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The Dirichlet problem in an unbounded cone-like domain for second order elliptic quasilinear equations with variable nonlinearity exponent

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Abstract. In this paper we consider the Dirichlet problem for quasi-linear second-order elliptic equation with the $m(x)$-Laplacian and the strong nonlinearity on the right side in an unbounded cone-like domain. We study the behavior of weak solutions to the problem at infinity and we find the sharp exponent of the solution decreasing rate. We show that the exponent is related to the least eigenvalue of the eigenvalue problem for the Laplace–Beltrami operator on the unit sphere.

Keywords: $m(x)$-Laplacian, elliptic equation, unbounded domain, cone-like domain.

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1 Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent, see e.g. [4, 16, 17, 21–23, 28, 29] and references therein. The basic properties of variable exponent function spaces were derived by O. Kováčik and J. Rákosník in [18] and (by different methods) by X.-L. Fan and D. Zhao in [14]. For a comprehensive survey concerning Lebesgue and Sobolev spaces with variable exponent we refer to [12].

Differential equations and variational problems with $m(x)$-growth conditions arise from the study of elastic mechanics, oscillation problem, electrorheological fluids [11, 24, 25], image restoration [10], thermistor problem [31] and other. Moreover, the motion of a compressible fluid in a nonhomogeneous anisotropic porous medium obeys to nonlinear the Darcy law [3]. The model of electrorheological fluids considered in [25] includes an integral of the symmetric part of gradient in a variable power which is caused by the action of an electromagnetic field. A similar structure of energy is also presented in the thermorheological model proposed in [30] for fluids with the stress tensor depending on the temperature. This system can be referred to as a coupled Boussinesq type sytem for a non-Newtonian fluid.

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Our interest is in the studying of the behavior of weak solutions to the Dirichlet problem with boundary condition on the lateral surface of a cone-like unbounded domain at infinity. For other results in unbounded and bounded cone-like domains we refer to [5–8,27]. We refer also to some very recent works dealing with complementary aspects [20,26]. These works can provide some ideas for further investigations in the cone-like domain too. For putting more emphasis on the effects of a gradient dependent reaction in the principal equation we refer to [15,19].

This paper is organized as follows. At first, we formulate the Dirichlet problem in an unbounded cone-like domain for second order elliptic quasilinear equations with variable nonlinearity exponent. Then, we introduce notations and function spaces that are used in the following sections. The main result, Theorem 1.2, is also formulated. In Section 2 we formulate an eigenvalue problem for the Laplace–Beltrami operator on the unit sphere, a Friedrichs–Wirtinger type inequality and some auxiliary inequalities and lemmas. In the next sections local estimate of the weighted Dirichlet integral and local estimate of weak solutions at infinity are investigated. Finally in Section 5 the power modulus of continuity near the infinity for weak solutions is considered.

Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$ with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded domain with the smooth boundary ∂G. We assume that $G \supset G_R$, where G_R is a cone-like domain, $G_R = \{x = (r, \omega) \in \mathbb{R}^n \mid r \in (R, \infty),\ \omega \in \Omega \subset S^{n-1},\ n \geq 2\}$, $R \gg 1$, S^{n-1} is the unit sphere (see Figure 1.1).

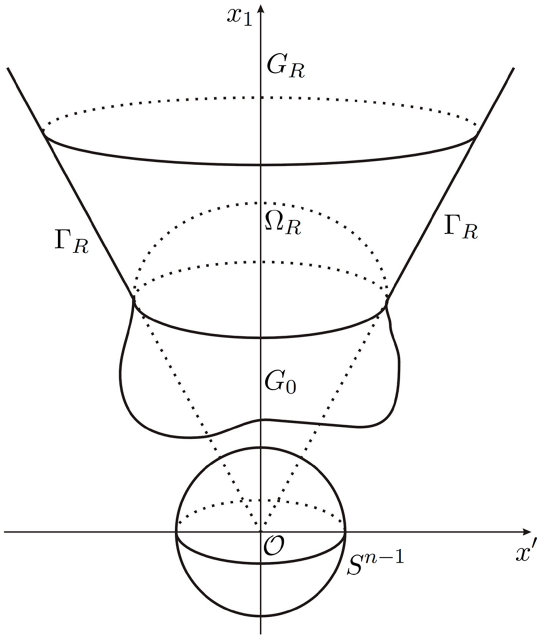

Figure 1.1: An unbounded cone-like domain

We consider the following Dirichlet problem for a quasi-linear elliptic equation **with the variable growth exponent**:

$$\begin{cases} -\frac{d}{dx_i}(|\nabla u|^{m(x)-2}u_{x_i}) + b(x,u,\nabla u) = 0, & x \in G_R, \\ u(x) = 0, & x \in \Gamma_R, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{QL}$$

The following conditions will be needed throughout the paper:

(i) $1 < \inf\{m(x) : x \in G_R\} = m_- \le m(x) \le m_+ = \sup\{m(x) : x \in G_R\} < \infty$;

(ii) the function $m(x)$ is Hölder continuous in $\overline{G_R}$, i.e. there exist a positive constant M and an exponent $\alpha \in (0,1)$ such that

$$|m(x) - m(+\infty)| \le M|x|^{-\alpha}, \quad \forall x \in \overline{G_R},$$

where $m(+\infty) = \lim_{|x|\to+\infty} m(x) = 2$;

(iii) $b(x,u,\xi)$ is a Carathéodory function $G_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and

$$|b(x,u,\xi)| \le \mu(|u|+1)^{-1}|\xi|^{m(x)}, \quad 0 \le \mu < \frac{1}{m_+} < 1;$$

(iv) $\partial\Omega \in C^{1+\gamma}$, $\gamma \in (0,1)$.

We introduce the following notations:

- $\mathcal{C}$: a rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$;
- $\partial\mathcal{C}$: the lateral surface of $\mathcal{C}$: $\{x_1 = r\cos\frac{\omega_0}{2}\}$;
- Ω : a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathcal{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathcal{C} \cap S^{n-1}$;
- $G_a^b = \{(r,\omega) \mid a < r < b;\ \omega \in \Omega\} \cap G$: the layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \{(r,\omega) \mid a < r < b;\ \omega \in \partial\Omega\} \cap \partial G$: the lateral surface of layer G_a^b, $\Gamma_\varrho = \Gamma_\varrho^\infty$

and the class of functions

$$W_{\text{loc}}(G_R) = \{u : u \in W_0^{1,1}(G_R,\Gamma_R),\ |\nabla u|^{m(x)} \in L_1(G_R),\ \forall R \gg 1\},$$

where $W_0^{1,1}(G_R,\Gamma_R)$ is the Sobolev space of those functions with zero trace on Γ_R that, together with all their first order distributional derivatives, are L^1-integrable in G_R.

We denote $W_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$.

Definition 1.1. A function $u(x) \in W_{\text{loc}}(G_R)$ such that $u(x) \to 0$ as $|x| \to \infty$ is said to be a weak solution of problem (QL) provided the integral identity

$$\int_{G_R} \left(|\nabla u|^{m(x)-2} u_{x_i}\eta_{x_i} + b(x,u,\nabla u)\eta(x) \right) dx = 0 \tag{II}$$

holds for all test functions $\eta(x) \in W_{\text{loc}}(G_R)$ such that $\eta(x) \to 0$ as $|x| \to \infty$.

We use the Sobolev embedding theorem for functions $\varphi \in W_0^{1,q}\left(G_1^2\right)$:

$$\left(\int_{G_1^2} |\varphi|^{\tilde{n}q} dx' \right)^{\frac{1}{\tilde{n}}} \le C \int_{G_1^2} |\nabla'\varphi|^q dx', \quad \tilde{n} = \frac{n}{n-1}, \ \forall q \ge 1, \tag{1.1}$$

where $x' = \frac{1}{\varrho}x$, $\varrho > R$. Our main theorem is the following:

Theorem 1.2. *Let u be a weak solution of problem (QL), $l = \max\left\{m(x) : x \in \overline{G_\varrho^{2\varrho}}\right\}$, λ_- be as in (2.4) and assumption (i)–(iv) be satisfied. Then there exist $R \gg 1$ and a positive constant C such that*

$$|u(x)| \le C \cdot |x|^{\lambda_-(1-\mu)} \quad \forall x \in G_R. \tag{1.2}$$

2 Preliminaries

2.1 Eigenvalue problem

We consider the eigenvalue problem for the Laplace–Beltrami operator Δ_ω on the unit sphere

$$\begin{cases} \Delta_\omega \psi + \vartheta\psi = 0, & \omega \in \Omega; \\ \psi(\omega) = 0, & \omega \in \partial\Omega, \end{cases} \tag{EVP}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has non-zero weak solutions $\psi(\omega) \neq 0$ (eigenfunctions).

Definition 2.1. A function ψ is said to be a weak solution of problem (EVP) provided that $\psi \in W_0^1(\Omega)$ and satisfies the integral identity

$$\int_\Omega \left(\frac{1}{q_i} \frac{\partial\psi}{\partial\omega_i} \frac{\partial\eta}{\partial\omega_i} - \vartheta\psi\eta \right) d\Omega = 0$$

for all $\eta(\omega) \in W_0^1(\Omega)$.

Throughout the paper we need only the least positive eigenvalue:

$$\vartheta_* := \inf_{\psi \in W_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_\omega \psi|^2 d\Omega}{\int_\Omega |\psi|^2 d\Omega}.$$

For the existence problem of the least positive eigenvalue to problem (EVP) see for example Section 8.2.3 [9].

2.2 The Friedrichs–Wirtinger type inequality

From the definition of $\vartheta_*(\Omega)$ we obtain the following Friedrichs–Wirtinger type inequality:

Theorem 2.2. *For all $\psi \in W_0^1(\Omega)$ the inequality*

$$\int_\Omega |\psi|^2 d\Omega \leq \frac{1}{\vartheta_*} \int\limits_\Omega |\nabla_\omega \psi|^2 d\Omega \tag{2.1}$$

holds with the sharp constant $\frac{1}{\vartheta_*}$.

Corollary 2.3. *Let $v(x) \in W_0^1(G_R)$. Then for any $\varrho > R$ and for all α*

$$\int_{G_\varrho} r^\alpha |v|^2 dx \leq \frac{1}{\vartheta_*} \int_{G_\varrho} r^{\alpha+2} |\nabla v|^2 dx \tag{2.2}$$

provided that the integral on the right is finite.

Proof. Consider the inequality (2.1) for the function $u(r, \omega)$. Multiplying it by $r^{\alpha+n-1}$ and integrating over $r \in (\varrho, \infty)$, we obtain the desired inequality. □

2.3 Auxiliary integro-differential inequalities

Lemma 2.4 (see Lemma 2.9 in [27]). *Let G_R be an unbounded cone-like domain and $\nabla u(\varrho, \cdot) \in L_2(\Omega)$ for almost all $\varrho \in (R, \infty)$. Suppose also that*

$$U(\varrho) = \int_{G_\varrho} r^{2-n} |\nabla u|^2 dx < \infty.$$

Then

$$\int_\Omega \left(\varrho u \frac{\partial u}{\partial r} + \frac{n-2}{2} u^2 \right) \bigg|_{r=\varrho} d\Omega \geq -\frac{\varrho}{2\lambda_-} U'(\varrho), \tag{2.3}$$

where λ_- is a negative number connected with ϑ_* by the equality

$$\lambda_- = \frac{2-n-\sqrt{(n-2)^2 + 4\vartheta_*}}{2}. \tag{2.4}$$

Theorem 2.5 (see Theorem 2.10 in [27]). *Suppose that $U(\varrho)$ is a monotonically decreasing, nonnegative differentiable function defined on $[R, \infty)$, $R \gg 1$, satisfying*

$$\begin{cases} U'(\varrho) + P(\varrho)U(\varrho) - Q(\varrho) \leq 0, & \varrho > R, \\ U(R) \leq U_0, \end{cases} \tag{CP}$$

where $P(\varrho), Q(\varrho)$ are nonnegative continuous functions defined on $[R, \infty)$ and U_0 is a constant. Then

$$U(\varrho) \leq U_0 \exp\left(-\int_R^\varrho P(\sigma) d\sigma \right) + \int_R^\varrho Q(t) \exp\left(-\int_t^\varrho P(\sigma) d\sigma \right) dt.$$

Now our aim is to estimate the gradient modulus of the problem (QL) solutions at infinity.

Lemma 2.6. *Let $u(x)$ be a weak solution of (QL) and assumptions (i)–(iv) hold. Then*

$$|\nabla u(x)| \leq M_1' |x|^{-1}, \quad \forall x \in G_R, \ R \gg 1. \tag{2.5}$$

We consider the solution u to the problem (QL) in the domain $G^{\varrho}_{\frac{\varrho}{2}} \subset G_R$, $\varrho > R$. We make the change of variables $x = \varrho x'$. Then the function $z(x') = u(\varrho x')$ satisfies the problem

$$\begin{cases} -\frac{d}{dx_i'}(\varrho^{m_- - m(\varrho x')} |\nabla' z|^{m(\varrho x') - 2} z_{x_i'}) + \varrho^{m_-} b(\varrho x', z, \varrho^{-1} \nabla' z) = 0, & x' \in G^1_{\frac{1}{2}}, \\ z(x') = 0, & x' \in \Gamma^1_{\frac{1}{2}}. \end{cases} \tag{QL'}$$

We verify that function $d(x') = \varrho^{m_- - m(\varrho x')}$ is Hölder continuous at infinity.

First of all, by the mean value Lagrange theorem, we have

$$|\varrho^{m_- - m(\varrho x')} - \varrho^{m_- - m(+\infty)}| = |m(+\infty) - m(\varrho x')| \cdot \varrho^t \ln \varrho,$$

where t is a negative number between $m_- - m(\varrho x')$ and $m_- - m(+\infty)$. Hence and by the Hölder assumption (ii), we get

$$|d(x') - d(+\infty)| = |\varrho^{m_- - m(\varrho x')} - \varrho^{m_- - m(+\infty)}| \leq M |x'|^{-\alpha} \varrho^{-\alpha} \ln \varrho.$$

Now, using first derivative test, we can conclude that

$$|a|^\delta |\ln|a|| \leq \frac{1}{\delta e}, \quad |a| < 1, \ \forall \delta > 0. \tag{2.6}$$

Thus, we obtain the required

$$|d(x') - d(+\infty)| \le \frac{M}{\alpha e}|x'|^{-\alpha}.$$

Further, assumptions (i), (iii) yield:

$$\varrho^{m_-}|b(\varrho x', z, \varrho^{-1}\nabla' z)| \le \mu|\nabla' z|^{m(\varrho x')}, \quad \varrho \gg 1$$

and therefore we can apply the X. Fan Theorem 1.2 and Remark 5.2 [13] about a priori estimate of the gradient modulus of the problem (QL') solution

$$\max_{x' \in G^1_{\frac{1}{2}}} |\nabla' z| \le M_1'.$$

Returning to variable x and function $u(x)$, we obtain

$$|\nabla u| \le M_1' \varrho^{-1}, \quad x \in G^{\varrho}_{\frac{\varrho}{2}},\ \varrho > R.$$

Setting now $|x| = \frac{2}{3}\varrho$ we obtain the required (2.5).

Lemma 2.7. *Let u be a weak solution of problem (QL) and assumptions (i)–(iv) be satisfied. Then we have:*

$$\int_{G_{2R}} r^{2-n}|\nabla u|^{m(x)}dx < \infty, \qquad \int_{G_{2R}} r^{2-n}|\nabla u|^2 dx < \infty \tag{2.7}$$

$$\lim_{\mathcal{N}\to+\infty} \mathcal{N}^{-1}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = 0. \tag{2.8}$$

Proof. At first we will show the convergence of the first integral. We set $r_k = 2^k \cdot R$, $k = 0, 1, 2, \ldots$ and let $\eta_k \in C_0^\infty(G_{r_k})$ with the following properties:

$$\begin{cases} 0 \le \eta_k \le 1,\ |\nabla \eta_k| \le c \cdot r_k^{-1} & x \in G_{r_k} \\ \eta_k = 1 & x \in G_{r_{k+1}}. \end{cases}$$

We choose $\eta = u\eta_k^{m^+}$ as a test function in (II). Then we obtain:

$$\int_{G_{r_k}} |\nabla u|^{m(x)}\eta_k^{m_+}dx = -\int_{G_{r_k}} \left(m_+ u|\nabla u|^{m(x)-2}\nabla u \nabla\eta_k \cdot \eta_k^{m_+-1} + b(x,u,u_x)\cdot u \cdot \eta_k^{m_+}\right)dx. \tag{2.9}$$

Next, using the Young inequality, with $q = \frac{m(x)}{m(x)-1}$, $q' = m(x)$, we get

$$m_+|u||\nabla u|^{m(x)-1}\cdot|\nabla\eta_k|\cdot\eta_k^{m_+-1} = \left(m_+|u||\nabla\eta_k|\eta_k^{\frac{m_+-m(x)}{m(x)}}\right)\cdot\left(|\nabla u|^{m(x)-1}\eta_k^{\frac{m_+(m(x)-1)}{m(x)}}\right)$$
$$\le \frac{m_+^{m(x)}}{m(x)}|u|^{m(x)}|\nabla\eta_k|^{m(x)}\eta_k^{m_+-m} + \frac{m(x)-1}{m(x)}|\nabla u|^{m(x)}\eta_k^{m_+}.$$

Thus, from (2.9) we get

$$\int_{G_{r_k}} |\nabla u|^m \eta_k^{m_+}dx \le c(m_-, m_+)\int_{G_{r_k}} |u|^m|\nabla\eta_k|^m\eta_k^{m_+-m(x)}dx + m_+\int_{G_{r_k}} |b(x,u,u_x)||u|\eta_k^{m_+}dx.$$

Next, using assumption (iii), the inequality above yields

$$(1-m_+\mu)\int_{G_{r_k}}|\nabla u|^m\eta_k^{m_+}dx\le c(m_-,m_+)\int_{G_{r_k}}|u|^m|\nabla\eta_k|^{m(x)}\eta_k^{m_+-m(x)}dx.$$

In view of the choice of η_k, we get

$$(1-m_+\mu)\int_{G_{r_{k+1}}^{r_{k+2}}}|\nabla u|^m dx\le\widetilde{c}_1(m_-,m_+)\int_{G_{r_k}^{r_{k+1}}}|u|^m r^{-m}dx. \tag{2.10}$$

We use the fact [2] that any solution u is Hölder continuous in G_R:

$$|u|\le H_0|x|^{-\alpha_0},\quad \forall x\in G_R.$$

Hence, by assumption (ii) and because

$$\lim_{r\to+\infty}r^{r^{-\alpha}}=1, \tag{2.11}$$

we can estimate

$$\begin{aligned}|u|^{m(x)}&\le(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0(2-m)}=(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0(m(+\infty)-m(x))}\\&\le(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0Mr^{-\alpha}}\le C(H_0,M,\alpha_0,\alpha,m_+)\cdot r^{-2\alpha_0},\quad x\in G_R;\\ r^{-m(x)}&=r^{-2}\cdot r^{2-m(x)}\le r^{-2}\cdot r^{Mr^{-\alpha}}\le C(M,\alpha)r^{-2}.\end{aligned} \tag{2.12}$$

Hence

$$|u|^{m(x)}\cdot r^{-m(x)}\le C(H_0,M,\alpha,\alpha_0,m_+)r^{-2-2\alpha_0},\quad x\in G_R. \tag{2.13}$$

In this way, from (2.10)

$$\int_{G_{r_{k+1}}^{r_{k+2}}}|\nabla u|^m dx\le C_2(H_0,M,\alpha,\alpha_0,m_\pm)\int_{G_{r_k}^{r_{k+1}}}r^{-2(\alpha_0+1)}dx. \tag{2.14}$$

Multiplying both sides of (2.14) by r_k^{2-n}, by the definition of r_k, we find

$$\int_{G_{r_{k+1}}^{r_{k+2}}}r^{2-n}|\nabla u|^m dx\le C_2\int_{G_{r_k}^{r_{k+1}}}r^{-2\alpha_0-n}dx.$$

Summing up above inequalities for all $k=0,1,2,\dots$, we obtain

$$\int_{G_{2R}}r^{2-n}|\nabla u|^{m(x)}\le C_2\int_{G_R}r^{-2\alpha_0-n}dx\le C_2|\Omega|\int_R^\infty r^{-2\alpha_0-1}dr=C_3\cdot R^{-2\alpha_0}. \tag{2.15}$$

Thus, the convergence of the first integral in (2.7) is proved.

Now we observe that, in virtue of (2.5), (ii) and (2.11), we get

$$|\nabla u|^2=|\nabla u|^{m(x)}|\nabla u|^{2-m(x)}\le C|\nabla u|^{m(x)}r^{Mr^{-\alpha}}\le C|\nabla u|^{m(x)},$$

which, by (2.15), yields the convergence of the second integral in (2.7).

We shall prove (2.8). Applying the Young inequality with $q=\frac{m(x)}{m(x)-1}$, $q'=m(x)$ we have

$$\begin{aligned}
&\left|\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx\right| \\
&\quad \le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-1}|u|dx \\
&\quad = \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} \left(r^{(3-n)\frac{m(x)-1}{m(x)}}|\nabla u|^{m(x)-1}\right)\cdot\left(r^{\frac{3-n-m(x)}{m(x)}}|u|\right)dx \\
&\quad \le \left(\mathcal{N}+\frac{1}{\mathcal{N}}\right)\left(\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx+\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m(x)}|u|^{m(x)}dx\right).
\end{aligned}$$

We can estimate the first integral using (2.5) and (2.12) in the following way:

$$\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx \le c\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m}dx \le C(M,M_1',\alpha,|\Omega|)\int_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}\frac{1}{r}dr = C\ln\left(1+\frac{1}{\mathcal{N}^2}\right),$$

while the second integral using (2.13):

$$\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m(x)}|u|^{m(x)}dx \le C\int_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}} r^{2-n}\cdot r^{-2-2\alpha_0}\cdot r^{n-1}dr \le C\mathcal{N}^{-2\alpha_0}.$$

From above inequalities we get

$$\begin{aligned}
&\lim_{\mathcal{N}\to+\infty}\mathcal{N}^{-1}\left|\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx\right| \\
&\quad \le \lim_{\mathcal{N}\to+\infty} C\cdot\left(1+\frac{1}{\mathcal{N}^2}\right)\cdot\left\{\ln\left(1+\frac{1}{\mathcal{N}^2}\right)+\mathcal{N}^{-2\alpha_0}\right\}=0,
\end{aligned}$$

which is the required (2.8). □

We indicate another consequence of the integral identity (*II*) for solutions u to the problem (*QL*) which is essentially used in the further consideration.

Lemma 2.8. *If assumptions* (i)–(iv) *are satisfied, then*

$$\begin{aligned}
&\int_{G_\varrho} r^{2-n}|\nabla u|^{m(x)}+(2-n)\int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx \\
&\quad +\int_{G_\varrho} r^{2-n}ub(x,u,u_x)dx = -\varrho^{2-n}\int_{\Omega_\varrho}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}d\Omega_\varrho,\ \forall\varrho\ge 4R\gg 1. \quad (2.16)
\end{aligned}$$

Proof. Let $\mathcal{N}>\varrho\ge 4R$. On $[R,\infty)$ we consider a Lipschitz piecewise linear function $\eta_{\mathcal{N}}(t)$ defined by

$$\eta_{\mathcal{N}}(t)=\begin{cases}0, & \text{if } t\in[4R,\varrho]\cup[\mathcal{N}+\frac{1}{\mathcal{N}},\infty),\\ 1, & \text{if } t\in[\varrho+\frac{1}{\mathcal{N}},\mathcal{N}],\\ \mathcal{N}(t-\varrho), & \text{if } t\in[\varrho,\varrho+\frac{1}{\mathcal{N}}],\\ \mathcal{N}(\mathcal{N}-t)+1, & \text{if } t\in[\mathcal{N},\mathcal{N}+\frac{1}{\mathcal{N}}]\end{cases}$$

$$\Longrightarrow\quad \eta_{\mathcal{N}}'(t)=\begin{cases}0, & \text{if } t\in[4R,\varrho)\cup(\varrho+\frac{1}{\mathcal{N}},\mathcal{N})\cup(\mathcal{N}+\frac{1}{\mathcal{N}},\infty),\\ \mathcal{N}, & \text{if } t\in(\varrho,\varrho+\frac{1}{\mathcal{N}}),\\ -\mathcal{N}, & \text{if } t\in(\mathcal{N},\mathcal{N}+\frac{1}{\mathcal{N}})\end{cases}$$

and take a test function $\eta(x) = r^{2-n}\eta_{\mathcal{N}}(r)u(x)$ in the integral identity (II). Calculating

$$\eta_{x_i} = r^{2-n}\eta_{\mathcal{N}}(r)u_{x_i} + u(x)\cdot\left((2-n)r^{1-n}\frac{x_i}{r}\eta_{\mathcal{N}}(r) + r^{2-n}\frac{x_i}{r}\eta'_{\mathcal{N}}(r)\right),$$

we arrive at the equality

$$\begin{aligned}
&\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)\mathcal{N}(r-\varrho)dx\\
&\quad+\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)dx\\
&\quad+\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)\cdot\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx\\
&= -\mathcal{N}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx + \mathcal{N}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx.
\end{aligned}$$

First of all we observe that by assumption (iii) we have $ub(x,u,u_x) \le \mu|\nabla u|^{m(x)}$.
In virtue of (2.7) it is clearly that

$$\begin{aligned}
\lim_{\mathcal{N}\to+\infty}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}\cup G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx &= 0,\\
\lim_{\mathcal{N}\to+\infty}\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx &= \int_{G_\varrho} r^{2-n}|\nabla u|^{m(x)}dx.
\end{aligned}\tag{2.17}$$

Since

$$\begin{aligned}
0 &\le \int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\mathcal{N}(r-\varrho)dx \le \int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx,\\
0 &\le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx \le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx,
\end{aligned}$$

by (2.17), we get

$$\lim_{\mathcal{N}\to+\infty}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\mathcal{N}(r-\varrho)dx = \lim_{\mathcal{N}\to+\infty}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx = 0.$$

Applying now the Young inequality with $q = \frac{m(x)}{m(x)-1}$, $q' = m(x)$ we have

$$\begin{aligned}
\left|\int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx\right| &\le \int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-1}|u|dx\\
&= \int_{G_\varrho}\left(r^{(2-n)\frac{m(x)-1}{m(x)}}|\nabla u|^{m(x)-1}\right)\cdot\left(r^{\frac{2-m(x)-n}{m(x)}}|u|\right)dx\\
&\le \int_{G_\varrho} r^{2-n}|\nabla u|^{m(x)}dx + \int_{G_\varrho} r^{2-m(x)-n}|u|^{m(x)}dx\\
&\le C\varrho^{-2\alpha_0}, \qquad \varrho \in (R,\infty),
\end{aligned}$$

by (2.13) and (2.15). Consequently

$$\lim_{\mathcal{N}\to+\infty}\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = \int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx.$$

Now we consider the integral

$$\left|\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx\right| \le \frac{1}{\mathcal{N}}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}|u|\left|\frac{\partial u}{\partial r}\right|dx$$

and hence, by (2.8)

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx = 0.$$

Next, because of (2.18),

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = 0,$$

and therefore we can apply the L'Hospital rule:

$$\begin{aligned}\varrho\cdot\lim_{\mathcal{N}\to+\infty}\mathcal{N}\cdot\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx &= \varrho\cdot\lim_{\mathcal{N}\to+\infty}\frac{\int_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}\left(\int_{\Omega}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\right)d\Omega dr}{\mathcal{N}^{-1}}\\ &= \varrho^{2-n}\int_{\Omega_{\varrho}}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}d\Omega\varrho\end{aligned}$$

and

$$\lim_{\mathcal{N}\to+\infty}\mathcal{N}\cdot\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = \varrho^{2-n}\int_{\Omega_{\varrho}}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}d\Omega\varrho.$$

Hence

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\mathcal{N}(r-\varrho)dx = 0.$$ □

3 Local estimate of the weighted Dirichlet integral

Theorem 3.1. *Let u be a weak solution of problem (QL) and assumptions (i)–(iv) be satisfied. Let λ_- be as in (2.4). Then there exist $R \gg 1$ and a constant $C > 0$ such that*

$$\int_{G_{\varrho}} r^{2-n}|\nabla u|^2dx \le C\varrho^{2\lambda_-(1-\mu)},\quad \forall\varrho > R.$$

Proof. We rewrite the inequality (2.16) in the form:

$$\begin{aligned}U(\varrho) = \int_{G_{\varrho}} r^{2-n}|\nabla u|^2dx &= \int_{G_{\varrho}} r^{2-n}(|\nabla u|^2-|\nabla u|^{m(x)})dx - \int_{G_{\varrho}} r^{2-n}ub(x,u,u_x)dx\\ &+(2-n)\int_{G_{\varrho}} r^{1-n}(1-|\nabla u|^{m(x)-2})uu_rdx + (n-2)\int_{G_{\varrho}} r^{1-n}uu_rdx\\ &-\varrho^{2-n}\int_{\Omega_{\varrho}}(|\nabla u|^{m(x)-2}-1)uu_rd\Omega_{\varrho} - \varrho^{2-n}\int_{\Omega_{\varrho}}uu_rd\Omega_{\varrho}.\end{aligned}\tag{3.1}$$

Now, we observe that

$$\int_{G_{\varrho}} r^{1-n}uu_rdx = -\frac{1}{2}\varrho^{1-n}\int_{\Omega_{\varrho}}u^2d\Omega\varrho.\tag{3.2}$$

In fact, we get

$$\int_{G_\varrho^{\mathcal{N}}} r^{1-n} u u_r dx = \int_\Omega \int_\varrho^{\mathcal{N}} u u_r dr d\Omega = \frac{1}{2}\int_\Omega \int_\varrho^{\mathcal{N}} \frac{\partial u^2}{\partial r} dr d\Omega = \frac{1}{2}\int_\Omega (u^2(\mathcal{N},\omega) - u^2(\varrho,\omega)) d\Omega$$
$$= \frac{1}{2}\int_\Omega u^2(\mathcal{N},\omega) d\Omega - \frac{1}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho.$$

Passing to the limit $\mathcal{N} \to +\infty$ we obtain (3.2).

By assumption (iii), we get

$$\left|\int_{G_\varrho} r^{2-n} u b(x,u,u_x) dx\right| \le \mu \int_{G_\varrho} r^{2-n} |\nabla u|^2 dx + \mu \int_{G_\varrho} r^{2-n} \left| |\nabla u|^{m(x)} - |\nabla u|^2 \right| dx.$$

Hence and from (3.1), (3.2) it follows that

$$\begin{aligned}(1-\mu)U(\varrho) \le (1+\mu) &\int_{G_\varrho} r^{2-n} \left| |\nabla u|^{m(x)} - |\nabla u|^2 \right| dx \\ &+ (n-2)\int_{G_\varrho} r^{1-n} \left| 1 - |\nabla u|^{m(x)-2} \right| |u||u_r| dx - \frac{n-2}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho \\ &+ \varrho^{2-n}\int_{\Omega_\varrho} \left| |\nabla u|^{m(x)-2} - 1 \right| |u||u_r| d\Omega_\varrho - \varrho^{2-n}\int_{\Omega_\varrho} u u_r d\Omega_\varrho. \qquad (3.3)\end{aligned}$$

Let us estimate the integrals:

$$I_1(\varrho) = \int_{G_\varrho} r^{2-n} \left| |\nabla u|^{m(x)} - |\nabla u|^2 \right| dx,$$
$$I_2(\varrho) = \int_{G_\varrho} r^{1-n} \left| 1 - |\nabla u|^{m(x)-2} \right| |u||u_r| dx,$$
$$I_3(\varrho) = \int_{\Omega_\varrho} \left| |\nabla u|^{m(x)-2} - 1 \right| |u||u_r| d\Omega_\varrho.$$

To estimate them we set

$$F_1 = \{x : \ x \in \overline{G_\varrho}, \ |\nabla u| < |x|^\gamma\},$$
$$F_2 = \{x : \ x \in \overline{G_\varrho}, \ |x|^\gamma \le |\nabla u| \le M_1'|x|^{-1}\},$$

where the constant $\gamma < -1$ will be defined above.

By assumption (ii) and (2.11) for any $x \in F_1$, we get

$$\begin{aligned}|\nabla u|^2 + |\nabla u|^m &< |x|^{2\gamma} + |x|^{\gamma(m-2)} \cdot |x|^{2\gamma} \qquad (3.4) \\ &\le |x|^{2\gamma} + |x|^{-\gamma M |x|^{-\alpha}} \cdot |x|^{2\gamma} \le C_1(M,\gamma,\alpha) \cdot |x|^{2\gamma}.\end{aligned}$$

In this way

$$\int_{F_1} r^{2-n} \left| |\nabla u|^2 - |\nabla u|^{m(x)} \right| dx \le C_2 \cdot \varrho^{2\gamma+2}.$$

Next, (ii) yields for $x \in F_2$, that

$$\begin{aligned}|\nabla u|^2 + |\nabla u|^{m(x)} &= |\nabla u|^2(1 + |\nabla u|^{m(x)-2}) \qquad (3.5) \\ &\le |\nabla u|^2(1 + |x|^{-M\gamma|x|^{-\alpha}}) \le C_3(M,\alpha)|\nabla u|^2,\end{aligned}$$

because

$$(m(x)-2)\ln|\nabla u| \le -M|x|^{-\alpha}\ln|\nabla u| \le -M|x|^{-\alpha}\ln|x|^{\gamma}.$$

Hence, once again in virtue of (ii) and by the inequality

$$\left||z|^{t_2}-|z|^{t_1}\right| \le \frac{1}{2}|t_2-t_1|(|z|^{t_1}+|z|^{t_2})|\ln|z||,\ z\in\mathbb{R}\setminus\{0\},\ t_1\ge 0,\ t_2\ge 0 \tag{3.6}$$

(see Proposition 2.1 in [1]), we obtain

$$\left||\nabla u|^2-|\nabla u|^{m(x)}\right| \le \frac{1}{2}|m(x)-2|(|\nabla u|^{m(x)}+|\nabla u|^2)|\ln|\nabla u|| \le \frac{MC_3}{2}|x|^{-\alpha}|\nabla u|^2|\ln|\nabla u||.$$

Applying inequality (2.6) with $\delta=-\frac{\alpha}{2\gamma}$, we get

$$|\ln|\nabla u|| \le |\ln|x|^{\gamma}| \le \frac{-2\gamma}{\alpha e}|x|^{\frac{\alpha}{2}} \quad x\in F_2. \tag{3.7}$$

Eventually, we find that

$$I_1 \le C_4\varrho^{-\frac{\alpha}{2}}U(\varrho)+C\varrho^{2\gamma+2}. \tag{3.8}$$

Integrals I_2 and I_3 are estimated similarly. Arguing as in (3.4), (3.5), we establish that

$$|\nabla u|+|\nabla u|^{m(x)-1} \le C|x|^{\gamma} \quad \forall x\in F_1, \tag{3.9}$$

$$|\nabla u|+|\nabla u|^{m(x)-1} \le C|\nabla u| \quad \forall x\in F_2. \tag{3.10}$$

From (3.9) and by our assumption about Hölder continuity we get

$$\int_{F_1} r^{1-n}\left||1-|\nabla u|^{m(x)-2}\right||u||u_r|dx \le C\int_{F_1} r^{1-n}|x|^{\gamma}|u|dx \le C\varrho^{\gamma-\alpha_0+1}, \tag{3.11}$$

$$\int_{\Omega_\varrho\cap F_1}\left||1-|\nabla u|^{m(x)-2}\right||u||u_r|d\Omega_\varrho \le C\varrho^{\gamma-\alpha_0+n-1}. \tag{3.12}$$

Repeating steps (3.6)–(3.7) and using (3.10), we have

$$\left||\nabla u|^{m(x)-1}-|\nabla u|\right| \le C_5|\nabla u||x|^{-\frac{\alpha}{2}} \tag{3.13}$$

on the set F_2. Thus

$$\begin{aligned}
\int_{F_2} r^{1-n}\left|1-|\nabla u|^{m(x)-2}\right||u_r||u|dx &\le C\int_{F_2} r^{1-n-\frac{\alpha}{2}}|\nabla u||u|dx \\
&\le C\varrho^{-\frac{\alpha}{2}}\int_{G_\varrho} r^{1-n}|\nabla u||u|dx = C\varrho^{-\frac{\alpha}{2}}\int_{G_\varrho}\left(r^{1-\frac{n}{2}}|\nabla u|\right)\cdot\left(r^{-\frac{n}{2}}|u|\right)dx \\
&\le C\varrho^{-\frac{\alpha}{2}}\left(\int_{G_\varrho} r^{2-n}|\nabla u|^2dx\right)^{1/2}\cdot\left(\int_{G_\varrho} r^{-n}u^2dx\right)^{1/2} \le C\varrho^{-\frac{\alpha}{2}}\cdot\frac{1}{\vartheta_*}\int_{G_\varrho} r^{2-n}|\nabla u|^2dx
\end{aligned}$$

in virtue of the Hardy–Wirtinger inequality (2.2), where ϑ_* is the smallest positive eigenvalue of the Dirichlet problem for the Laplace–Beltrami operator in the domain Ω. Using (3.11), we obtain the estimate

$$I_2 \le C\varrho^{-\frac{\alpha}{2}}\cdot\frac{1}{\vartheta_*}U(\varrho)+C\varrho^{\gamma-\alpha_0+1}. \tag{3.14}$$

Now, by (3.13), we have

$$\int_{\Omega_\varrho\cap F_2}\left||\nabla u|^{m(x)-2}-1\right||u||u_r|d\Omega_\varrho \le C\varrho^{-\frac{\alpha}{2}}\int_{\Omega_\varrho}|u_r||u|d\Omega_\varrho.$$

Taking into account (3.12) we find that

$$I_3 \le C\varrho^{-\frac{\alpha}{2}} \int_{\Omega_\varrho} |u_r||u| d\Omega_\varrho + C\varrho^{\gamma-\alpha_0+n-1}. \tag{3.15}$$

Thus, inserting (3.8), (3.14), (3.15) into (3.3), we obtain

$$\begin{aligned}\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le C\varrho^{1-\frac{\alpha}{2}} & \int_{\Omega} |u_r||u| d\Omega \\ & -\frac{n-2}{2}\int_{\Omega} u^2 d\Omega - \varrho\int_{\Omega} u u_r d\Omega + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).\end{aligned} \tag{3.16}$$

Now we can use Lemma 2.4. Hence, (3.16) takes the following form

$$\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le \frac{\varrho}{2\lambda_-}U'(\varrho) + C\varrho^{1-\frac{\alpha}{2}}\int_{\Omega}|\nabla u||u| d\Omega + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).$$

Applying the Cauchy inequality and (2.1), we have

$$\varrho\int_{\Omega}|u||\nabla u| d\Omega \le \frac{1}{2}\int_{\Omega}(\varrho^2|\nabla u|^2+|u|^2)d\Omega \le -c_1(\vartheta_*)\varrho U'(\varrho).$$

Thus we get

$$\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le \frac{\varrho}{2\lambda_-}(1+\tilde{C}\varrho^{-\frac{\alpha}{2}})U'(\varrho) + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})$$

or

$$U'(\varrho) - \frac{2\lambda_-}{\varrho}\cdot\frac{1-\mu-C\varrho^{-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}}U(\varrho) + 2\lambda_- C\cdot\frac{\varrho^{2\gamma+1}+\varrho^{\gamma-\alpha_0}}{1+\tilde{C}\varrho^{\frac{-\alpha}{2}}} \le 0.$$

In this way we have the Cauchy problem (CP) with

$$\begin{aligned}P(\varrho) &= -\frac{2\lambda_-}{\varrho}\cdot\frac{1-\mu-C\varrho^{-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}},\\ Q(\varrho) &= -2\lambda_- C\cdot\frac{\varrho^{2\gamma+1}+\varrho^{\gamma-\alpha_0}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}}.\end{aligned}$$

Now we show that $U(R) \le U_0 = \text{const}$. We can rewrite inequality (3.16) in the following form

$$\begin{aligned}\left((1-\mu)-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le (1+C\varrho^{-\frac{\alpha}{2}})\varrho^{2-n} & \int_{\Omega_\varrho}|\nabla u||u| d\Omega_\varrho \\ & +\frac{n-2}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).\end{aligned}$$

Hence

$$\frac{1-\widetilde{C}\varrho^{-\frac{\alpha}{2}}}{1+C\varrho^{-\frac{\alpha}{2}}}U(\varrho) \le \frac{1}{1-\mu}\varrho^{2-n}\int_{\Omega_\varrho}|\nabla u||u| d\Omega_\varrho + \frac{n-2}{2(1-\mu)}\varrho^{1-n}\int_{\Omega_\varrho}u^2 d\Omega_\varrho + \frac{\widetilde{C}(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})}{1+C\varrho^{-\frac{\alpha}{2}}}.$$

Since $\gamma < -1$ for sufficiently large $\varrho \ge 1$, we have

$$\frac{1-\widetilde{C}\varrho^{-\frac{\alpha}{2}}}{1+C\varrho^{-\frac{\alpha}{2}}} \ge 1-\varrho^{-\frac{\alpha}{4}} \quad \text{and} \quad \frac{\widetilde{C}(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})}{1+C\varrho^{-\frac{\alpha}{2}}} \le \varrho^{\gamma+1}.$$

In this way

$$(1-\varrho^{-\frac{\alpha}{4}})U(\varrho) \le \frac{1}{1-\mu}\int_{\Omega}(\varrho|\nabla u||u| + \frac{n-2}{2}u^2)d\Omega + \varrho^{\gamma+1}.$$

Hence, from (2.5) it follows that $U(R) < \infty$.

All assumptions of Theorem 2.5 are satisfied. Since

$$-P(\varrho) = \frac{2\lambda_-(1-\mu)}{\varrho} - \frac{2\lambda_-(1-\mu)c_2\varrho^{-1-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}} \le \frac{2\lambda_-(1-\mu)}{\varrho} - 2\lambda_-(1-\mu)c_2\varrho^{-1-\frac{\alpha}{2}}$$

it follows that

$$-\int_R^{\varrho} P(\sigma)d\sigma \le 2\lambda_-(1-\mu)\int_R^{\varrho}\left(\frac{1}{\sigma} - c_2\sigma^{-\frac{\alpha}{2}-1}\right)d\sigma \le \ln\left(\frac{\varrho}{R}\right)^{2\lambda_-(1-\mu)} + c_3(\lambda_-,\mu,R,\vartheta_*)$$

which yields

$$\exp\left(-\int_R^{\varrho} P(\sigma)d\sigma\right) \le c_4\cdot\left(\frac{\varrho}{R}\right)^{2\lambda_-(1-\mu)}.$$

Next, because

$$Q(\varrho) \le -2C\lambda_-(\varrho^{2\gamma+1} + \varrho^{\gamma-\alpha_0}),$$

choosing $\gamma = -1 + 2\lambda_-(1-\mu)$ we have:

$$\begin{aligned}&\int_R^{\varrho} Q(t)\exp\left(-\int_t^{\varrho} P(\sigma)d\sigma\right)dt\\ &\qquad \le -2\lambda_-\cdot c_5\cdot\varrho^{2\lambda_-(1-\mu)}\int_R^{\varrho}(t^{-1-\alpha_0} + t^{2\lambda_-(1-\mu)-1})dt \le c_6\varrho^{2\lambda_-(1-\mu)}.\end{aligned}$$

Eventually, by Theorem 2.5 we get

$$U(\varrho) \le C\varrho^{2\lambda_-(1-\mu)}.$$

□

4 Local estimate at infinity

The weak solution of problem (QL) is locally bounded at infinity. More precisely, we have

Theorem 4.1. *Let u be a weak solution of problem (QL) and assumptions (i)–(iv) be satisfied. Then for any $k < 0$, $\varkappa \in (1,2)$, $\varrho > R$ with $R \gg 1$ the inequality*

$$\sup_{x\in G^{2\varrho}_{\varrho\varkappa}}|u| \le C^*\left(\varrho^{-\frac{n}{t}}\|u\|_{t,G^{2\varrho}_{\varrho}} + \varrho^k\right),$$

holds, where constant C^* depends only $m_+, m_-, \mu, M, M_1', \alpha, R, k, n, \varkappa$.

Proof. Set

$$l = \max_{\overline{G}^{2\varrho}_{\varrho}} m(x).$$

Let us consider the case $t \ge l > 1$. We make the coordinate transformation $x = \varrho x'$, $\varrho > R$ in the integral identity (II). Let $v(x') = u(\varrho x')$. We choose a test function η as

$$\eta(\varrho x') = v(x')\overline{v}^{t-l}(x')\zeta^l(|x'|),$$

where $\overline{v} = |v| + \varrho^k$ with a certain $k < 0$, $\zeta(|x'|) \in C_0^\infty([1,2])$ with the property that $0 \le \zeta(x') \le 1$ for $x' \in [1,2]$. Then (II) takes the following form

$$\int_{G_1^2} \Big[\overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \varrho^{-m(\varrho x')} \left(1 + (t-l)\frac{|v|}{\overline{v}} \right) \zeta^l \\ + l v \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')-2} \varrho^{-m(\varrho x')} \zeta^{l-1} v_{x_i'} \zeta_{x_i'} + v\overline{v}^{t-l} b(\varrho x', v, \varrho^{-1} v_{x'}) \zeta^l \Big] dx' = 0.$$

Now, in virtue of $(t-l)\frac{|v|}{\overline{v}} \ge 0$, it follows that

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \varrho^{-m(\varrho x')} \zeta^l \le l \int_{G_1^2} |v| \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')-1} \varrho^{-m(\varrho x')} \zeta^{l-1} |\nabla' \zeta| dx' \\ + \int_{G_1^2} |v| \overline{v}^{t-l} |b(\varrho x', v, \varrho^{-1} v_{x'})| \zeta^l dx'.$$

Now, by assumption (iii) regarding that $|v| < \overline{v}$ and in virtue of $\varrho^{-m(\varrho x')} \ge \varrho^{-l}$ we obtain

$$(1-\mu) \int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le l \int_{G_1^2} \overline{v}^{t-l+1} |\nabla' v|^{m(\varrho x')-1} \varrho^{l-m(\varrho x')} \zeta^{l-1} |\nabla' \zeta| dx'. \tag{4.1}$$

Next, by assumption (ii) we can estimate for all $x', x_2' \in G_1^2$:

$$l - m(\varrho x') = m(\varrho x_2') - m(\varrho x') \le M\varrho^{-\alpha}(|x_2'|^{-\alpha} + |x'|^{-\alpha}) \le 2M\varrho^{-\alpha}. \tag{4.2}$$

This estimation, with regard to (2.11) implies that

$$\varrho^{l-m(\varrho x')} \le \varrho^{2M\varrho^{-\alpha}} \le C.$$

For estimating the integral from the right-hand side of (4.1), we apply the Young inequality with $p = \frac{m(\varrho x')}{m(\varrho x')-1}$, $q = m(\varrho x')$, $\delta = \frac{\tilde{\delta}}{l}$:

$$\overline{v} |\nabla' v|^{m(\varrho x')-1} \zeta^{-1} |\nabla' \zeta| = \left(|\nabla' v|^{m(\varrho x')-1} \right) \left(\overline{v} \zeta^{-1} |\nabla' \zeta| \right) \\ \le \frac{\tilde{\delta}}{l} |\nabla' v|^{m(\varrho x')} + \left(\frac{\tilde{\delta}}{l} \right)^{1-m(\varrho x')} \cdot \overline{v}^{m(\varrho x')} \zeta^{-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')}.$$

Hence, (4.1) takes the following form:

$$(1-\mu-\tilde{\delta}) \int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le \int_{G_1^2} \tilde{\delta}^{1-m(\varrho x')} \cdot l^{m(\varrho x')} \cdot \overline{v}^{t-l+m(\varrho x')} \zeta^{l-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx'.$$

Choosing $\tilde{\delta} = \frac{1-\mu}{2}$, we get

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le \int_{G_1^2} \left(\frac{2l}{1-\mu} \right)^{m(\varrho x')} \overline{v}^{t-l+m(\varrho x')} \zeta^{l-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx'.$$

Now we observe that $\zeta^{l-m(\varrho x')} \le 1$ for $x' \in G_1^2$, because $0 \le \zeta \le 1$ and $\left(\frac{2l}{1-\mu}\right)^{m(\varrho x')} \le \left(\frac{2l}{1-\mu}\right)^l$. By these arguments, we obtain

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le C_1 \int_{G_1^2} \overline{v}^{t-l+m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx', \tag{4.3}$$

where $C_1 = \left(\frac{2l}{1-\mu}\right)^l$. Now our aim is to estimate the integral from the left hand side. For this purpose we write

$$|\nabla' v|^l = |\nabla' v|^{m(\varrho x')} \cdot |\nabla' v|^{l-m(\varrho x')}.$$

If $|\nabla' v| \leq 1$, then $|\nabla' v|^l \leq |\nabla' v|^{m(\varrho x')}$. Let $1 < |\nabla' v| \leq M_1'$. Hence, by (4.2):

$$|\nabla' v|^{l-m(\varrho x')} \leq |\nabla' v|^{2M\varrho^{-\alpha}} \leq M_1'^{2M\varrho^{-\alpha}} \leq C(M, M_1', \alpha, R).$$

Thus

$$|\nabla' v|^l \leq C|\nabla' v|^{m(\varrho x')}. \tag{4.4}$$

Further, in virtue of $\overline{v} \geq \varrho^k, k < 0$, by (2.11), (4.2):

$$\overline{v}^{m(\varrho x')-l} \leq \varrho^{k(m-l)} \leq \varrho^{-2M\varrho^{-\alpha}k} \leq C(M, k, \alpha). \tag{4.5}$$

From (4.3), (4.4) and (4.5) it follows that

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^l \zeta^l dx' \leq C \int_{G_1^2} \overline{v}^t |\nabla' \zeta|^{m(\varrho x')} dx'. \tag{4.6}$$

Applying now the Sobolev embedding theorem's formula (1.1) for $\varphi = \overline{v}^{\frac{t}{l}} \zeta,\ q = l$, we obtain

$$\|\overline{v}^t \zeta^l\|_{\tilde{n}, G_1^2} \leq C \int_{G_1^2} \left(t^l \overline{v}^{t-l} |\nabla' v|^l \zeta^l + \overline{v}^t |\nabla' \zeta|^l \right) dx', \quad \tilde{n} = \frac{n}{n-1}. \tag{4.7}$$

Eventually, from (4.6), (4.7):

$$\|\overline{v}^t \zeta^l\|_{\widetilde{n}, G_1^2} \leq Ct^l \int_{G_1^2} \overline{v}^t (|\nabla' \zeta|^{m(\varrho x')} + |\nabla' \zeta|^l) dx'. \tag{4.8}$$

For any $\varkappa \in (1, 2)$ we define sets $G'_{(j)} \equiv G^2_{\varkappa-(\varkappa-1)2^{-j}}$, $j = 0, 1, 2, \ldots$ We see at once that

$$G^2_\varkappa \equiv G'_{(\infty)} \subset \ldots \subset G'_{(j+1)} \subset G'_{(j)} \subset \ldots \subset G'_{(0)} \equiv G^2_1.$$

Now we consider the sequence of cut-off functions $\zeta_j(x') \in C^\infty(G'_{(j)})$ such that

$$\begin{gathered}
0 \leq \zeta_j(x') \leq 1 \text{ in } G'_{(j)} \quad \text{and} \quad \zeta_j(x') \equiv 1 \text{ in } G'_{(j+1)}, \\
\zeta_j(x') \equiv 0 \quad \text{for } 1 < |x'| < \varkappa - 2^{-j}(\varkappa - 1); \\
|\nabla \zeta_j'| \leq \frac{2^{j+1}}{\varkappa - 1} \quad \text{for } \varkappa - 2^{-j}(\varkappa - 1) < |x'| < \varkappa - 2^{-j-1}(\varkappa - 1)
\end{gathered}$$

and the number sequence $t_j = t\widetilde{n}^j,\ j = 0, 1, 2, \ldots$. We rewrite the inequality (4.8) replacing ζ by ζ_j and t by t_j. As a result, by virtue of properties of functions ζ_j, we obtain

$$\left(\int_{G'_{(j+1)}} \overline{v}^{\widetilde{n} t_j} dx' \right)^{\frac{1}{\widetilde{n}}} \leq Ct_j^l \int_{G'_{(j)}} \overline{v}^{t_j} \left(\frac{2^{j+1}}{\varkappa - 1} \right)^l dx'.$$

Hence, taking t_j-th root we get

$$\|\overline{v}\|_{t_{j+1}, G'_{(j+1)}} \leq \left(\frac{C}{\varkappa - 1} \right)^{\frac{l}{t_j}} t_j^{\frac{l}{t_j}} 2^{\frac{(j+1)l}{t_j}} \|\overline{v}\|_{t_j, G'_{(j)}}.$$

After iteration process we find

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \leq \left(\frac{Ct}{\varkappa-1}\right)^{l\sum_{j=0}^{\infty}\frac{1}{t_j}} \left(\frac{n}{n-1}\right)^{l\sum_{j=0}^{\infty}\frac{j}{t_j}} 2^{l\sum\limits_{j=0}^{\infty}\frac{j+1}{t_j}} \|\overline{v}\|_{t,G_1^2}.$$

The series $\sum_{j=0}^{\infty}\frac{j}{t_j}, \sum_{j=0}^{\infty}\frac{j+1}{t_j}$ are convergent according to the d'Alembert ratio test, while the series $\sum_{j=0}^{\infty}\frac{1}{t_j} = \frac{1}{t}\cdot\sum_{j=0}^{\infty}\left(\frac{n-1}{n}\right)^j = \frac{n}{t}$ as a geometric series. Hence, letting $j \to \infty$, we obtain

$$\sup_{x\in G_{\varkappa}^2} \overline{v} \leq \frac{C^*}{(\varkappa-1)^{\frac{l\cdot n}{t}}}\|\overline{v}\|_{t,G_1^2}.$$

Thus, by the definition of $\overline{v}$, we obtain the required estimate.

5 The power modulus of continuity near infinity for weak solutions

By Theorem 4.1 with $t = 2$, we have

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u| \leq C^*\left(\varrho^{-\frac{n}{2}}\|u\|_{2,G_{\varrho}^{2\varrho}} + \varrho^k\right).$$

We can observe that

$$\varrho^{-\frac{n}{2}}\|u\|_{2,G_{\varrho}^{2\varrho}} \leq 2^{\frac{n}{2}}\left(\int_{G_{\varrho}^{2\varrho}} r^{-n}u^2dx\right)^{\frac{1}{2}}.$$

Then, by (2.2) we get

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u| \leq C^*\cdot\left\{\left(\int_{G_{\varrho}^{2\varrho}} r^{-n}u^2dx\right)^{\frac{1}{2}} + \varrho^k\right\} \leq \widetilde{C^*}\cdot\left\{\left(\int_{G_{\varrho}^{2\varrho}} r^{2-n}|\nabla u|^2dx\right)^{\frac{1}{2}} + \varrho^k\right\}.$$

Next, by Theorem 3.1, choosing $k = \lambda_-(1-\mu)$ we obtain

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u(x)| \leq C\varrho^{\lambda_-(1-\mu)}.$$

Putting now $|x| = \frac{7}{4}\varrho$ we eventually obtain the desired estimate (1.2). □

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