Electronic Journal of Qualitative Theory of Differential Equations
2023, No. **54**, 1–12; https://doi.org/10.14232/ejqtde.2023.1.54 www.math.u-szeged.hu/ejqtde/

Structural stability for scalar reaction-diffusion equations

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Received 22 May 2023, appeared 21 December 2023
Communicated by Hans-Otto Walther

Abstract. In this paper, we prove the structural stability for a family of scalar reaction-diffusion equations. Our arguments consist of using invariant manifold theorem to reduce the problem to a finite dimension and then, we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors and estimate the Gromov–Hausdorff distance of the attractors using continuous ε-isometries.

Keywords: Morse–Smale semiflows, rate of convergence of attractors, structural stability, invariant manifolds, Gromov–Hausdorff distance.

2020 Mathematics Subject Classification: 37D15, 34D30, 35B41, 35B42.

1 Introduction and statement of the results

The continuity of attractors is an important feature to study the stability of the semilinear evolution equations. For a family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ the continuity at $\varepsilon = 0$ means that the symmetric Hausdorff distance $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0$ as $\varepsilon \to 0$. The work [8] obtained positive results in the class of gradient systems, assuming structural conditions on the unperturbed attractor, together with information on the continuity of unstable manifolds of equilibria. In particular, if $\{u_*^\varepsilon\}_{\varepsilon\in[0,1]}$ is the family of equilibrium points then $d(u_*^\varepsilon, u_*^0) \to 0$ as $\varepsilon \to 0$ for the phase space metric d.

There is a natural question, as follows.

Question 1. Is the order in which $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ goes to zero the same as $d(u_*^\varepsilon, u_*^0)$?

There are many works concerning the rate of convergence of attractors to different situations, as we can see in [1, 3, 6] and [7]. The case of reaction-diffusion equation in a smooth domain, [1] has been shown that

$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon^\beta, \quad 0 < \beta < 1. \tag{1.1}$$

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In [3], the authors have analyzed the reaction-diffusion equation in a thin domain under perturbations, where they have obtained

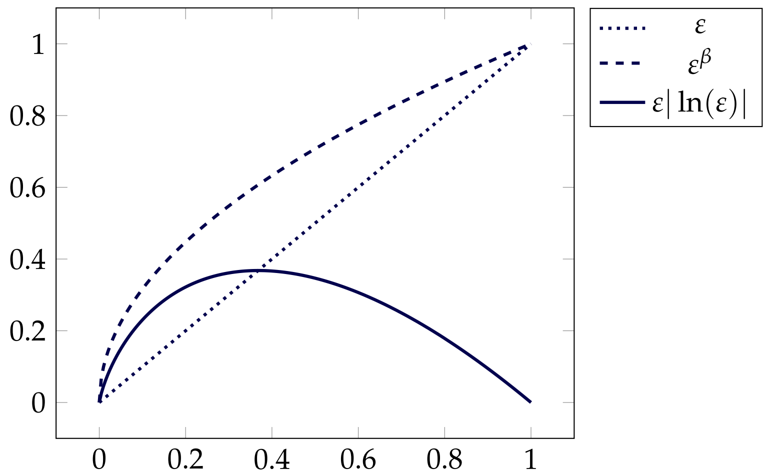
$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon|\ln(\varepsilon)|. \tag{1.2}$$

Notice that booth above problems does not provide an answer to Question 1 because the rate of convergence of attractors is worse than equilibria.

The work [6] was able to answer Question 1 considering the reaction-diffusion equation where the diffusion coefficient becomes large in all domains when $\varepsilon \to 0$. The optimal rate state

$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon. \tag{1.3}$$

The figure below shows (1.2) is better than (1.1) and (1.3) improves (1.2) as the parameter ε goes to zero.



The main argument to obtain (1.2) and (1.3) is the existence of a finite-dimensional invariant manifold that allows us to reduce the problem to finite dimension and, then we can use properties of Morse–Smale dynamical systems in finite-dimensional closed manifolds. For instance, [3] have used that in a neighborhood of the attractor, a Morse–Smale flow has the Lipschitz Shadowing property to estimate $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ by the continuity of the solution $T_\varepsilon(\cdot) \to T_0(\cdot)$ in a neighborhood of the $\cup_\varepsilon \mathcal{A}_\varepsilon$.

The purpose of this paper is to prove that the rate of convergence of the attractors for the scalar reaction-diffusion equations is optimal. Inspired by the optimal rate obtained in [6] and using the framework proposed by [3] we can reduce the problem to Morse–Smale flows in finite dimension and we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors. We observe that our arguments can be carried over to the problem addressed in [3] under appropriate adaptations. Another consequence of the structural stability is the estimate of the Gromov–Hausdorff distance of the attractors $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$. This subject has been introduced by reaction-diffusion equation under perturbation of the domain in the paper [10]. Since structural stability means that there is a topological equivalence $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ close to identity conjugating the flows, we have κ_ε a continuous ε-isometry between the attractors. This is enough requirement that we need to estimate $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$.

Consider the following family of scalar reaction-diffusion equations

$$\begin{cases} u_t^\varepsilon - (a_\varepsilon(x)u_x^\varepsilon)_x = f(u^\varepsilon), & (t,x) \in (0,\infty) \times (0,\pi) \\ u^\varepsilon(t,0) = 0 = u^\varepsilon(t,\pi), & t \in (0,\infty), \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), & x \in (0,\pi), \end{cases} \tag{1.4}$$

where $\varepsilon \in [0, \varepsilon_0]$ is a parameter, $0 < \varepsilon_0 < 1$, the diffusion coefficients $a_\varepsilon \in C^1([0,\pi],[m_0,M_0])$, $m_0, M_0 > 0$, are continuous functions satisfying

$$\|a_\varepsilon - a\|_\infty := \|a_\varepsilon - a\|_{L^\infty(0,\pi)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that,

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{1.6}$$

It follows from [5, Theorem 14.2] that for each $\varepsilon \in [0, \varepsilon_0]$, the solutions of (1.4) defines a nonlinear gradient semigroup $T_\varepsilon(\cdot)$ having a global attractor $\mathcal{A}_\varepsilon$ such that

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{w\in\mathcal{A}_\varepsilon} \|w\|_{H_0^1(0,\pi)} < \infty \quad \text{and} \quad \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{w\in\mathcal{A}_\varepsilon} \|w\|_{L^\infty(0,\pi)} < \infty. \tag{1.7}$$

Moreover, we assume that the equilibrium points of (1.4) with $\varepsilon = 0$ is hyperbolic. Hence, there are finitely many equilibrium points and we denote them by $\mathcal{E}_0 = \{u_*^{1,0}, \ldots, u_*^{p,0}\}$.

Under the above assumption, we have from [5, Chapter 14] that, for ε_0 sufficiently small, the semigroup $T_\varepsilon(\cdot)$ has exactly p equilibria that we denote $\mathcal{E}_\varepsilon = \{u_*^{1,\varepsilon}, \ldots, u_*^{p,\varepsilon}\}$ and the global attractors are given by $\mathcal{A}_\varepsilon = \cup_{i=1}^p W^u(u_*^{i,\varepsilon})$ and $\mathcal{A}_0 = \cup_{i=1}^p W^u(u_*^{i,0})$, where W^u denotes the unstable manifold. The main results of [5, Chapter 14] and [1] state that the convergence of equilibria can be estimate by

$$\|u_*^{i,\varepsilon} - u_*^{i,0}\|_{H_0^1(0,\pi)} \leq C\|a_\varepsilon - a_0\|_\infty \tag{1.8}$$

and the continuity of the global attractors can be estimated by

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C\|a_\varepsilon - a_0\|_\infty^\beta, \tag{1.9}$$

where $C > 0$ and $0 < \beta < 1$ are constants independent of ε and d_H denotes the Hausdorff distance in $H_0^1(0,\pi)$, that is,

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) = \max\left\{ \sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \inf_{u^0\in\mathcal{A}_0} \|u^\varepsilon - u^0\|_{H_0^1(0,\pi)}, \sup_{u^0\in\mathcal{A}_0} \inf_{u^\varepsilon\in\mathcal{A}_\varepsilon} \|u^\varepsilon - u^0\|_{H_0^1(0,\pi)} \right\}. \tag{1.10}$$

Finally, we assume that $T_\varepsilon(\cdot)|_{\mathcal{A}_\varepsilon}$ is a group. It is well-known that under standard conditions the solutions of (1.4) are backward uniquely defined inside the attractor.

The main result of this paper states as follows.

Theorem 1.1. *The equation (1.4) is structurally stable at $\varepsilon = 0$. That is, given $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0, \varepsilon_\eta]$, there is a homeomorphism $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ such that*

$$\sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \|\kappa_\varepsilon(u^\varepsilon) - u^\varepsilon\|_{H_0^1(0,\pi)} < C(\|a_\varepsilon - a_0\|_\infty + \eta) \quad \textit{and} \quad \kappa_\varepsilon(T_\varepsilon(\tau_\varepsilon(t,u^\varepsilon))u^\varepsilon) = T_0(t)\kappa_\varepsilon(u^\varepsilon),$$

where $t \in \mathbb{R}$, $u^\varepsilon \in \mathcal{A}_\varepsilon$, $C > 0$ is a constant independent of ε and $\tau_\varepsilon : \mathbb{R} \times \mathcal{A}_\varepsilon \to \mathbb{R}$ is a function such that, $\tau_\varepsilon(0, u^\varepsilon) = 0$ and $\tau_\varepsilon(\cdot, u^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

As an immediate consequence of the Theorem 1.1 and (1.10) we have the following result.

Corollary 1.2. *For $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0, \varepsilon_\eta]$, the Hausdorff distance between the attractors can be estimated by*

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C(\|a_\varepsilon - a_0\|_\infty + \eta), \tag{1.11}$$

where C is a constant independent of ε.

We say that a map $i_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ is a ε-isometry between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ if

$$\left| \|i_\varepsilon(u^\varepsilon) - i_\varepsilon(v^\varepsilon)\|_{H_0^1(0,\pi)} - \|u^\varepsilon - v^\varepsilon\|_{H_0^1(0,\pi)} \right| \leq \varepsilon, \qquad u^\varepsilon, v^\varepsilon \in \mathcal{A}_\varepsilon \tag{1.12}$$

and $B(i_\varepsilon(\mathcal{A}_\varepsilon), \varepsilon) = \mathcal{A}_0$, where $B(i_\varepsilon(\mathcal{A}_\varepsilon), \varepsilon) = \{u^0 \in \mathcal{A}_0 : \|i_\varepsilon(u^\varepsilon) - u^0\|_{H_0^1(0,\pi)} \leq \varepsilon$, for some $u^\varepsilon \in \mathcal{A}_\varepsilon\}$. Analogously we can define a ε-isometry between $\mathcal{A}_0$ and $\mathcal{A}_\varepsilon$. The Gromov–Hausdorff distance $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ is defined as the infimum of $\varepsilon > 0$ for which there are ε-isometries $i_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ and $l_\varepsilon : \mathcal{A}_0 \to \mathcal{A}_\varepsilon$.

We have the following result as an immediate consequence of the Theorem 1.1.

Corollary 1.3. *For $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0, \varepsilon_\eta]$, the Gromov–Hausdorff distance between the attractors can be estimated by*

$$d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C(\|a_\varepsilon - a_0\|_\infty + \eta), \tag{1.13}$$

where C is a constant independent of ε.

This paper is organized as follows. In Section 2 we introduce the functional setting to deal with (1.4). In Section 3 we use invariant manifolds to reduce the problem to finite dimension. In Section 4 we prove the Theorem 1.1.

2 Functional setting and technical results

Let $\varepsilon \in [0, \varepsilon_0]$. We define the operator $A_\varepsilon : D(A_\varepsilon) \subset L^2(0,\pi) \to L^2(0,\pi)$ by

$$\begin{cases} D(A_\varepsilon) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ A_\varepsilon u = -(a_\varepsilon(x) u_x)_x, \quad u \in D(A_\varepsilon). \end{cases} \tag{2.1}$$

It is well-known that A_ε is a self-adjoint operator with compact resolvent. Hence, we can define the fractional power spaces X_ε^α, $0 < \alpha \leq 1$, where $X_\varepsilon^0 = L^2(0,\pi)$, $X_\varepsilon^1 = D(A_\varepsilon)$ and $X_\varepsilon^{\frac{1}{2}} = H_0^1(0,\pi)$ with the inner product

$$\langle u, v \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_0^\pi a_\varepsilon(x) u_x v_x \, dx \tag{2.2}$$

which produces norms uniformly equivalent to the standard $H_0^1(0,\pi)$ norm, since a_ε is uniformly bounded in ε. Therefore, estimates on $X_\varepsilon^{\frac{1}{2}}$ are transported to $H_0^1(0,\pi)$ uniformly in ε.

Since there are many estimates in the paper, we will let C be a generic constant which is independent of ε, but which may depend on $m_0, M_0, u_0^0, \mathcal{E}_0$.

We summarize in the next theorem some useful estimates that can be proved as in [1] and [5, Chapter 14].

Theorem 2.1. *Let $\varepsilon \in [0, \varepsilon_0]$. The operators A_ε satisfy the following properties.*

(i) $\sup_{\varepsilon\in[0,\varepsilon_0]} \|A_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,\pi),H_0^1(0,\pi))} \leq C.$

(ii) $\|A_\varepsilon^{-1}u^\varepsilon - A_0^{-1}u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \quad u^\varepsilon, u^0 \in L^2(0,\pi).$

(iii) $\|(\mu + A_\varepsilon)^{-1}u^\varepsilon - (\mu + A_0)^{-1}u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty)$*, for μ in the resolvent set of A_ε and A_0 and $u^\varepsilon, u^0 \in L^2(0,\pi)$.*

Here, $C > 0$ is a constant independent of ε.

Proof. The proof has been done in [1, Section 3] and [5, Chapter 14]. Since there is a difference between these works due to the presence of an exponent $\frac{1}{2}$, we outline the proof of item (ii) here.

Let $u^\varepsilon, u^0 \in L^2(0,\pi)$ and let v^ε, v^0 be the respective solution of $A_\varepsilon v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$. Then,

$$\int_0^\pi a_\varepsilon v_x^\varepsilon \varphi_x \, dx = \int_0^\pi u^\varepsilon \varphi \, dx, \quad \text{and} \quad \int_0^\pi a_0 v_x^0 \varphi_x \, dx = \int_0^\pi u^0 \varphi \, dx, \quad \varphi \in H_0^1(0,\pi). \tag{2.3}$$

Taking $\varphi = v^\varepsilon - v^0$, we obtain

$$\int_0^\pi a_\varepsilon v_x^\varepsilon (v_x^\varepsilon - v_x^0) \, dx - \int_0^\pi a_0 v_x^0 (v_x^\varepsilon - v_x^0) \, dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) \, dx.$$

which implies

$$\int_0^\pi a_\varepsilon (v_x^\varepsilon - v_x^0)^2 \, dx + \int_0^\pi (a_\varepsilon - a_0) v_x^0 (v_x^\varepsilon - v_x^0) \, dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) \, dx.$$

By (2.2) and the uniformity between the $X_\varepsilon^{\frac{1}{2}}$ norm and $H_0^1(0,\pi)$ norm, we get

$$\|v^\varepsilon - v^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty),$$

for some positive constant C independent of ε.

Finishing we notice that $A_\varepsilon v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$ implies $v^\varepsilon = A_\varepsilon^{-1} u^\varepsilon$ and $v^0 = A_0^{-1} u^0$. □

We write (1.4) as an evolution equation in $L^2(0,\pi)$ in the following way

$$\begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon = f(u^\varepsilon), \\ u^\varepsilon(0) = u_0^\varepsilon, \end{cases} \tag{2.4}$$

where we have used the same notation f for the nonlinearity of (1.4) and its functional f_I : $H_0^1(0,\pi) \to L^2(0,\pi)$ given by $f_I(u)(x) = f(u(x))$.

We denote the spectra of the divergence operator $-A_\varepsilon$, $\varepsilon \in [0,\varepsilon_0]$, ordered and counting multiplicity by

$$\cdots < -\lambda_m^\varepsilon < -\lambda_{m-1}^\varepsilon < \cdots < -\lambda_1^\varepsilon$$

and we let $\{\varphi_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions.

The resolvent convergence $\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L^2(0,\pi),H^1(0,\pi))} \to 0$ as $\varepsilon \to 0$ imply the convergence of eigenvalues, that is, $\lambda_m^\varepsilon \to \lambda_m^0$ as $\varepsilon \to 0$, $m = 1, 2, \ldots$ as we can see in [1, Proposition 3.3]. Moreover, by [1, Corollary 3.6], we obtain a constant $C > 0$ independent of ε such that,

$$|\lambda_m^\varepsilon - \lambda_m^0| \leq C\|a_\varepsilon - a\|_\infty, \qquad m = 1, 2, \ldots \tag{2.5}$$

We take a closed curve Γ_m contained in the resolvent set of $-A_0$ around $\{-\lambda_1^0, \dots, -\lambda_m^0\}$. By (2.5) we can take ε sufficiently small for that Γ_m be contained in the resolvent set of $-A_\varepsilon$ around $\{-\lambda_1^\varepsilon, \dots, -\lambda_m^\varepsilon\}$. Thus, we can define

$$P_\varepsilon^m = \frac{1}{2\pi i} \int_{\Gamma_m} (\mu + A_\varepsilon)^{-1}\, d\mu, \qquad \varepsilon \in [0, \varepsilon_0], \tag{2.6}$$

which is the spectral projection onto the space generated by the first m eigenfunctions of A_ε. It follows from (2.6) and Theorem 2.1 that there is a constant $C > 0$ independent of ε such that,

$$\|P_\varepsilon^m u^\varepsilon - P_0^m u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \qquad u^\varepsilon, u^0 \in H_0^1(0,\pi) \tag{2.7}$$

and

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{u^\varepsilon \in L^2(0,\pi)} \|P_\varepsilon^m u^\varepsilon\|_{H_0^1(0,\pi)} \leq C.$$

In the next section, we will fix m sufficiently large to obtain conditions for the invariant manifold theorem. Thus, to avoid heavy notation, we omit the dependency of m on P_ε^m and we denote $Q_\varepsilon = (I - P_\varepsilon)$ the projection over its orthogonal complement.

3 Invariant manifold and reduction of the dimension

The resolvent convergence $\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L^2(0,\pi),H^1(0,\pi))} \to 0$ as $\varepsilon \to 0$ guarantees the spectral convergence of the eigenvalues $\lambda_m^\varepsilon \to \lambda_m^0$ as $\varepsilon \to 0$, $m = 1, 2, \dots$. But, the operator A_0 has a gap on its eigenvalues, that is, $\lambda_{m+1}^0 - \lambda_m^0 \to \infty$ as $m \to \infty$. Thus, for ε_0 sufficiently small, we have a similar gap on the eigenvalues of A_ε. This fact, enables us to construct inertial manifolds of the same dimension given by $\mathrm{rank}(P_\varepsilon) = \mathrm{span}[\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon]$, where according with the previous section, φ_i^ε is the associated eigenfunction to the eigenvalue λ_i^ε, $m = 1, 2, \dots$.

For each $\varepsilon \in [0, \varepsilon_0]$, we decompose $H_0^1(0,\pi) = Y_\varepsilon \oplus Z_\varepsilon$, where $Y_\varepsilon = P_\varepsilon(H_0^1(0,\pi))$ and $Z_\varepsilon = Q_\varepsilon(H_0^1(0,\pi))$ and we define $A_\varepsilon^+ = A_\varepsilon|_{Y_\varepsilon}$ and $A_\varepsilon^- = A_\varepsilon|_{Z_\varepsilon}$. Using this decomposition we rewrite (2.4) as the following coupled equation

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = P_\varepsilon f(v^\varepsilon + z^\varepsilon) := H_\varepsilon(v^\varepsilon, z^\varepsilon), \\ z_t^\varepsilon + A_\varepsilon^- z^\varepsilon = Q_\varepsilon f(v^\varepsilon + z^\varepsilon) := G_\varepsilon(v^\varepsilon, z^\varepsilon). \end{cases} \tag{3.1}$$

The invariant manifold theorem whose proof can be found in [5, Chapter 8], states as follows.

Theorem 3.1. *For sufficiently large m and $\varepsilon_0 > 0$ small, there is an invariant manifold $\mathcal{M}_\varepsilon$ for (2.4) given by*

$$\mathcal{M}_\varepsilon = \{u^\varepsilon \in H_0^1(0,\pi)\,;\, u^\varepsilon = P_\varepsilon u^\varepsilon + s_*^\varepsilon(P_\varepsilon u^\varepsilon)\}, \qquad \varepsilon \in [0, \varepsilon_0],$$

where $s_*^\varepsilon : Y_\varepsilon \to Z_\varepsilon$ is a Lipschitz continuous map satisfying

$$\|s_*^\varepsilon(\tilde{v}^\varepsilon) - s_*^0(\tilde{v}^0)\|_{H_0^1(0,\pi)} \leq C(\|\tilde{v}^\varepsilon - \tilde{v}^0\|_{H_0^1(0,\pi)} + \|a_\varepsilon - a_0\|_\infty |\log(\|a_\varepsilon - a_0\|_\infty)|), \tag{3.2}$$

where $\tilde{v}^\varepsilon \in Y_\varepsilon$, $\tilde{v}^0 \in Y_0$ and C is a positive constant independent of ε. The invariant manifold $\mathcal{M}_\varepsilon$ is exponentially attracting and the global attractor $\mathcal{A}_\varepsilon$ of the problem (2.4) lies in $\mathcal{M}_\varepsilon$. The flow of $u_0^\varepsilon \in \mathcal{M}_\varepsilon$ is given by

$$T_\varepsilon(t) u_0^\varepsilon = v^\varepsilon(t) + s_*^\varepsilon(v^\varepsilon(t)), \qquad t \in \mathbb{R}, \tag{3.3}$$

where $v^\varepsilon(t)$ satisfies

$$\begin{cases} v^\varepsilon_t + A^+_\varepsilon v^\varepsilon = H_\varepsilon(v^\varepsilon, s^\varepsilon_*(v^\varepsilon)), \quad t \in \mathbb{R}, \\ v^\varepsilon(0) = P_\varepsilon u^\varepsilon_0 \in Y_\varepsilon. \end{cases} \tag{3.4}$$

For the proof of Theorem 3.1 we refer [5]. To see how obtain the estimate (3.2), we refer [1,4].

Now, we use the theory developed in [4] to identify (3.4) as an ordinary differential equation in $\mathbb{R}^m$. This identification is made by an isomorphism between Y_ε and $\mathbb{R}^m$. Since our aim in the next section will be to construct a ε-isometry between the attractors, it is convenient to make the isomorphism $Y_\varepsilon \approx \mathbb{R}^m$ an isometry. To accomplish this we follow the ideas of [4] that modify the basis of Y_ε.

Let $\varepsilon \in [0, \varepsilon_0]$. We consider in Y_ε the following set $\{P_\varepsilon \varphi^0_1, \dots, P_\varepsilon \varphi^0_m\}$. It has been proved in [4] that this set is a basis for Y_ε. We define $L_\varepsilon : Y_\varepsilon \to \mathbb{R}^m$ by $L_\varepsilon(\sum_{i=1}^m \alpha_i P_\varepsilon \varphi^0_i) = \sum_{i=1}^m \alpha_i e_i$, where $\{e_i\}_{i=1}^m$ is the canonical basis of $\mathbb{R}^m$. This choices make L_ε a isometry between Y_ε and $\mathbb{R}^m$ and if we denote $\mathbb{R}^m_\varepsilon$ the $\mathbb{R}^m$ with the norm $\|x\|_{\mathbb{R}^m_\varepsilon} = (\sum_{i=1}^m x_i^2 \lambda_i^\varepsilon)^{\frac{1}{2}}$, then $\|\tilde{u}^0\|_{H^1_0(0,\pi)} = \|L_0 \tilde{u}^0\|_{\mathbb{R}^m_0}$.

Proposition 3.2. *The following statements hold true:*

(i) If $\tilde{u}^\varepsilon \in Y_\varepsilon$ and $\tilde{u}_0 \in Y_0$ are such that $\|\tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} < \bar{C}$ and $\|\tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} < \bar{C}$, where $\bar{C}$ is a constant independent of ε. Then $\|L_\varepsilon \tilde{u}^\varepsilon - L_0 \tilde{u}^0\|_{\mathbb{R}^m} \le C(\|\tilde{u}^\varepsilon - \tilde{u}_0\|_{H^1_0(0,\pi)} + \|a_\varepsilon - a_0\|_\infty)$, for a constant $C > 0$ independent of ε.

(ii) If $\bar{u}^\varepsilon, \bar{u}^0 \in \mathbb{R}^m$ are such that $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$ and $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$, where $\bar{C}$ is a constant independent of ε. Then $\|L_\varepsilon^{-1} \bar{u}^\varepsilon - L_0^{-1} \bar{u}^0\|_{H^1_0(0,\pi)} \le C(\|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m} + \|a_\varepsilon - a_0\|_\infty)$, for a constant $C > 0$ independent of ε.

Proof. The proof of item (i) follows as Lemma 5.4 of [4]. We prove (ii) using similar arguments.

Let $\bar{u}^\varepsilon = (\alpha^\varepsilon_1, \dots, \alpha^\varepsilon_m)$ and $\bar{u}^0 = (\alpha^0_1, \dots, \alpha^0_m)$ in $\mathbb{R}^m$. Then,

$$\begin{aligned} L_\varepsilon^{-1} \bar{u}^\varepsilon - L_0^{-1} \bar{u}^0 &= \sum_{i=1}^m \alpha^\varepsilon_i P_\varepsilon \varphi^0_i - \sum_{i=1}^m \alpha^0_i P_0 \varphi^0_i \\ &= (P_\varepsilon - P_0) \sum_{i=1}^m \alpha^\varepsilon_i \varphi^0_i + \sum_{i=1}^m (\alpha^\varepsilon_i - \alpha^0_i) P_0 \varphi^0_i \end{aligned}$$

which implies,

$$\|L_\varepsilon^{-1} \bar{u}^\varepsilon - L_0^{-1} \bar{u}^0\|_{H^1_0(0,\pi)} \le C\|a_\varepsilon - a_0\|_\infty + \|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m_0}$$

□

The map $s^\varepsilon_* : Y_\varepsilon \to Z_\varepsilon$ is obtained as the fixed point of the contraction $\Phi_\varepsilon : \Sigma_\varepsilon \to \Sigma_\varepsilon$ given by

$$\begin{cases} \Sigma_\varepsilon = \Big\{ s^\varepsilon : Y_\varepsilon \to Z_\varepsilon \,;\, \|s^\varepsilon\|_\infty \le D \text{ and } \|s^\varepsilon(v) - s^\varepsilon(\tilde{v})\|_{H^1_0(0,\pi)} \le \Delta \|v - \tilde{v}\|_{H^1_0(0,\pi)} \Big\}, \\ \Phi_\varepsilon(s^\varepsilon)(\eta) = \displaystyle\int_{-\infty}^0 e^{-A^-_\varepsilon r} G_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r)))\, dr, \end{cases}$$

where D and Δ are positive constants independent of ε and $v^\varepsilon(r) \in Y_\varepsilon$ is the global solution of (3.4) with $\eta = P_\varepsilon u^\varepsilon_0$. With the aid of L_ε, we can define new invariant manifolds $\mathcal{N}_\varepsilon$, given by

$$\mathcal{N}_\varepsilon = \{L_\varepsilon^{-1}(x) + \theta_\varepsilon(x) : x \in \mathbb{R}^m\},$$

where $\theta : \mathbb{R}^m \to Z_\varepsilon$ is given by $\theta_*^\varepsilon = s_*^\varepsilon \circ L_\varepsilon^{-1}$. Therefore, θ_*^ε is a fixed point of

$$\theta_*^\varepsilon(x) = \int_{-\infty}^0 e^{-A_\varepsilon^- r} G_\varepsilon(v^\varepsilon(r), \theta_*^\varepsilon(L_\varepsilon v^\varepsilon(r)))\, dr,$$

such that

$$\|\theta_*^\varepsilon - \theta_*^0\|_\infty \leq C\|a_\varepsilon - a_0\|_\infty |\log(\|a_\varepsilon - a_0\|_\infty)|,$$

for some constant $C > 0$ independent of ε.

By Theorem 3.1 the semigroup $T_\varepsilon(\cdot)$ restrict to $\mathcal{M}_\varepsilon$ is a flow whose behavior is dictate by solutions of (3.4) that can be transposed to $\mathbb{R}^m$ as

$$\begin{cases} x_t^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}(x^\varepsilon) = L_\varepsilon H_\varepsilon(L_\varepsilon^{-1}(x^\varepsilon), \theta_*^\varepsilon(x^\varepsilon)), \quad t \in \mathbb{R}, \\ x^\varepsilon(0) = L_\varepsilon P_\varepsilon u_0^\varepsilon := x_0^\varepsilon \in \mathbb{R}^m. \end{cases} \tag{3.5}$$

Theorem 3.3. *The solutions of (3.5) generate a Morse–Smale flow in $\mathbb{R}^m$.*

Proof. Since all equilibrium points of (1.4) are hyperbolic, the author in [9] has proved that the semigroup $T_\varepsilon(\cdot)$ is Morse–Smale. Therefore, $T_\varepsilon(\cdot)|_{\mathcal{N}_\varepsilon}$ is a Morse–Smale semigroup. Following [12, Chapter 3] we obtain that the projected semiflow $\bar{T}_\varepsilon(\cdot)$ of $T_\varepsilon(\cdot)$ in $\mathbb{R}^m$ is Morse–Smale. □

In what follows we prove several technical results that will be essential to prove the results in the next section. Here is the moment that we take a different way of [3].

Proposition 3.4. *The projection P_ε restrict to $\mathcal{M}_\varepsilon$ is an injective map and $P_\varepsilon^{-1}|_{\mathcal{M}_\varepsilon}$ restrict to the set $\tilde{\mathcal{A}}_\varepsilon := P_\varepsilon \mathcal{A}_\varepsilon$ is uniformly bounded in ε and*

$$\|P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1}q_\varepsilon(\tilde{u}^\varepsilon)\| \leq C(\|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \qquad \tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon, \tag{3.6}$$

for any homeomorphism $q_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$.

Proof. Let $u^\varepsilon, v^\varepsilon \in \mathcal{M}_\varepsilon$ such that $P_\varepsilon u^\varepsilon = P_\varepsilon v^\varepsilon$, then $u^\varepsilon = P_\varepsilon u^\varepsilon + s_*^\varepsilon(P_\varepsilon u^\varepsilon) = P_\varepsilon v^\varepsilon + s_*^\varepsilon(P_\varepsilon v^\varepsilon) = v^\varepsilon$. By (1.7), we have a positive constant C independent of ε such that,

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{\tilde{u}^\varepsilon \tilde{\mathcal{A}}_\varepsilon} \|P_\varepsilon^{-1}\tilde{u}^\varepsilon\|_{H^1(0,\pi)} \leq \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \|u^\varepsilon\|_{H^1(0,\pi)} \leq C.$$

Finally, if $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ and $q_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ is a homeomorphism, then

$$\begin{aligned} &\|P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1}q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} = \|P_0^{-1}P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1}q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_\varepsilon P_\varepsilon^{-1}\tilde{u}^\varepsilon + P_\varepsilon P_\varepsilon^{-1}\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|(P_0 - P_\varepsilon)P_\varepsilon^{-1}\tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} + \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &\quad \leq C(\|a_\varepsilon - a_0\|_\infty + \|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{L^2(0,\pi)}), \end{aligned}$$

where we have used (2.7) to obtain a positive constant C independent of ε. □

In what follows, we denote P_ε^{-1} the inverse of $P_\varepsilon|_{\mathcal{M}_\varepsilon} : \mathcal{M}_\varepsilon \to Y_\varepsilon$.

Proposition 3.5. *Let $\bar{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.5) and $\tilde{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.4). Then, it is valid the following properties* *(i)* $L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon = \tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon, \quad \bar{u}^\varepsilon \in \mathbb{R}^m, \quad t \in \mathbb{R}.$

(ii) $\bar{T}_\varepsilon(t)L_\varepsilon\tilde{u}^\varepsilon = L_\varepsilon\tilde{T}_\varepsilon(t)\tilde{u}^\varepsilon, \quad \tilde{u}^\varepsilon \in Y_\varepsilon, \quad t \in \mathbb{R}.$

(iii) $P_\varepsilon T_\varepsilon(t)u^\varepsilon = \tilde{T}_\varepsilon(t)P_\varepsilon u^\varepsilon, \quad u^\varepsilon \in H_0^1(0,\pi), \quad t \geq 0.$

(iv) $P_\varepsilon^{-1}\tilde{T}_\varepsilon(t)\tilde{u}^\varepsilon = T_\varepsilon(t)P_\varepsilon^{-1}\tilde{u}^\varepsilon, \quad \tilde{u}^\varepsilon \in Y_\varepsilon, \quad t \geq 0.$

(v) Given a function $\tau_\varepsilon : \mathbb{R} \times \mathcal{A}_\varepsilon \to \mathbb{R}$ *such that,* $\tau_\varepsilon(0,u^\varepsilon) = 0$ *and* $\tau_\varepsilon(\cdot,u^\varepsilon)$ *is a increasing function mapping* $\mathbb{R}$ *onto* $\mathbb{R}$*, there exist a function* $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ *such that,* $\tilde{\tau}_\varepsilon(0,P_\varepsilon u^\varepsilon) = 0$ *and* $\tilde{\tau}_\varepsilon(\cdot,P_\varepsilon u^\varepsilon)$ *is a increasing function mapping* $\mathbb{R}$ *onto* $\mathbb{R}$ *such that*

$$P_\varepsilon T_\varepsilon(\tau_\varepsilon(t,u^\varepsilon))u^\varepsilon = \tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t,P_\varepsilon u^\varepsilon))P_\varepsilon u^\varepsilon, \qquad u^\varepsilon \in \mathcal{A}_\varepsilon,\ t \in \mathbb{R}.$$

(vi) Given a function $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ *such that,* $\tilde{\tau}_\varepsilon(0,\tilde{u}^\varepsilon) = 0$ *and* $\tilde{\tau}_\varepsilon(\cdot,\tilde{u}^\varepsilon)$ *is a increasing function mapping* $\mathbb{R}$ *onto* $\mathbb{R}$*, there exist a function* $\bar{\tau}_\varepsilon : \mathbb{R} \times \bar{\mathcal{A}}_\varepsilon \to \mathbb{R}$ *such that,* $\bar{\tau}_\varepsilon(0,L_\varepsilon\tilde{u}^\varepsilon) = 0$ *and* $\bar{\tau}_\varepsilon(\cdot,L_\varepsilon\tilde{u}^\varepsilon)$ *is a increasing function mapping* $\mathbb{R}$ *onto* $\mathbb{R}$ *such that*

$$L_\varepsilon\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t,\tilde{u}^\varepsilon))\tilde{u}^\varepsilon = \bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t,L_\varepsilon\tilde{u}^\varepsilon))L_\varepsilon\tilde{u}^\varepsilon, \qquad \tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon,\ t \in \mathbb{R}.$$

Proof. Let $\bar{u}^\varepsilon \in \mathbb{R}^m$, then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in Y_\varepsilon$ and $\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon$ is a solution of

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = H_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)), & t \in \mathbb{R}, \\ v^\varepsilon(0) = L_\varepsilon^{-1}\bar{u}^\varepsilon \in Y_\varepsilon. \end{cases} \tag{3.7}$$

Defining $\varphi^\varepsilon(t) = L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon$, we have $\varphi^\varepsilon(0) = L_\varepsilon^{-1}\bar{T}_\varepsilon(0)\bar{u}^\varepsilon = L_\varepsilon^{-1}\bar{u}^\varepsilon$ and

$$\begin{aligned} \varphi_t^\varepsilon + A_\varepsilon^+\varphi^\varepsilon(t) &= L_\varepsilon^{-1}\frac{\partial}{\partial t}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon + A_\varepsilon^+ L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon \\ &= L_\varepsilon^{-1}(\frac{\partial}{\partial t}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon). \end{aligned}$$

Since $x_\varepsilon(t) := \bar{T}_\varepsilon(t)\bar{u}^\varepsilon$ is a solution of

$$\begin{cases} x_t^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}(x^\varepsilon) = L_\varepsilon H_\varepsilon(L_\varepsilon^{-1}(x^\varepsilon), \theta_*^\varepsilon(x^\varepsilon)), & t \in \mathbb{R}, \\ x^\varepsilon(0) = \bar{u}^\varepsilon \in \mathbb{R}^m, \end{cases} \tag{3.8}$$

we obtain

$$\varphi_t^\varepsilon + A_\varepsilon^+\varphi^\varepsilon(t) = H_\varepsilon(\varphi^\varepsilon(t), \theta_*^\varepsilon(\varphi^\varepsilon(t))).$$

The bijection between θ_*^ε and s_*^ε enables us to conclude that $\varphi_\varepsilon(t)$ is also a solution of (3.7). The result follows from the well-posedness of (3.7).

In the same way, we proof item (ii).

Item (iii) is immediate from (3.3) and (3.4) by noticing that $P_\varepsilon T_\varepsilon(t)u^\varepsilon = v^\varepsilon(t)$ and we are denoting $v^\varepsilon(t) = \tilde{T}_\varepsilon(t)P_\varepsilon u^\varepsilon$. Item (iv) follows from (iii) using that $P_\varepsilon u^\varepsilon = \tilde{u}^\varepsilon$ if only if $u^\varepsilon = P_\varepsilon^{-1}\tilde{u}^\varepsilon$, for some $\tilde{u}^\varepsilon \in Y_\varepsilon$. Item (v) follows from (iii) defining $\tilde{\tau}_\varepsilon(t,P_\varepsilon u^\varepsilon) = \tau_\varepsilon(t,u^\varepsilon)$. In the same way, we obtain (vi). □

Proposition 3.6. *The set* $\tilde{\mathcal{A}}_\varepsilon = P_\varepsilon\mathcal{A}_\varepsilon$ *is the global attractor for the semigroup* $\tilde{T}_\varepsilon(\cdot)$ *given by solutions of* (3.4).

Proof. Since $\mathcal{A}_\varepsilon$ is compact and P_ε is continuous, we have $\tilde{A}_\varepsilon = P_\varepsilon \mathcal{A}_\varepsilon$ a compact set in Y_ε. Proving the attraction, let $B \subset Y_\varepsilon$ a bounded set and let $v^\varepsilon \in B$. Then $v^\varepsilon + s_*^\varepsilon(v^\varepsilon) \in \mathcal{M}_\varepsilon$ and $T_\varepsilon(t)w^\varepsilon = \tilde{T}_\varepsilon(t)v^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon)$, for $t > 0$ and $w^\varepsilon \in P_\varepsilon^{-1}(v^\varepsilon)$. But $T_\varepsilon(t)$ is a gradient semigroup, then there is $u^\varepsilon \in \mathcal{A}_\varepsilon$ such that, $\|T_\varepsilon(t)w^\varepsilon - u^\varepsilon\|_{H_0^1(0,\pi)} \to 0$ as $t \to \infty$. In fact, the attraction property of the global attractor is uniform for the solutions starting at B. Hence, there is a neighborhood of $\mathcal{A}_\varepsilon$ containing all trajectory starting at B after a time t_B. We take $\tilde{u}^\varepsilon \in \tilde{A}_\varepsilon$ such that $\tilde{u}^\varepsilon = P_\varepsilon u^\varepsilon$. Thus,

$$\begin{aligned}\|\tilde{T}_\varepsilon(t)v^\varepsilon - \tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} &\le \|\tilde{T}_\varepsilon(t)v^\varepsilon - \tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} + \|s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon) - s_*^\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &= C\|\tilde{T}_\varepsilon(t)v^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon) - P_\varepsilon u^\varepsilon - s_*^\varepsilon(P_\varepsilon u^\varepsilon)\|_{H_0^1(0,\pi)} \\ &= C\|T_\varepsilon(t)w^\varepsilon - u^\varepsilon\|_{H_0^1(0,\pi)} \to 0 \quad \text{as } t \to \infty,\end{aligned}$$

for a constant $C > 0$ independent of ε, where the attraction property is also uniform for the solutions starting at bounded sets.

It remains to prove that $\tilde{A}_\varepsilon$ is invariant. Let $\tilde{u}^\varepsilon \in \tilde{A}_\varepsilon$ and $t \ge 0$. Writing $w^\varepsilon = P_\varepsilon \tilde{u}^\varepsilon$ for some $w^\varepsilon \in \mathcal{A}_\varepsilon$, we have by the invariance of $\mathcal{A}_\varepsilon$, that there is $\hat{w}^\varepsilon \in \mathcal{A}_\varepsilon$ such that $T_\varepsilon(\bar{t})\hat{w}^\varepsilon = w^\varepsilon$, for some $\bar{t} \ge 0$. Thus,

$$\tilde{u}^\varepsilon + s_*^\varepsilon(\tilde{u}^\varepsilon) = P_\varepsilon w^\varepsilon + s_*^\varepsilon(P_\varepsilon w^\varepsilon) = w^\varepsilon = T_\varepsilon(\bar{t})\hat{w}^\varepsilon = \tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon),$$

which implies $\tilde{u}^\varepsilon = \tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon$, where $P_\varepsilon\hat{w}^\varepsilon \in \tilde{A}_\varepsilon$. □

Proposition 3.7. *The set $\bar{A}_\varepsilon = L_\varepsilon P_\varepsilon \mathcal{A}_\varepsilon$ is the global attractor for the semigroup $\bar{T}_\varepsilon(\cdot)$ given by solutions of* (3.5)*.*

Proof. Since L_ε is continuous and $P_\varepsilon \mathcal{A}_\varepsilon$ is compact, we have $\bar{A}_\varepsilon = L_\varepsilon P_\varepsilon \mathcal{A}_\varepsilon$ a compact set in $\mathbb{R}^m$. Let B a bounded set in $\mathbb{R}^m$ and $\bar{u}^\varepsilon \in B$, then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in L_\varepsilon^{-1}B$ which is a bounded set in Y_ε. Since $\tilde{T}_\varepsilon(\cdot)$ is gradient, there is $\tilde{w}^\varepsilon \in \tilde{A}_\varepsilon$ such that, $\|\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \to 0$ as $t \to \infty$, where the attraction property is uniform for the solutions starting at bounded sets. Hence, $L_\varepsilon\tilde{w}^\varepsilon \in \bar{A}_\varepsilon$ is such that,

$$\begin{aligned}\|\bar{T}_\varepsilon(t)\bar{u}^\varepsilon - L_\varepsilon\tilde{w}^\varepsilon\|_{\mathbb{R}_0^m} &= \|L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \\ &= \|\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \to 0, \quad \text{as } t \to \infty,\end{aligned}$$

where we have used that L_ε is a isometry and Proposition 3.5.

It remains to prove that $\bar{A}_\varepsilon$ is invariant. Let $\bar{u}^\varepsilon \in \bar{A}_\varepsilon$. Then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in \tilde{A}_\varepsilon$ which is invariant. Thus, there is $\tilde{w}^\varepsilon \in \tilde{A}_\varepsilon$ and $\bar{t} > 0$ such that $\tilde{T}_\varepsilon(\bar{t})\tilde{w}^\varepsilon = L_\varepsilon^{-1}\bar{u}^\varepsilon$. Thus, $L_\varepsilon\tilde{T}_\varepsilon(\bar{t})\tilde{w}^\varepsilon = \bar{u}^\varepsilon$ and by Proposition 3.5, we have $\bar{T}_\varepsilon(\bar{t})L_\varepsilon\tilde{w}^\varepsilon = \bar{u}^\varepsilon$. □

4 Proof of Theorem 1.1

In this section, we prove the main result of this paper, the Theorem 1.1.

Theorem 4.1. *The equation* (3.5) *is structurally stable at $\varepsilon = 0$. That is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0, \varepsilon_\eta]$ there is a homeomorphism $h_\varepsilon : \bar{A}_\varepsilon \to \bar{A}_0$ such that,*

$$\sup_{\bar{u}^\varepsilon \in \bar{A}_\varepsilon} \|h_\varepsilon(\bar{u}^\varepsilon) - \bar{u}^\varepsilon\|_{\mathbb{R}^m} < \eta \quad \text{and} \quad h_\varepsilon(\bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t, \bar{u}^\varepsilon))\bar{u}^\varepsilon) = \bar{T}_0(t)h_\varepsilon(\bar{u}^\varepsilon), \tag{4.1}$$

where $\bar{u}^\varepsilon \in \bar{A}_\varepsilon$, $t \in \mathbb{R}$ and $\bar{\tau}_\varepsilon : \mathbb{R} \times \bar{A}_\varepsilon \to \mathbb{R}$ is function such that, $\bar{\tau}_\varepsilon(0, \bar{u}^\varepsilon) = 0$ and $\bar{\tau}_\varepsilon(\cdot, \bar{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

Proof. The works [1] and [5, Chapter 14] have obtained the continuity of the semigroups $T_\varepsilon(\cdot) \to T_0(\cdot)$ as $\varepsilon \to 0$ in the $H^1_0(0,\pi)$ norm. Following [2] we obtain $\bar{T}_\varepsilon(\cdot) \to \bar{T}_0(\cdot)$ as $\varepsilon \to 0$ in the C^1 norm, since the invariant manifolds $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$ are close in the C^1 topology. Thus, $\bar{T}_\varepsilon(\cdot)$ is a small C^1 perturbation of $\bar{T}_0(\cdot)$ which is a Morse–Smale semigroup $\mathbb{R}^m$. The main property of Morse–Smale flows in finite dimension stated in [11, 14] and [13] is the structural stability, that is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0, \varepsilon_\eta]$ there is a homeomorphism $h_\varepsilon : \bar{\mathcal{A}}_\varepsilon \to \bar{\mathcal{A}}_0$ such that, (4.1) is valid. □

Theorem 4.2. *The equation (3.4) is structurally stable at $\varepsilon = 0$. That is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0, \varepsilon_\eta]$ there is a homeomorphism $j_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ such that,*

$$\sup_{\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon} \|j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} < C(\|a_\varepsilon - a_0\|_\infty + \eta) \quad \text{and} \quad j_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) = \tilde{T}_0(t) j_\varepsilon(\tilde{u}^\varepsilon), \tag{4.2}$$

where $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$, $t \in \mathbb{R}$ and $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ is function such that, $\tilde{\tau}_\varepsilon(0, \tilde{u}^\varepsilon) = 0$ and $\tilde{\tau}_\varepsilon(\cdot, \tilde{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

Proof. We define the map $j_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ by $j_\varepsilon = L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon$. Then, for $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ it follows from Proposition 3.2 and (4.1) that

$$\begin{aligned} \|j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} &= \|L_0^{-1} h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} \\ &= \|L_0^{-1} h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon^{-1} L_\varepsilon \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} \\ &\le C(\|h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon \tilde{u}^\varepsilon\|_{\mathbb{R}^m} + \|a_\varepsilon - a_0\|_\infty) \\ &\le C(\eta + \|a_\varepsilon - a_0\|_\infty). \end{aligned}$$

Moreover, by (4.1) and Proposition 3.5, we obtain

$$\begin{aligned} j_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) &= L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) \\ &= L_0^{-1}(h_\varepsilon(\bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t, L_\varepsilon \tilde{u}^\varepsilon)) L_\varepsilon \tilde{u}^\varepsilon)) \\ &= L_0^{-1} \bar{T}_0(t) h_\varepsilon(L_\varepsilon \tilde{u}^\varepsilon) \\ &= \tilde{T}_0(t) L_0^{-1} h_\varepsilon(L_\varepsilon \tilde{u}^\varepsilon) \\ &= \tilde{T}_0(t) j_\varepsilon(\tilde{u}^\varepsilon). \end{aligned}$$

Hence, j_ε is a homeomorphism between $\tilde{\mathcal{A}}_\varepsilon$ and $\tilde{\mathcal{A}}_0$ satisfying (4.2). □

Now, we are in a condition to prove the Theorem 1.1.

Proof. of Theorem 1.1. We define the map $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ by $\kappa_\varepsilon = P_0^{-1} \circ j_\varepsilon \circ P_\varepsilon$. Similarly to the proof of Theorem 4.2, we can prove that κ_ε is a homeomorphism between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ satisfying

$$\|\kappa_\varepsilon(u^\varepsilon) - u^\varepsilon\|_{H^1_0(0,\pi)} \le C(\eta + \|a_\varepsilon - a_0\|_\infty)$$

and

$$\kappa_\varepsilon(T_\varepsilon(\tau_\varepsilon(t, u^\varepsilon))u^\varepsilon) = T_0(t)\kappa_\varepsilon(u^\varepsilon).$$

□

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