

$$\begin{aligned}(a+b\|u_n\|^p) &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\ &\times [(v-u_n)(x)-(v-u_n)(y)]dxdy \\ &- \int_\Omega |u_n|^{q-2}u_n(v-u_n)\ln|u_n|^2dx + |DI_\lambda|(u_n)\|v-u_n\| \\ \geq \lambda &\int_\Omega \frac{(v-u_n)}{u_n^\gamma}dx.\end{aligned} \tag{2.7}$$

Choosing $v = 2u_n \in P$ in (2.7), we obtain that

$$(a+b\|u_n\|^p)\|u_n\|^p - \int_\Omega |u_n|^q \ln|u_n|^2dx + |DI_\lambda|(u_n)\|u_n\| \geq \lambda\int_\Omega u_n^{1-\gamma}dx. \tag{2.8}$$

Combining with (2.6), (2.8) and the Hölder inequality, there exists a constant $C > 0$, we get

$$\begin{aligned}c+1+o(\|u_n\|) &\geq I_\lambda(u_n) + \frac{1}{q}|DI_\lambda|(u_n)\|u_n\| \\ &\geq a\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p + b\left(\frac{1}{2p}-\frac{1}{q}\right)\|u_n\|^{2p} + \frac{2}{q^2}\int_\Omega |u_n|^q dx \\ &\quad - \lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)\int_\Omega |u_n|^{1-\gamma}dx \\ &\geq a\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p - \lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\|u_n\|^{1-\gamma}.\end{aligned}$$

Since $1-\gamma < 1 < p$, we deduce that $\{u_n\}$ is bounded in X_0. Therefore, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in X_0$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X_0, \\ u_n \to u, & \text{strongly in } L^r(\Omega)\ (1 \leq r < p_s^*), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.9}$$

as $n \to \infty$. Taking $v = u_m$ in (2.7), we have

$$\begin{aligned}(a+b\|u_n\|^p) &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\ &\times [(u_m-u_n)(x)-(u_m-u_n)(y)]dxdy \\ &- \int_\Omega |u_n|^{q-2}u_n(u_m-u_n)\ln|u_n|^2dx + o(1)\|u_m-u_n\| \\ \geq \lambda &\int_\Omega \frac{(u_m-u_n)}{u_n^\gamma}dx.\end{aligned} \tag{2.10}$$

By changing the role of u_m and u_n in (2.10), we have a similar inequality. By adding the two inequalities, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}}\\ &\quad\times[(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy\\ &\le\int_\Omega\left(\frac{|u_n|^{q-2}u_n\ln|u_n|^2}{a+b\|u_n\|^p}-\frac{|u_m|^{q-2}u_m\ln|u_m|^2}{a+b\|u_m\|^p}\right)(u_n-u_m)dx\\ &\quad+\lambda\int_\Omega\left(\frac{u_n^{-\gamma}}{a+b\|u_n\|^p}-\frac{u_m^{-\gamma}}{a+b\|u_m\|^p}\right)(u_n-u_m)dx+o(1)\|u_m-u_n\|\\ &\le\int_\Omega\left(\frac{|u_n|^{q-2}u_n\ln|u_n|^2}{a+b\|u_n\|^p}-\frac{|u_m|^{q-2}u_m\ln|u_m|^2}{a+b\|u_m\|^p}\right)(u_n-u_m)dx+o(1)\|u_m-u_n\|.\end{aligned}\tag{2.11}$$

With the help of (2.4), (2.9) and $\{u_n\}$ is bounded in X_0, for all $r\in(q,p_s^*)$, we have

$$\begin{aligned}&\left|\int_\Omega\frac{|u_n|^{q-2}u_n\ln|u_n|^2}{a+b\|u_n\|^p}(u_n-u_m)dx\right|\\ &\le C\left|\int_\Omega|u_n|^{q-2}u_n\ln|u_n|^2(u_n-u_m)dx\right|\\ &\le C\varepsilon\int_\Omega|u_n|^{p-1}|u_n-u_m|dx+C_\varepsilon\int_\Omega|u_n|^{r-1}|u_n-u_m|dx\\ &\le C\varepsilon\left(\int_\Omega|u_n|^pdx\right)^{\frac{p-1}{p}}\left(\int_\Omega|u_n-u_m|^pdx\right)^{\frac{1}{p}}\\ &\quad+C_\varepsilon\left(\int_\Omega|u_n|^rdx\right)^{\frac{r-1}{r}}\left(\int_\Omega|u_n-u_m|^rdx\right)^{\frac{1}{r}}\\ &\le C\varepsilon\|u_n-u_m\|_p+C_\varepsilon\|u_n-u_m\|_r\to0,\end{aligned}\tag{2.12}$$

as $n\to\infty$. By a similar calculation in (2.12), one has

$$\left|\int_\Omega\frac{|u_m|^{q-2}u_m\ln|u_m|^2}{a+b\|u_m\|^p}(u_n-u_m)dx\right|\le C\varepsilon\|u_n-u_m\|_p+C_\varepsilon\|u_n-u_m\|_r\to0,\tag{2.13}$$

as $n\to\infty$. It follows from (2.12) and (2.13) that

$$\lim_{n\to\infty}\int_\Omega\left(\frac{|u_n|^{q-2}u_n\ln|u_n|^2}{a+b\|u_n\|^p}-\frac{|u_m|^{q-2}u_m\ln|u_m|^2}{a+b\|u_m\|^p}\right)(u_n-u_m)dx=0.\tag{2.14}$$

Therefore, by (2.11) and (2.14), we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}}\\ &\quad\times[(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy=0.\end{aligned}\tag{2.15}$$

Let us now recall the well-known Simon inequalities, for all $\xi,\zeta\in\mathbb{R}$ such that

$$|\xi-\zeta|^p\le\begin{cases}c_p(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta), & \text{for } p\ge2,\\ C_p[(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta)]^{\frac{p}{2}}(|\xi|^p+|\zeta|^p)^{\frac{2-p}{2}}, & \text{for } 1<p<2,\end{cases}\tag{2.16}$$

where $c_p, C_p > 0$ depending only on p. From which we distinguish two cases:
Case (i): if $p \geq 2$, it follows from (2.15) and (2.16) as $n \to \infty$ that

$$\begin{aligned}
&\|u_n - u_m\|^p \\
&\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_n - u_m)(x) - (u_n - u_m)(y)|^p}{|x-y|^{N+ps}} dxdy \\
&\quad \leq c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u_m(x) - u_m(y)|^{p-2}(u_m(x) - u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n - u_m)(x) - (u_n - u_m)(y)] dxdy \to 0.
\end{aligned}$$

Case (ii): if $1 < p < 2$, since $\|u_n\|^p$ and $\|u_m\|^p$ are bounded in X_0, by the subadditivity inequality, for all $\xi, \zeta \geq 0$, we have

$$(\xi + \zeta)^{\frac{2-p}{2}} \leq \xi^{\frac{2-p}{2}} + \zeta^{\frac{2-p}{2}}.$$

Letting $\xi = u_n(x) - u_n(y)$ and $\zeta = u_m(x) - u_m(y)$ in (2.16) as $n \to \infty$, we obtain

$$\begin{aligned}
&\|u_n - u_m\|^p \\
&\quad \leq C_p \Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u_m(x) - u_m(y)|^{p-2}(u_m(x) - u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n - u_m)(x) - (u_n - u_m)(y)] dxdy \Bigg]^{\frac{p}{2}} (\|u_n\|^p + \|u_m\|^p)^{\frac{2-p}{2}} \\
&\quad \leq C_p \Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u_m(x) - u_m(y)|^{p-2}(u_m(x) - u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n - u_m)(x) - (u_n - u_m)(y)] dxdy \Bigg]^{\frac{p}{2}} (\|u_n\|^{\frac{p(2-p)}{2}} + \|u_m\|^{\frac{p(2-p)}{2}}) \\
&\quad \leq C \Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u_m(x) - u_m(y)|^{p-2}(u_m(x) - u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n - u_m)(x) - (u_n - u_m)(y)] dxdy \Bigg]^{\frac{p}{2}} \to 0,
\end{aligned}$$

where the constant $C > 0$. Thus, we can deduce that $u_n \to u$ in X_0. The proof is complete. □

Lemma 2.6. *If $|DI_\lambda|(u) = 0$, then u is a weak solution of (1.1). That is, $u^{-\gamma}\varphi \in L^1(\Omega)$ for all $\varphi \in X_0$ such that*

$$\begin{aligned}
&(a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} dxdy \\
&\quad = \int_\Omega |u|^{q-2}u\varphi \ln|u|^2 dx + \lambda \int_\Omega u^{-\gamma}\varphi dx.
\end{aligned} \tag{2.17}$$

Proof. Since $|DI_\lambda|(u) = 0$, by Lemma 2.3, for all $v \in P$, we have

$$\begin{aligned}
&(a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[(v-u)(x) - (v-u)(y)]}{|x-y|^{N+ps}} dxdy \\
&\quad - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx \geq \lambda \int_\Omega \frac{(v-u)}{u^\gamma} dx.
\end{aligned} \tag{2.18}$$

Letting $t \in \mathbb{R}$, $\varphi \in X_0$, and taking $v = (u + t\varphi)^+ \in P$ in (2.18), for any $\varphi \in X_0$, we get

$$\begin{aligned}
0 \le\ & (a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \\
& \times [((u + t\varphi)^+ - u)(x) - ((u + t\varphi)^+ - u)(y)]dxdy \\
& - \int_\Omega |u|^{q-2}u((u + t\varphi)^+ - u)\ln|u|^2 dx - \lambda \int_\Omega \frac{((u + t\varphi)^+ - u)}{u^\gamma} dx \\
\le\ & t\Big[(a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& - \int_\Omega |u|^{q-2}u\varphi \ln|u|^2 dx - \lambda \int_\Omega \frac{\varphi}{u^\gamma} dx\Big] \\
& - (a + b\|u\|^p) \int_{u+t\varphi<0}\int_{u+t\varphi<0} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \\
& \times [(u + t\varphi)(x) - (u + t\varphi)(y)]dxdy \\
& + \int_{u+t\varphi<0} |u|^{q-2}u(u + t\varphi)\ln|u|^2 dx + \lambda \int_{u+t\varphi<0} \frac{u + t\varphi}{u^\gamma} dx \\
\le\ & t\Big[(a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& - \int_\Omega |u|^{q-2}u\varphi \ln|u|^2 dx - \lambda \int_\Omega \frac{\varphi}{u^\gamma} dx\Big] \\
& - t(a + b\|u\|^p) \int_{u+t\varphi<0}\int_{u+t\varphi<0} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& + \int_{u+t\varphi<0} |u|^{q-2}u(u + t\varphi)\ln|u|^2 dx.
\end{aligned}$$

Since $u(x) = 0$ for a.e. $x \in \Omega$ and

$$\text{meas}\{x \in \Omega | u(x) + t\varphi(x) < 0, u(x) > 0\} \to 0, \quad \text{as } t \to 0,$$

we have

$$\begin{aligned}
& (a + b\|u\|^p) \int_{u+t\varphi<0}\int_{u+t\varphi<0} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& = (a + b\|u\|^p) \int_{u+t\varphi<0,u>0}\int_{u+t\varphi<0,u>0} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& \to 0,
\end{aligned}$$

and

$$\int_{u+t\varphi<0} |u|^{q-2}u(u + t\varphi)\ln|u|^2 dx = \int_{u+t\varphi<0,u>0} |u|^{q-2}u(u + t\varphi)\ln|u|^2 dx \to 0,$$

as $t \to 0$. Therefore, we have that

$$\begin{aligned}
0 \le\ & t\Big[(a + b\|u\|^p) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[\varphi(x) - \varphi(y)]}{|x - y|^{N+ps}} dxdy \\
& - \int_\Omega |u|^{q-2}u\varphi \ln|u|^2 dx - \lambda \int_\Omega \frac{\varphi}{u^\gamma} dx\Big] + o(t).
\end{aligned}$$

Consequently, one has

$$(a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy$$
$$-\int_\Omega |u|^{q-2}u\varphi\ln|u|^2dx-\lambda\int_\Omega\frac{\varphi}{u^\gamma}dx\geq 0.$$

By the arbitrariness of the sign of φ, we can obtain that (2.18) holds. The proof is complete. □

3 Proof of Theorem 1.1

Theorem 3.1. *Suppose that* $0<\lambda<\Lambda_0$ *(*Λ_0 *is as in Lemma 2.4), then equation (1.1) has a positive solution* u_* *satisfying* $I_\lambda(u_*)<0$.

Proof. According to Lemma 2.4 and the definition of d in (2.5), there exists a minimizing sequence $\{u_n\}\subset B_\rho\subset P$ such that $\lim_{n\to\infty}I_\lambda(u_n)=d<0$. Obviously, $\{u_n\}$ is bounded in B_ρ, up to a subsequence, still denoted by $\{u_n\}$, there exists $u_*\in X_0$ such that

$$\begin{cases} u_n\rightharpoonup u_*, & \text{weakly in } X_0,\\ u_n\to u_*, & \text{strongly in } L^r(\Omega),\ 1\leq r<p_s^*,\\ u_n(x)\to u_*(x), & \text{a.e. in } \Omega,\end{cases}$$

as $n\to\infty$. Next, we prove that $u_n\to u_*$ as $n\to\infty$ in X_0. Let $w_n=u_n-u_*$, by the Brézis–Lieb lemma, there holds

$$\|u_n\|^p=\|w_n\|^p+\|u_*\|^p+o(1).$$

Therefore, by Lemma 2.5, we have

$$\begin{aligned} d&=\lim_{n\to\infty}I_\lambda(u_n)\\ &=I_\lambda(u_*)+\lim_{n\to\infty}\left[\frac{a}{p}\|w_n\|^p+\frac{b}{2p}(\|w_n\|^{2p}+2\|w_n\|^p\|u_*\|^p)\right]\\ &\geq I_\lambda(u_*)\geq d,\end{aligned}$$

which implies that $\|w_n\|\to 0$ as $n\to\infty$. Since B_ρ is closed and convex, we have $u_*\in B_\rho$. Thus, we can deduce that $I_\lambda(u_*)=d<0$, which implies that u_* is a local minimizer of I_λ and $u\not\equiv 0$ in Ω. For $v\in P$ and $t>0$ small enough such that $u_*+t(v-u_*)\in B_\rho$, similar to the proof of Lemma 2.6, we get

$$(a+b\|u_*\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_*(x)-u_*(y)|^{p-2}(u_*(x)-u_*(y))[(v-u_*)(x)-(v-u_*)(y)]}{|x-y|^{N+ps}}dxdy$$
$$-\int_\Omega|u_*|^{q-2}u_*(v-u_*)\ln|u_*|^2dx$$
$$\geq\lambda\int_\Omega\frac{(v-u_*)}{u_*^\gamma}dx.$$

Therefore, u_* is a critical point of I_λ, by Lemma 2.6, we obtain that $u_*\in P$ is a solution of (1.1) with $I_\lambda(u_*)=d<0$, which implies that $u_*\geq 0$ and $u_*\not\equiv 0$. We claim that

$$g(t)=2\ln t+\frac{\lambda}{t^{q-1+\gamma}}.$$

Notice that

$$\lim_{t\to 0^+} g(t) = +\infty, \quad \text{and} \quad \lim_{t\to +\infty} g(t) = +\infty.$$

Therefore, g achieves its minimum at

$$t_* = \left[\frac{\lambda(q-1+\gamma)}{2}\right]^{\frac{1}{q-1+\gamma}},$$

which implies that

$$\min_{t>0} g(t) = g(t_*) = \frac{2}{q-1+\gamma}\ln\frac{\lambda(q-1+\gamma)}{2} + \frac{2}{q-1+\gamma} \triangleq C.$$

Consequently, we obtain that

$$(-\Delta)_p^s u_* = \frac{1}{a+b\|u_*\|^p}\left(u_*^{q-1}\ln u_*^2 + \frac{\lambda}{u_*^\gamma}\right) \geq \frac{Cu_*^{q-1}}{a+b\|u_*\|^p} \geq 0,$$

where $a > 0, b \geq 0$. By using the strong maximum principle in [5,18], we deduce that $u_* \in P$ is a positive solution of (1.1). The proof is complete. □

Theorem 3.2. *Suppose that $0 < \lambda < \Lambda_0$, then equation (1.1) has a positive solution v_* such that $I_\lambda(v_*) > 0$.*

Proof. Applying the mountain pass lemma in [2] and Lemma 2.4, there exists a sequence $\{u_n\} \subset X_0$ such that

$$I_\lambda(u_n) \to c, \quad \text{and} \quad |DI_\lambda|(u_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], X_0) : \gamma(0) = 0, \gamma(1) = e\}.$$

According to Lemma 2.5, we know that $\{u_n\} \subset X_0$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to v_*$ in X_0 as $n \to \infty$, we have

$$I_\lambda(v_*) = \lim_{n\to\infty} I_\lambda(u_n) \geq \alpha > 0,$$

which implies that $v_* \not\equiv 0$. It is similar to Theorem 3.1 that $v_* > 0$, we obtain that v_* is a positive solution of equation (1.1) such that $I_\lambda(v_*) > 0$. Combining the above facts with Theorem 3.1 the proof of Theorem 1.1 is complete. □

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