

A class of nonlinear oscillators with non-autonomous first integrals and algebraic limit cycles

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Abstract. In this paper, we present a class of autonomous nonlinear oscillators with non-autonomous first integral. We prove explicitly the existence of a global sink which is, under some conditions, an algebraic limit cycle. For that class, we draw the possible phase portraits in the Poincaré disk.

Keywords: algebraic limit cycles, global sink, non-autonomous first integrals.

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1 Introduction and the main result

In this paper, we consider the class of second-order nonlinear ordinary differential equations of the form

$$x_{tt} + f_3(x)x_t^3 + f_2(x)x_t^2 + f_1(x)x_t + f_0(x) = 0, \quad (1.1)$$

where $f_i \neq 0, i = 0, 1, 2, 3$ are smooth real functions of the variable $x = x(t)$. In the (x, \dot{x}) phase plane, equation (1.1) is equivalent to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \end{cases} \quad (1.2)$$

where the dot is the derivative with respect to the independent variable t . This kind of oscillator arises in modeling physical, chemical or electronic processes [3, 13]. The qualitative behavior of the solutions of such oscillators is very important and complicated. Various methods have been proposed in the literature to examine the global dynamics of these solutions. Analytical methods, such as the integrability method, attempt to transform the differential system (1.2) into a known differential equation (linear, Bernoulli, Riccati, Abel). This method is used to obtain the solutions explicitly. However, this method may not be sufficient to characterize all the features of the system, especially when the solutions are not analytically known.

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On the other hand, some mathematicians have introduced new tools allowing to obtain the maximum qualitative information about the dynamics of planar differential systems in general. The tool relies on geometric characteristics is called the classification of phase portraits in the Poincaré disk.

A significant number of papers regarding limit cycles, first integrals and invariants curves [9, 16, 18, 20, 22, 24, 31] has been published, where the main goal was studying the qualitative behavior of these solutions.

Starting with [7], Chandrasekar et al. investigated the integrability of a class of oscillators, described by the generalized second-order nonlinear ordinary differential equation

$$\ddot{x} + (k_1x^q + k_2)\dot{x} + k_3x^{2q+1} + k_4x^{q+1} + \lambda_1x = 0, \quad (1.3)$$

where the parameters λ_1, q and $k_i, i = 1, 2, 3, 4$ are real. Using the extended Prelle–Singer method, the authors were able to determine the first integrals and general solutions for the integrable cases.

In [27], Sinelshchikov proved that two subfamilies of the following family of oscillators

$$y_{zz} + k(y)y_z^3 + h(y)y_z^2 + f(y)y_z + g(y) = 0, \quad (1.4)$$

with k, h, f and $g \neq 0$ are arbitrary sufficiently smooth functions, are integrable and each subfamily possesses an autonomous parametric first integral and two autonomous invariant curves.

In [28], the same author, along with Guha and Choudhury, studied a family of non-autonomous second-order differential equations of the type

$$y_{zz} + a_3(z, y)y_z^3 + a_2(z, y)y_z^2 + a_1(z, y)y_z + a_0(z, y) = 0, \quad (1.5)$$

where $a_i, i = 0, 1, 2, 3$ are smooth functions such that $a_3 \neq 0$ and $|a_2|^2 + |a_1|^2 + |a_0|^2 \neq 0$. The authors showed that equations from (1.5) with a Lax representation admit a quadratic rational first integral.

As a continuation of [7], Jibin and Han [17] showed that the oscillator considered in [7] has a unique and stable limit cycle and they gave its exact parametric representation. In fact, this limit cycle was obtained explicitly a long time before (see [1, 4] and references therein).

Naturally, the following question arises: is there an integrable polynomial planar oscillator of the form (1.1) with an explicit algebraic limit cycle? To the best of our knowledge, we have not encountered such an example in the literature. In this paper, we provide the answer to this question. Moreover, since autonomous rational first integrals and limit cycles are incompatible, in the sense that a planar vector field may have at most one of them. We think that it is interesting to provide an example in which algebraic limit cycles and non-autonomous first integrals can coexist. We consider the class of autonomous oscillators of the form (1.2), where

$$\begin{aligned} f_3(x) &= -\frac{w}{2h}, \\ f_2(x) &= -\frac{3w^2(x^3 - hx)}{4h^2}, \\ f_1(x) &= -\frac{3}{8}\left(\frac{w}{h}\right)^3 x^6 + \frac{3w^3}{4h^2}x^4 - \frac{1w}{8h}(3w^2 + 20)x^2 + w, \\ f_0(x) &= -\frac{x}{16h^4}h_1(x)h_2(x), \end{aligned}$$

with

$$\begin{aligned} h_1(x) &= w^2 x^4 - hw^2 x^2 + 4h^2, \\ h_2(x) &= w^2 x^4 - 2hw^2 x^2 + h^2 w^2 + 4h^2, \end{aligned}$$

are smooth real functions of the variable $x \in \mathbb{R}$, while $h \in \mathbb{R}^*$ and $w \in \mathbb{R}$ are parameters.

Our main result is the following.

Theorem 1.1. *Let X be the vector field given by (1.2) and let Γ be the set*

$$\left\{ (x, y) \in \mathbb{R}^2 : x^2 + \left(y + \frac{w}{2h}(x^2 - h)x \right)^2 = h \right\}.$$

Then the following statements hold:

(a) X has a non-autonomous first integral given by

$$H(x, y, t) = \frac{x^2 + \left(y + \frac{w}{2h}(x^2 - h)x \right)^2 - h}{x^2 + \left(y + \frac{w}{2h}(x^2 - h)x \right)^2} e^{wt}, \quad \forall (x, y, t) \in \mathbb{R}^3.$$

(b) If $h > 0$ and $w > 0$ (resp. $w < 0$), then Γ is a global sink (resp. source) of X .

(c) If $h > 0$ and $0 < w < 4$ (resp. $-4 < w < 0$), then Γ is a hyperbolic stable (resp. unstable) algebraic limit cycle of X . Moreover, Γ is the unique limit cycle of X and is a global sink (resp. source) of X .

Moreover, the phase portraits of X in the Poincaré disk are topologically equivalent to those given in Figure 1.1.

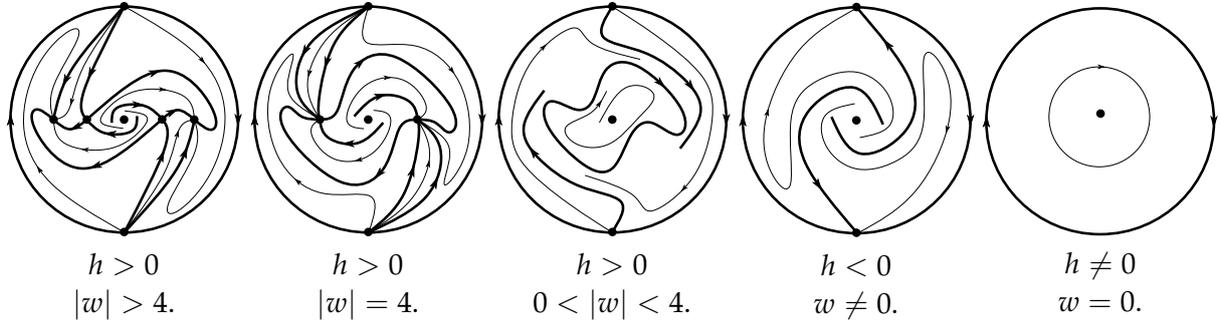


Figure 1.1: The topological distinct phase portraits of X .

The paper is organized as follows. In Section 2, we introduce some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminary results

2.1 First integrals and invariant algebraic curves

Let $X = (P, Q)$ be a polynomial vector field. We say that X is integrable if and only if there exists a non-constant C^1 function $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$P(x, y) \frac{\partial H}{\partial x}(x, y) + Q(x, y) \frac{\partial H}{\partial y}(x, y) = 0, \quad (2.1)$$

for all $(x, y) \in \mathbb{R}^2$. Therefore the function H is constant along the trajectories $(x(t), y(t))$ of X , i.e., if $I \subset \mathbb{R}$ is an interval, then there exists $c \in \mathbb{R}$: $H(x(t), y(t)) = c$, for all $t \in I$. In such a case the function H is called *first integral* and the trajectories of X are contained in the level sets of H . If the first integral depends on the time t , i.e., $H = H(x, y, t)$, thus we say that H is a *non-autonomous first integral* of X if

$$P(x, y) \frac{\partial H}{\partial x}(x, y, t) + Q(x, y) \frac{\partial H}{\partial y}(x, y, t) + \frac{\partial H}{\partial t}(x, y, t) = 0, \quad (2.2)$$

for all $(x, y, t) \in \mathbb{R}^2 \times I$. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a real polynomial. We say that F is an *invariant* for X if it satisfies

$$P(x, y) \frac{\partial F}{\partial x}(x, y) + Q(x, y) \frac{\partial F}{\partial y}(x, y) = K(x, y)F(x, y), \quad (2.3)$$

for all $(x, y) \in \mathbb{R}^2$. Here, $K: \mathbb{R}^2 \rightarrow \mathbb{R}$, which is the *cofactor* of F , is a real polynomial and its degree is at most $n - 1$, where n represents the maximum of the degrees of P and Q . It can be observed that the set defined by the equation $F(x, y) = 0$ is invariant under the flow of X . In this case, this set may contain ovals, which can be *algebraic limit cycles*. For more details about first integrals, invariant algebraic curves and algebraic limit cycles, see [4,6,8,10,21,23,32] and Chapter 8 of [11] and the references therein.

2.2 Singular points

Let $X = (P, Q)$ be a polynomial vector field. We say $q \in \mathbb{R}^2$ is a *singularity* of X if $P(q) = Q(q) = 0$. The Jacobian matrix J of the vector field X at q is given by

$$J(q) = \begin{pmatrix} \frac{\partial P}{\partial x}(q) & \frac{\partial P}{\partial y}(q) \\ \frac{\partial Q}{\partial x}(q) & \frac{\partial Q}{\partial y}(q) \end{pmatrix}. \quad (2.4)$$

Let $D(q) = \lambda_1 \lambda_2$ be the determinant and $T(q) = \lambda_1 + \lambda_2$ the trace of $J(q)$, where λ_1 and λ_2 are the eigenvalues of $J(q)$, that are the solutions of the characteristic equation

$$\lambda^2 - T(q)\lambda + D(q) = 0.$$

The singularity q is:

- (a) *Hyperbolic* if both eigenvalues have real parts different from zero. Here, we distinguish:
 - (i) If $D(q) < 0$, then q is a saddle.
 - (ii) If $D(q) > 0$ and $T(q) > 0$, then q is an unstable focus/node.
 - (iii) If $D(q) > 0$ and $T(q) < 0$, then q is a stable focus/node.
- (b) *Degenerate monodromic* if $D(q) > 0$ and $T(q) = 0$. In this case, q is a weak focus or a center.
- (c) *Semi-hyperbolic* if $D(q) = 0$ and $T(q) \neq 0$.
- (d) *Nilpotent* if $D(q) = T(q) = 0$ and $J(q)$ is not identically zero.
- (e) *Degenerate* if $D(q) = T(q) = 0$ and $J(q)$ is identically zero.

We characterize the local phase portraits at hyperbolic, semi-hyperbolic and nilpotent singular points using Theorems 2.15, 2.19 and 3.5 of [11], respectively. For the degenerate singularities, we employ the blow-up technique, see [2] for details.

2.3 The blow-up technique

Consider X a planar polynomial vector field with an isolated singularity at the origin, then we can apply the change of coordinates $\phi : \mathbb{S}^1 \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$ given by $\phi(\theta, r) = (r \cos \theta, r \sin \theta) = (x, y)$, where $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$. Consequently, we can induce the vector field X_0 in $\mathbb{S}^1 \times \mathbb{R}_+$ by pullback, i.e., $X_0 = D\phi^{-1}X$. One can see that if the k -jet of X (i.e., the Taylor expansion of order k of X , denoted by j_k) is zero at the origin, then the k -jet of X_0 is also zero at every point in $\mathbb{S}^1 \times \{0\}$. Thus, taking the first $k \in \mathbb{N}$ satisfying $j_k(X(0,0)) = 0$ and $j_{k+1}(X(0,0)) \neq 0$, we can define the vector field $\hat{X} = \frac{1}{r^k}X_0$. Therefore, it follows that the behavior of \hat{X} near \mathbb{S}^1 is the same as that of X near the origin. One can also see that \mathbb{S}^1 is invariant under the flow of \hat{X} . For a more detailed study of this technique, see [2] or Chapter 3 of [11]. The vector field \hat{X} can be also expressed as

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r^{k+1}}, \quad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^{k+2}}.$$

The blow-up technique has a generalization called the *quasihomogeneous blow-up*. In this case, we consider the change of coordinates $\psi(\theta, r) = (r^\alpha \cos \theta, r^\beta \sin \theta) = (x, y)$ for $(\alpha, \beta) \in \mathbb{N}^2$. In a similar way, we can induce the vector field X_0 in $\mathbb{S}^1 \times \mathbb{R}_+$. For some $k \in \mathbb{N}$ maximal, one can define $X_{\alpha,\beta} = \frac{1}{r^k}X_0$ and such a vector field is given by

$$\dot{r} = \zeta(\theta) \frac{\cos \theta r^\beta \dot{x} + \sin \theta r^\alpha \dot{y}}{r^{\alpha+\beta+k-1}}, \quad \dot{\theta} = \zeta(\theta) \frac{\alpha \cos \theta r^\alpha \dot{y} - \beta \sin \theta r^\beta \dot{x}}{r^{\alpha+\beta+k}},$$

where $\zeta(\theta) = (\beta \sin^2 \theta + \alpha \cos^2 \theta)^{-1}$. Since $\zeta(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, therefore it can be eliminated by a change in the time variable. Thus, it follows then

$$\dot{r} = \frac{\cos \theta r^\beta \dot{x} + \sin \theta r^\alpha \dot{y}}{r^{\alpha+\beta+k-1}}, \quad \dot{\theta} = \frac{\alpha \cos \theta r^\alpha \dot{y} - \beta \sin \theta r^\beta \dot{x}}{r^{\alpha+\beta+k}},$$

As in the previous technique, the behavior of $X_{\alpha,\beta}$ near \mathbb{S}^1 (which is invariant) is similar to the behavior of X near the origin.

2.4 The Poincaré compactification

To study the behavior of the trajectories of a planar vector field near infinity, we will employ the *Poincaré compactification* (for more details, see [30] or Chapter 5 of [11]).

Let $X = (P, Q)$ be a planar polynomial vector field of degree $n \in \mathbb{N}$. We identify \mathbb{R}^2 with the plane $(x_1, x_2, 1)$ in \mathbb{R}^3 and define the *Poincaré sphere* as $\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$. We denote the *northern hemisphere*, the *southern hemisphere* and the *equator* by $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$, $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$ and $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$, respectively. The *Poincaré compactified vector field* $p(X)$ associated with X is an analytic vector field generated on \mathbb{S}^2 by the central projections $f_\pm : \mathbb{R}^2 \rightarrow H_\pm$, given by $f_\pm(x_1, x_2) = \pm \Delta(x_1, x_2)(x_1, x_2, 1)$, where $\Delta(x_1, x_2) = (x_1^2 + x_2^2 + 1)^{-\frac{1}{2}}$. These two maps define two symmetric copies of X , one copy X^+ in H_+ and the other copy X^- in H_- . In brief, we obtain the vector field $X' = X^+ \cup X^-$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$. Note that the equator \mathbb{S}^1 of the sphere \mathbb{S}^2 corresponds with the *infinity* of \mathbb{R}^2 . The analytic extension of X' from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to \mathbb{S}^2 , given by $y_3^{n-1}X'$, is the Poincaré compactified vector field $p(X)$. The *Poincaré disk* \mathbb{D} is the projection of the closed northern hemisphere on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ (the vector field given by this projection will also be denoted by $p(X)$). The behavior of $p(X)$ near \mathbb{S}^1 is the same as the behavior of X near infinity

of \mathbb{R}^2 . We define the local charts of \mathbb{S}^2 by $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i \in \{1, 2, 3\}$ and their corresponding local maps by $\phi_i : U_i \rightarrow \mathbb{R}^2$, $\psi_i : V_i \rightarrow \mathbb{R}^2$ with $\phi_i(y_1, y_2, y_3) = -\psi_i(y_1, y_2, y_3) = \left(\frac{y_m}{y_i}, \frac{y_n}{y_i}\right)$, where $m \neq i$, $n \neq i$ and $m < n$. Denoting by (u, v) the image of ϕ_i and ψ_i , for $i = 1, 2$, in each chart. The expression of $p(X)$ in the local chart U_1 is

$$\dot{u} = v^n \left[Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{n+1}P \left(\frac{1}{v}, \frac{u}{v} \right),$$

and in the local chart U_2 , it is given by

$$\dot{u} = v^n \left[P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right) \right], \quad \dot{v} = -v^{n+1}Q \left(\frac{u}{v}, \frac{1}{v} \right).$$

The expression of $p(X)$ in V_1 and V_2 is the same as that for U_1 and U_2 , except by a multiplicative factor of $(-1)^{n-1}$. In these local charts for $i \in \{1, 2\}$, the coordinate $v = 0$ represents the points of \mathbb{S}^1 . Thus, the singularities at infinity of \mathbb{R}^2 . Note that \mathbb{S}^1 is invariant under the flow of $p(X)$.

2.5 The Markus–Neumann theorem

Let X be a polynomial vector field and $p(X)$ be its compactification defined on \mathbb{D} . Consider ϕ the flow associated to $p(X)$. The separatrices of $p(X)$ are orbits, which can be:

1. All the orbits contained in \mathbb{S}^1 , i.e., at infinity;
2. All the singular points;
3. All the trajectories that are located on the boundaries of the hyperbolic sectors of the finite and infinite singular points;
4. All the limit cycles of X .

The set of all separatrices, denoted by \mathcal{S} is closed. Each connected component of $\mathbb{D} \setminus \mathcal{S}$ is called a *canonical region* of the flow (\mathbb{D}, ϕ) .

The *separatrix configuration* \mathcal{S}_c of the flow (\mathbb{D}, ϕ) , is the union of all the separatrices \mathcal{S} of the flow, together with one orbit from each canonical region.

Two separatrix configurations \mathcal{S}_c and \mathcal{S}_c^* of the flow (\mathbb{D}, ϕ) are *topologically equivalent* if there exists a homeomorphism from \mathbb{D} to \mathbb{D} that transforms orbits of \mathcal{S}_c into those of \mathcal{S}_c^* while preserving or reversing the orientation of all these orbits.

Theorem 2.1 (Markus–Neumann). *Let $p(X)$ and $p(Y)$ be two Poincaré compactifications in the Poincaré disk \mathbb{D} of two polynomial vector fields X and Y , with finitely many singularities. Then the phase portraits of $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

Proof. See [5, 25, 26] and Section 1.9 of [11]. □

2.6 The solutions of the quartic algebraic equation of degree four

It is well known that the quartic equation

$$ax^4 + bx^3 + cx^2 + d = 0, \tag{2.5}$$

where $a \neq 0$, can be transformed via the change of variable $x \mapsto x - \frac{b}{4a}$ into the equation

$$x^4 + px^2 + qx + r = 0. \quad (2.6)$$

The discriminant of equation (2.6) is given by

$$\Delta = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

Suppose $\Delta > 0$. Then the following statements hold (see [19] or Chapter 12 of [12]).

- (i) If $p < 0$ and $4r < p^2$, then all roots of (2.6) are simple and real.
- (ii) If $p \geq 0$ or $4r \geq p^2$, then all roots of (2.6) are simple and complex.

Moreover, we observe that if $q = 0$, then $\Delta = 16(p^2 - 4r)^2r$.

3 Proof of Theorem 1.1

Let us look at statement (a). To see that

$$H(x, y, t) = \frac{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2 - h}{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2} e^{wt}, \quad (3.1)$$

is a non-autonomous first integral of X , it is sufficient to observe that the equation

$$P(x, y) \frac{\partial H}{\partial x}(x, y, t) + Q(x, y) \frac{\partial H}{\partial y}(x, y, t) + \frac{\partial H}{\partial t}(x, y, t) = 0,$$

is satisfied. We now look at statement (b). Suppose $w > 0$ and let

$$H_1(x, y) = \frac{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2 - h}{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2}.$$

We want to prove that

$$\Gamma = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + \left(y + \frac{w}{2h}(x^2 - h)x\right)^2 = h \right\},$$

is a global sink of X . It follows from (3.1) that $H(x, y, t) = H_1(x, y)e^{wt}$.

Let $(x(t), y(t)) \in \mathbb{R}^2$ be an orbit of X . Since $w > 0$, observe that if $t \rightarrow +\infty$, then $e^{wt} \rightarrow +\infty$. However, it follows from statement (a) that $H(x(t), y(t), t)$ is constant, for every $t \in \mathbb{R}$. Therefore, we have $H_1(x(t), y(t)) \rightarrow 0$ as $t \rightarrow +\infty$. The statement now follows from the fact that Γ coincides with the set $\{(x, y) \in \mathbb{R}^2 : H_1(x, y) = 0\}$. For $w < 0$, the proof follows straightforwardly from the fact that X is invariant under the change of variables and parameters $(x, t, w) \mapsto (-x, -t, -w)$.

Let us look at statement (c). First, let

$$F(x, y) = x^2 + \left(y + \frac{w}{2h}(x^2 - h)x\right)^2 - h. \quad (3.2)$$

Notice that if $h > 0$, then

$$P(x, y) \frac{\partial F}{\partial x}(x, y) + Q(x, y) \frac{\partial F}{\partial y}(x, y) = K(x, y)F(x, y), \quad (3.3)$$

where

$$K(x, y) = -\frac{w}{4h^3}(w^2x^6 - 2hw^2x^4 + 4hwx^3y + h^2(w^2 + 4)x^2 - 4h^2wxy + 4h^2y^2). \quad (3.4)$$

Therefore, if $h > 0$, the curve $F = 0$ is an invariant algebraic curve of X .

We claim that if $0 < |w| < 4$, then the origin is the unique finite singularity of X . Indeed, it follows from (1.2) that the finite singularities of X , other than the origin, are of the form $(x_i, 0)$, $i \in \{1, 2, 3, 4\}$ where x_i are the real solutions of $h_1(x)h_2(x) = 0$, with

$$h_1(x) = w^2x^4 - hw^2x^2 + 4h^2, \quad h_2(x) = w^2x^4 - 2hw^2x^2 + h^2w^2 + 4h^2.$$

Let Δ_i denote the discriminant of h_i , $i \in \{1, 2\}$. It follows from Subsection 2.6 that

$$\Delta_1 = 64h^6w^6(w^2 - 16)^2, \quad \Delta_2 = 4096h^6w^6(w^2 + 4).$$

Therefore, we conclude that if $w \neq 0$ and $h \neq 0$, then h_2 always has a positive discriminant. Hence, all its singularities are either real or complex. Since h_2 satisfies statement (ii) of Subsection 2.6, we conclude that h_2 never has real solutions.

Similarly, it can be seen that if $h > 0$ and $0 < |w| < 4$, then h_1 also does not have real solutions. Thus, if $h > 0$ and $0 < |w| < 4$, then the origin is the unique finite singularity of X . Since it does not lie on the curve $F^{-1}(0) = \Gamma$, we conclude that Γ is an algebraic limit cycle. Moreover, the limit cycle Γ is hyperbolic (for more details, see [14]) if only if

$$I(\Gamma) = \int_0^T K(\gamma(t))dt \neq 0,$$

where $T > 0$ is the period of Γ , $\gamma(t)$ is the parameterization of Γ and the cofactor K is given by (3.3) and (3.4), hence

- if $I(\Gamma) < 0$, Γ is a stable limit cycle;
- if $I(\Gamma) > 0$, Γ is an unstable limit cycle.

It follows from (3.4) that $K(x, y) < 0$ (resp. $K(x, y) > 0$) if $w > 0$ (resp. $w < 0$). Consequently, Γ is a hyperbolic limit cycle. In particular, it follows from statement (b) that Γ is the unique limit cycle of X and that it is stable if $w > 0$ and unstable if $w < 0$.

We now look to the phase portraits of X . If $w = 0$ then

$$\dot{x} = y, \quad \dot{y} = -x,$$

and thus X has a global center. In the sequel, we assume $w \neq 0$. Since X is invariant under the change of variables $(x, t, w) \mapsto (-x, -t, -w)$, it is enough to assume $w > 0$. Similarly to the previous analysis on the roots of h_1 and h_2 , one can see that:

- (a) All the roots of h_2 are complex;
- (b) If $h < 0$, then all roots of h_1 are complex;
- (c) If $h > 0$ and $0 < w < 4$, then all the roots of h_1 are complex;
- (d) If $h > 0$ and $w = 4$, then h_1 has two real solutions of multiplicity two, given by $x^\pm = \pm\sqrt{\frac{h}{2}}$;

(e) If $h > 0$ and $w > 4$, then h_1 has four distinct real solutions, given by

$$x_1 = -\frac{1}{\sqrt{2}}\sqrt{h\left(1 + \frac{\sqrt{w^2 - 16}}{w^2}\right)}, \quad x_2 = -\frac{1}{\sqrt{2}}\sqrt{h\left(1 - \frac{\sqrt{w^2 - 16}}{w^2}\right)},$$

$$x_3 = \frac{1}{\sqrt{2}}\sqrt{h\left(1 - \frac{\sqrt{w^2 - 16}}{w^2}\right)}, \quad x_4 = \frac{1}{\sqrt{2}}\sqrt{h\left(1 + \frac{\sqrt{w^2 - 16}}{w^2}\right)}.$$

Let $p_i = (x_i, 0)$, $i \in \{1, 2, 3, 4\}$ be the singularities associated to x_i and let \mathcal{O} denote the origin. Calculations show that the origin is always a hyperbolic unstable focus. Moreover, if $w > 4$ then p_1 and p_4 are hyperbolic stable nodes, while p_2 and p_3 are hyperbolic saddles. Furthermore, if $w = 0$ then $p_1 = p_2$ and $p_3 = p_4$ are semi-hyperbolic saddle-nodes.

We now look at the infinity. The unique singularity at infinity is the origin of the second chart of the Poincaré compactification. In this case, after performing two quasihomogeneous blow-ups, with weights $(\alpha_1, \beta_1) = (2, 3)$ and $(\alpha_2, \beta_2) = (2, 1)$ respectively, we obtain the local phase portraits as illustrated in Figure 3.1.

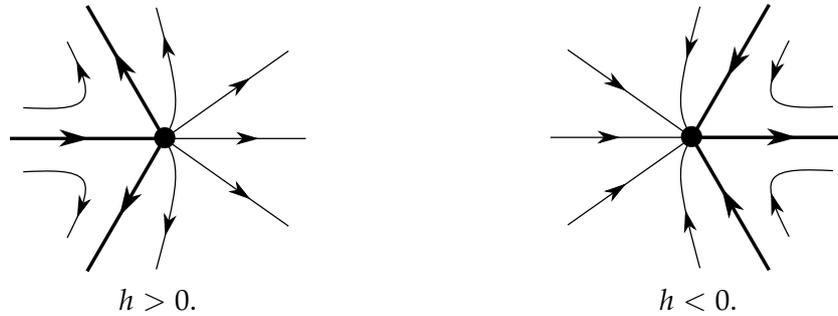


Figure 3.1: Local phase portrait at the origin of the second chart of the Poincaré compactification.

We now study the phase portrait for the case $w > 4$. In this case, the local phase portrait is shown in Figure 3.2.

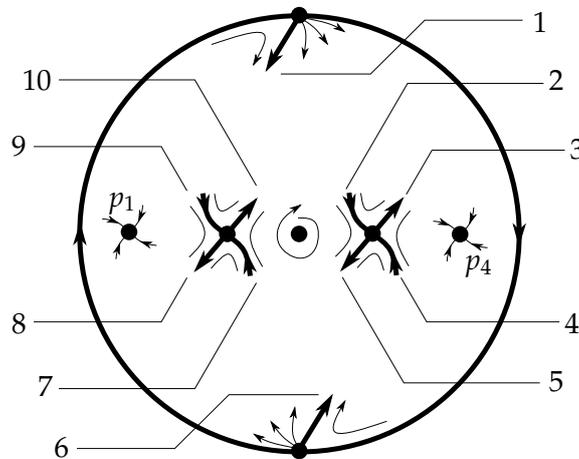


Figure 3.2: Uncompleted phase portrait for $w > 4$.

Observe that the invariant algebraic curve $F(x, y) = 0$ is given by the union of the curves

$$y^\pm = \frac{w}{2h}(h - x^2)x \pm \sqrt{h - x^2}, \quad (3.5)$$

for $|x| < \sqrt{h}$. It follows that separatrix 10 goes to the stable node p_4 , while separatrix 8 goes to the stable node p_1 . Since X is invariant under the change of variables $(x, y) \mapsto (-x, -y)$, it follows that separatrix 5 goes to p_1 and separatrix 3 goes to p_4 . Separatrices 7 and 2 are now enclosed in the bounded region delimited by Γ and thus have no other option than to be generated at the origin. See Figure 3.3.

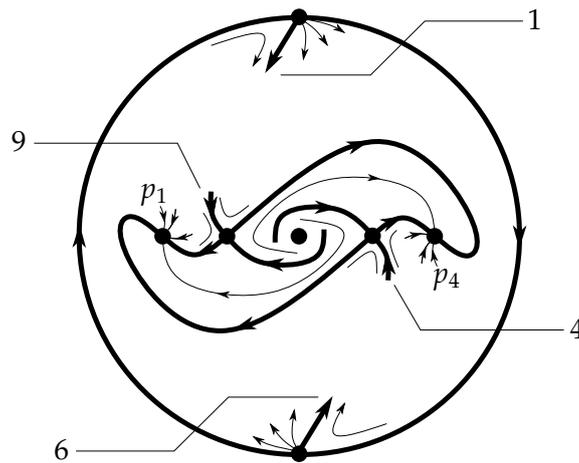


Figure 3.3: Uncompleted phase portrait for $w > 4$.

We now have numerical evidence, according to software $P4$ (see Chapters 9 and 10 of [11]), that separatrix 1 goes to p_1 . Therefore, it follows from the invariance of X under the change of variables $(x, y) \mapsto (-x, -y)$ that separatrix 6 goes to p_4 . Hence, separatrix 9 must be born at the north pole, while separatrix 4 must be born at the south pole. The other phase portraits can be obtained in a similar manner.

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Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

References

- [1] M. A. ABDELKADER, Relaxation oscillators with exact limit cycles, *J. Math. Anal. Appl.* **218**(1998), No. 1, 308–312. <https://doi.org/10.1006/jmaa.1997.5746>; MR1601905
- [2] M. ALVAREZ, A. FERRAGUT, J. XAVIER, A survey on the blow up technique, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **21**(2011), No. 11, 3103–3118. <https://doi.org/10.1142/S021812741103041>; MR2871964
- [3] A. A. ANDRONOV, A. A. F. VITT, S. E. KHAIKIN, *Theory of oscillators*, International Series of Monographs in Physics, Vol. 4, Pergamon Press, Oxford, 1966. <https://doi.org/10.1002/zamm.19670470720>; MR198734; Zbl 0188.56304
- [4] A. BENDJEDDOU, R. CHEURFA, On the exact limit cycle for some class of planar differential systems, *NoDEA Nonlinear Differential Equations Appl.* **14**(2007), No. 5, 491–498. <https://doi.org/10.1007/s00030-007-4005-8>; MR2374196
- [5] J. G. E. BUENDÍA, V. J. LÓPEZ, On the Markus–Neumann theorem, *J. Differential Equations* **265**(2018), No. 11, 6036–6047. <https://doi.org/10.1016/j.jde.2018.07.021>; MR3857505
- [6] J. CAO, H. JIANG, Planar polynomial vector fields having first integrals and algebraic limit cycles, *J. Math. Anal. Appl.* **361**(2010), No. 1, 177–186. <https://doi.org/10.1016/j.jmaa.2009.09.007>; MR2567292
- [7] V. K. CHANDRASEKAR, S. N. PANDEY, M. SENTHILVELAN, M. LAKSHMANAN, A simple and unified approach to identify integrable nonlinear oscillators and systems, *J. Math. Phys.* **47**(2006), No. 2, 37 pp. <https://doi.org/10.1063/1.2171520>; MR2208170
- [8] J. CHAVARRIGA, H. GIACOMINI, J. LLIBRE, Uniqueness of algebraic limit cycles for quadratic systems, *J. Math. Anal. Appl.* **261**(2001), No. 1, 85–99. <https://doi.org/10.1006/jmaa.2001.7476>
- [9] V. CHERESIZ, E. VOLOKITIN, The algebraic curves of planar polynomial differential systems with homogeneous nonlinearities, *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 51, 1–12. <https://doi.org/10.14232/ejqtde.2021.1.51>; MR4287686
- [10] C. CHRISTOPHER, J. LLIBRE, G. ŚWIRSZCZ, G., Invariant algebraic curves of large degree for quadratic system, *J. Math. Anal. Appl.* **303**(2005), No. 2, 450–461. <https://doi.org/10.1016/j.jmaa.2004.08.042>; MR2122228
- [11] F. DUMORTIER, J. LLIBRE, J. ARTÉS, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, New York, 2006. <https://doi.org/10.1007/978-3-540-32902-2>; MR2256001
- [12] I. M. GELFAND, M. M. KAPRANOV, A. V. ZELEVINSKY, *Discriminants, resultants, and multidimensional determinants*, Modern Birkhäuser Classics, U. S. A., 1994. <https://doi.org/10.1007/978-0-8176-4771-1>; MR1264417
- [13] S. GHOSH, D. S. RAY, Chemical oscillator as a generalized Rayleigh oscillator, *J. Chem. Phys.* **139**(2013), No. 16, 164112, 7 pp. <https://doi.org/10.1063/1.4826169>

- [14] H. GIACOMINI, M. GRAU, On the stability of limit cycles for planar differential systems, *J. Differential Equations* **213**(2005), No. 2, 368–388. <https://doi.org/10.1016/j.jde.2005.02.010>; MR2142371
- [15] J. GINÉ, Conditions for the existence of a center for the Kukles homogeneous systems, *Comput. Math. Appl.* **43**(2002), No. 10–11, 1261–1269. [https://doi.org/10.1016/S0898-1221\(02\)00098-6](https://doi.org/10.1016/S0898-1221(02)00098-6); MR1906353
- [16] J. GINÉ, C. VALLS, Liouvillian integrability of a general Rayleigh–Duffing oscillator, *J. Non-linear Math. Phys.* **26**(2019), No. 2, 169–187. <https://doi.org/10.1080/14029251.2019.1591710>; MR3937134
- [17] L. JIBIN, M. HAN, Planar integrable nonlinear oscillators having a stable limit cycle, *J. Appl. Anal. Comput.* **12**(2022), No. 2, 862–867. <https://doi.org/10.11948/20210504;MR4398697>
- [18] A. KINA, A. BERBACHE, A. BENDJEDDOU, Limit cycles for a class of generalized Liénard polynomial differential systems via averaging theory, *Acta Math. Univ. Comenian. (N.S.)* **90**(2021), No. 4, 437–455. MR4359818
- [19] D. LAZARD, Quantifier elimination: optimal solution for two classical examples, *J. Symbolic Comput.* **5**(1988), No. 1–2, 261–266. [https://doi.org/10.1016/S0747-7171\(88\)80015-4](https://doi.org/10.1016/S0747-7171(88)80015-4); MR949122
- [20] J. LLIBRE, Integrability and limit cycles via first integrals, *Symmetry* **13**(2021), No. 9, 1736, 21 pp. <https://doi.org/10.3390/sym13091736>
- [21] J. LLIBRE, R. RAMÍREZ, N. SADOVSKAIA, On the 16th Hilbert problem for algebraic limit cycles, *J. Differential Equations* **248**(2010), No. 6, 1401–1409. <https://doi.org/10.1016/j.jde.2010.06.009;MR2737821>
- [22] J. LLIBRE, T. SALHI, On a class of polynomial differential systems of degree 4: phase portraits and limit cycles, *Topol. Methods Nonlinear Anal.* **57**(2021), No. 2, 441–463. <https://doi.org/10.12775/TMNA.2020.042;MR4359721>
- [23] J. LLIBRE, G. ŚWIRSZCZ, Relationships between limit cycles and algebraic invariant curves for quadratic systems, *J. Differential Equations* **229**(2006), No. 2, 529–537. <https://doi.org/10.1016/j.jde.2006.03.013;MR2263566>
- [24] J. LLIBRE, X. ZHANG, A survey on algebraic and explicit non-algebraic limit cycles in planar differential systems, *Expo. Math.* **39**(2021), No. 1, 48–61. <https://doi.org/10.1016/j.exmath.2020.03.001;MR4229369>
- [25] L. MARKUS, Global structure of ordinary differential equations in the plane, *Trans. Amer. Math. Soc.* **76**(1954), No. 1, 127–148. <https://doi.org/10.2307/1990747;MR60657>
- [26] A. D. NEUMANN, Classification of continuous flows on 2-manifolds, *Proc. Amer. Math. Soc.* **48**(1975), No. 1, 73–81. <https://doi.org/10.2307/2040695;MR356138>
- [27] D. I. SINELSHCHIKOV, On an integrability criterion for a family of cubic oscillators, *AIMS Math.* **6**(2021), No. 11, 12902–12910. <https://doi.org/10.3934/math.2021745;MR4311391>

- [28] D. I. SINELSHCHIKOV, P. GUHA, A. G. CHOUDHURY, Lax representation and a quadratic rational first integral for second-order differential equations with cubic nonlinearity, *Commun. Nonlinear Sci. Numer. Simul.* **112**(2022), Paper No. 106553, 5 pp. <https://doi.org/10.1016/j.cnsns.2022.106553>; MR4421680
- [29] S. H. STROGATZ, *Nonlinear dynamics and chaos. With applications to physics, biology, chemistry, and engineering*, Studies in Nonlinearity, Westview Press, Boulder, CO, 2015. MR3837141
- [30] E. A. G. VELASCO, Generic properties of polynomial vector fields at infinity, *Trans. Amer. Math. Soc.* **143**(1969), 201–222. <https://doi.org/10.2307/1995243>; MR252788
- [31] X. ZHANG, *Integrability of dynamical systems: algebra and analysis*, Developments in Mathematics, Vol. 47, Springer, Singapore, 2017. <https://doi.org/10.1007/978-981-10-4226-3>; MR3642375
- [32] X. ZHANG, The 16th Hilbert problem on algebraic limit cycles, *J. Differential Equations*, **251**(2011), No. 7, 1778–1789. <https://doi.org/10.1016/j.jde.2011.06.008>; MR2823674