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Multiple subharmonic solutions with prescribed minimal periods for a class of second order impulsive differential systems

Huaxin Ou ✉

School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P. R. China

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Abstract. In this paper, we obtain three subharmonic solutions with different prescribed minimal periods for a class of second order impulsive differential systems. Our proof is based on level estimates and the least action principle in critical point theory.

Keywords: subharmonic solution, impulsive system, minimal period, variational method.

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1 Introduction and main result

In many evolution processes, the states of systems are changed abruptly at certain instants, which leads to impulsive behaviors in dynamical systems [2, 11, 13]. In recent years, the investigation of differential equations with impulses got particular attention by a lot of scholars, because of the widespread application of these impulsive differential systems in biology, mechanics, engineering and chaos theory, etc. [5, 6, 13, 21, 22].

Some classical approaches, such as the method of upper and lower solutions with the monotone iterative technique, the coincidence degree theory of Mawhin and the fixed point theory, were used to study impulsive problems [2, 11]. Especially, in the remarkable work of Nieto and O'Regan [13], by constructing a variational structure, they converted the problem of finding solutions for a second order impulsive equation to that of the existence of critical points for the corresponding energy functional [19]. After that, the variational methods and critical point theory were applied to prove the existence and multiplicity of solutions for second order, fourth order and fractional order impulsive differential equations by more and more researchers, see [1–4, 7–11, 13, 14, 16–20].

✉ Email: ouhx2020@163.com

Our purpose is to investigate the existence of multiple subharmonic solutions for the following second order impulsive systems

$$\begin{cases} \ddot{x}(t) + \nabla V(t, x(t)) = 0, \\ \Delta\left(\dot{x}^i\left(t_j\right)\right) = \dot{x}^i\left(t_j^+\right) - \dot{x}^i\left(t_j^-\right) = I_{ij}\left(x^i\left(t_j\right)\right), i = 1, 2, \dots, N, j = 1, 2, \dots, l, \\ x(0) = x(pT),\ \dot{x}(0) = \dot{x}(pT), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $\nabla V \in C([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$ denotes the gradient of V in x, $I_{ij} \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and impulses occur at instants t_j with $j \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, $0 < t_1 < \cdots < t_l < T$ and $t_{j+l} = t_j$.

In [11], Luo, Xiao and Xu studied the existence of subharmonic solutions for the equation with non-negative impulses as follows

$$\begin{cases} \ddot{x}(t) + f(t, x(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}\backslash\{t_k \mid k \in \mathbb{Z}^*\}, \\ \Delta\dot{x}(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k \in \mathbb{Z}^*, \end{cases}$$

where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $I_k \geq 0$ are impulses that happen at instants t_k. Bai and Wang [2] generalized the results in [11] to allow a negative impulse term. Here, motivated by [2,11], we investigate the existence of multiple subharmonic solutions for system (1.1).

By a classical solution of (1.1), we mean a function

$$x \in \left\{ w \in C([0, pT], \mathbb{R}^N) : w|_{[t_j, t_{j+1}]} \in H^2([t_j, t_{j+1}], \mathbb{R}^N),\ j = 1, 2, \dots, l \right\},$$

which satisfies the differential equation in (1.1) and the boundary conditions $x(0) = x(pT)$, $\dot{x}(0) = \dot{x}(pT)$, the limits $\dot{x}^i(t_j^+), \dot{x}^i(t_j^-), i = 1, 2, \dots, N, j = 1, 2, \dots, l$, exist and verify the impulsive conditions in (1.1).

Now we state our main result.

Theorem 1.1. Suppose that $V(t, x)$ and $I_{ij}(x)$ satisfy the following conditions.

(H1) $V(t, x) = V(-t, x) = V(t, -x) = V(t + \frac{T}{2}, x)$ for every $(t, x) \in [0, T] \times \mathbb{R}^N$.

(H2) For every $x \in \mathbb{R}^N, t \in [0, T]$, there exist constants $\delta > 0$ and $A > \bar{A} > 0$ such that

$$V(t, x) \geq \frac{\bar{A}}{2}|x|^2, \qquad |x| \leq \delta$$

and

$$V(t, x) - (\nabla V(t, 0), x) \leq \frac{A}{2}|x|^2.$$

(H3) For $i = 1, 2, \dots, N, j = 1, 2, \dots, l$, there exist constants $d_{ij} \geq 0$ such that

$$I_{ij}(x) \leq d_{ij}|x|, \qquad x \in \mathbb{R}^N.$$

(H4) There exists an integer $p > 1$ such that

$$1 - 2\rho p^2 T > 0, \quad \frac{4\omega^2}{\bar{A} - 4D/T} < p^2 < \frac{\omega^2 s_p^2}{2\rho T \omega^2 s_p^2 + A + 2\rho/T}$$

and

$$\left(\frac{\omega^2}{2p^2} - \rho T \omega^2 - \frac{\rho}{T}\right)|x|^2 - V(t, x) \to +\infty, \quad \text{as } |x| \to \infty,$$

where $\omega = \frac{2\pi}{T}, \rho = \sum_{j=1}^{l}\sum_{j=1}^{N} d_{ij}, D = \sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}}$ and s_p is the smallest prime factor of p.

(H5) ∇V satisfies

$$\int_0^T |\nabla V(t,0)|^2 \,\mathrm{d}t < \delta^2 \min\left\{K_1(K_2-K_3), \left(K_1 - \frac{3\omega^2 T}{p^2}\right)(4K_2-K_3)\right\},$$

$$K_1 = T\left(\bar{A} - \frac{\omega^2}{p^2}\right) - 4D, \qquad K_2 = \frac{\omega^2 s_p^2}{2p^2}\left(1-2\rho p^2 T\right), \qquad K_3 = \frac{A}{2} + \frac{\rho}{T}.$$

(H6) Suppose that q_0 is rational. If both $x(t)$ and $\nabla V(t,x)$ have minimal period $\frac{q_0 T}{2}$, then q_0 is an integer.

Then the impulsive system (1.1) possesses at least three periodic solutions. Two of them have minimal period pT and the other one has minimal period $\frac{pT}{2}$.

Remark 1.2. In [11], Luo, Xiao and Xu investigated second order impulsive differential equations with a non-negative impulse term and obtained the existence of at least one solution with minimal period pT. Bai and Wang [2] generalized the results of [11] by proving the existence of at least one solution with prescribed minimal period for second order impulsive systems allowing negative impulse terms. Here, we also do not have to assume that the impulse term is non-negative. Giving a suitable range of p and $\int_0^T |\nabla V(t,0)|^2 \mathrm{d}t$, we find three solutions with prescribed minimal periods for system (1.1).

2 Proof of the theorem

In the first place, we recall some basic notations. Let $p > 1$ is an integer, $T > 0$. We denote the inner product on $\mathbb{R}^N$ by $(\cdot,\cdot)$. $H^1_{pT}(\mathbb{R}^N)$ is a Hilbert space, which defined as

$$H^1_{pT} = \{x : [0,pT] \to \mathbb{R}^N \mid x \text{ is absolutely continuous, } x(0) = x(pT),\ \dot{x} \in L^2(0,pT;\mathbb{R}^N)\}.$$

Let $\langle \cdot,\cdot \rangle$ be the inner product on H^1_{pT}, i.e.

$$\langle x,y \rangle = \int_0^{pT} (\dot{x},\dot{y})\mathrm{d}t + \int_0^{pT} (x,y)\mathrm{d}t, \qquad x,y \in H^1_{pT},$$

which induces the norm $\|x\| = \langle x,x \rangle$. Additionally, the energy functional corresponds to system (1.1) is

$$\varphi(x) = \int_0^{pT} \left[\frac{1}{2}|\dot{x}|^2 - V(t,x)\right] \mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N} \int_0^{x^i(t_j)} I_{ij}(s)\mathrm{d}s, \qquad x \in H^1_{pT}.$$

It follows that

$$\begin{aligned} \langle \varphi'(x), y \rangle = &\int_0^{pT} (\dot{x},\dot{y})\mathrm{d}t - \int_0^{pT} (\nabla V(t,x), y)\,\mathrm{d}t \\ &+ \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i(t_j)\right) y^i(t_j), \quad x,y \in H^1_{pT}. \end{aligned} \tag{2.1}$$

Definition 2.1. A function x is called a weak pT-periodic solution of (1.1) if and only if the following equation holds

$$\int_0^{pT} (\dot{x},\dot{y})\mathrm{d}t + \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i(t_j)\right) y^i(t_j) = \int_0^{pT} (\nabla V(t,x), y)\,\mathrm{d}t, \qquad \forall y \in H^1_{pT}.$$

The critical points of φ correspond to periodic solutions of impulsive system (1.1). Indeed, suppose x is a critical point of φ, by (2.1) and the Definition 2.1, x is a weak pT-periodic solution of (1.1). Moreover, for every $y \in H^1_{pT}$, we have

$$\begin{aligned}\langle \varphi'(x), y\rangle &= \int_0^{pT} (\dot{x}, \dot{y})\mathrm{d}t - \int_0^{pT} (\nabla V(t,x), y)\,\mathrm{d}t + \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i(t_j)\right) y^i\left(t_j\right) \\ &= -\int_0^{pT} (\ddot{x}, y)\mathrm{d}t - \int_0^{pT} (\nabla V(t,x), y)\,\mathrm{d}t.\end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$\ddot{x}(t) + \nabla V(t,x) = 0, \quad \text{a.e. } t \in [t_j, t_{j+1}].$$

Then we get $x \in H^2([t_j, t_{j+1}], \mathbb{R}^N)$ and

$$\ddot{x}(t) + \nabla V(t,x) = 0, \quad \text{a.e. } t \in [0, pT].$$

Multiplying the above equation by $y \in H^1_{pT}$ and integrating over $[0, pT]$, we obtain

$$\sum_{j=1}^{pl}\sum_{j=1}^{N} \Delta\left(\dot{x}^i\left(t_j\right)\right) y^i(t_j) = \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i\left(t_j\right)\right) y^i(t_j).$$

Thus, $\Delta\left(\dot{x}^i\left(t_j\right)\right) = I_{ij}\left(x^i\left(t_j\right)\right)$ for $i = 1, 2, \ldots, N, j = 1, 2, \ldots, l$, and the impulsive conditions in (1.1) are verified.

For the sake of convenience, let us define a couple of subspaces of H^1_{pT}. Set

$$X = \left\{x \in H^1_{pT} \mid x(t) = -x(-t)\right\}, \qquad Y = \left\{x \in H^1_{pT} \mid x\left(t + \frac{pT}{2}\right) = -x(t)\right\},$$

then we can define

$$\begin{aligned} X_1 &= X \cap Y, & X_2 &= X \cap Y^{\perp}, \\ Y_1 &= X^{\perp} \cap Y, & Y_2 &= X^{\perp} \cap Y^{\perp}. \end{aligned}$$

Clearly, we have $H^1_{pT} = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$. In the following, we denote the norm of x on L^2_{pT} and L^∞_{pT} by $\|x\|_{L^2}$ and $\|x\|_\infty$ respectively.

Lemma 2.2 ([12])**.** Suppose that W is a reflexive Banach space, $\varphi : W \to \mathbb{R}$ is weakly lower semi-continuous and coercive on W, then φ attains its minimum on W.

Lemma 2.3. Under condition (H1), critical points of φ on X_1 (or X_2, Y_1, Y_2) are also critical points of φ on H^1_{pT}. The minimal period of such a critical point is an integer multiple of $\frac{T}{2}$.

Proof. On the one hand, if x is a critical point of φ on X, then

$$\langle \varphi'(x), y\rangle = 0, \qquad \forall y \in X.$$

Let $y \in X^{\perp}$, it can be deduced from (H1) and (2.2) that

$$\begin{aligned}
\langle \varphi'(x), y \rangle &= -\int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&= -\int_{0}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&\quad -\int_{0}^{\frac{pT}{2}} (\ddot{x}(-t), y(-t))\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(-t,x(-t)), y(-t))\,\mathrm{d}t \\
&= -\int_{0}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&\quad -\int_{0}^{\frac{pT}{2}} (-\ddot{x}(t), y(t))\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (-\nabla V(t,x(t)), y(t))\,\mathrm{d}t \\
&= 0.
\end{aligned} \tag{2.3}$$

Thus, $\langle \varphi'(x), y \rangle = 0$, for all $y \in H^1_{pT}$.

On the other hand, providing that x is a critical point of φ on X_1, set $y \in X_2$, we find

$$\begin{aligned}
\langle \varphi'(x), y \rangle &= -\int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&= -\int_{0}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&\quad -\int_{0}^{\frac{pT}{2}} \left(\ddot{x}\left(t-\frac{pT}{2}\right), y\left(t-\frac{pT}{2}\right)\right)\mathrm{d}t \\
&\quad -\int_{0}^{\frac{pT}{2}} \left(\nabla V\left(t-\frac{pT}{2}, x\left(t-\frac{pT}{2}\right)\right), y\left(t-\frac{pT}{2}\right)\right)\mathrm{d}t \\
&= -\int_{0}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\
&\quad -\int_{0}^{\frac{pT}{2}} (-\ddot{x}(t), y(t))\mathrm{d}t - \int_{0}^{\frac{pT}{2}} (\nabla V(t,-x(t)), y(t))\,\mathrm{d}t \\
&= 0.
\end{aligned}$$

It follows that x is a critical point of φ on X. From (2.3) we know that x is a critical point of φ on H^1_{pT}.

By a similar discussion, one can prove the cases of X_2, Y_1, Y_2 alike.

If the minimal period of $x(t)$ is $\frac{pT}{2q}$, where q is an integer. From (1.1) we have

$$\ddot{x}\left(t+\frac{pT}{2q}\right) + \nabla V\left(t+\frac{pT}{2q}, x\left(t+\frac{pT}{2q}\right)\right) = 0. \tag{2.4}$$

It follows from (2.4) that $\nabla V(t, x(t))$ has minimal period $\frac{pT}{2q}$. Then by $(H6)$, $\frac{p}{q}$ is an integer, which means that the minimal period of $x(t)$ is an integer multiple of $\frac{T}{2}$. □

Lemma 2.4 ([15]). Suppose that $H(t,x) \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$ with $H(t,x) \to +\infty$ as $|x| \to \infty$ uniformly in $t \in [0,T]$, then there exist a real function $\gamma \in L^1([0,T], \mathbb{R})$ and a subadditive function $G : \mathbb{R}^N \to \mathbb{R}$, i.e.

$$G(x+y) \le G(x) + G(y), \qquad x, y \in \mathbb{R}^N, \tag{2.5}$$

such that

$$H(t,x) \geq G(x) + \gamma(t), \qquad x \in \mathbb{R}^N, \tag{2.6}$$

$$G(x) \to +\infty \quad \text{as } |x| \to \infty, \tag{2.7}$$

$$0 \leq G(x) \leq |x| + 1, \qquad x \in \mathbb{R}^N. \tag{2.8}$$

Lemma 2.5. Under condition (H4), φ is coercive on X_1 (or X_2, Y_1).

Proof. From Lemma 2.4 and (H4), there exist $G(x)$ and $\gamma(t) \in L^1([0,T], \mathbb{R})$ such that

$$\left(\frac{\omega^2}{2p^2} - \rho T\omega^2 - \frac{\rho}{T}\right)|x|^2 - V(t,x) \geq G(x) + \gamma(t). \tag{2.9}$$

To begin with, we claim that $\int_0^{pT} G(x)\mathrm{d}t$ is coercive on

$$X_1^1 = \left\{ r \sin\frac{\omega t}{p} \mid r \in \mathbb{R}^N \right\} \subset X_1.$$

Providing that $\{x_n\}$ is a sequence in X_1^1 with $\|x_n\| \to \infty$ as $n \to \infty$, we can set $x_n(t) = r_n \sin\frac{\omega t}{p}$, where $r_n \in \mathbb{R}^N$ and $|r_n| \to +\infty$ as $n \to \infty$. By (2.7), for every $L > 0$, there exists $M > 0$ such that

$$G(x) \geq L, \qquad |x| \geq M. \tag{2.10}$$

Since $|r_n| \to +\infty$ as $n \to \infty$, there exists $N_0 > 0$ such that $|r_n| > 2M$ for $n > N_0$. Furthermore, it is clear that

$$|x_n(t)| > M, \qquad \forall t \in \left[\frac{pT}{12}, \frac{5pT}{12}\right] \cup \left[\frac{7pT}{12}, \frac{11pT}{12}\right], \quad n > N_0. \tag{2.11}$$

From (2.10) and (2.11) we have

$$\int_0^{pT} G(x_n)\mathrm{d}t > \frac{2pLT}{3}, \qquad n > N_0.$$

The coercivity of $\int_0^{pT} G(x)\mathrm{d}t$ follows from the arbitrariness of L and $\{x_n\}$.

Let $x \in X_1$ and $x = x_1 + x_2$, where $x_1 \in X_1^1, x_2 \in (X_1^1)^{\perp} \cap X_1$. By the Parseval equality,

$$\|\dot{x}_1\|_{L^2}^2 = \frac{\omega^2}{p^2}\|x_1\|_{L^2}^2, \qquad \|\dot{x}_2\|_{L^2}^2 \geq \frac{9\omega^2}{p^2}\|x_2\|_{L^2}^2, \qquad \|\dot{x}_2\|_{L^2}^2 \geq \frac{9\omega^2}{9\omega^2 + p^2}\|x_2\|^2. \tag{2.12}$$

Additionally, (2.5) implies that

$$G(x_1) = G(x - x_2) \leq G(x) + G(-x_2). \tag{2.13}$$

It can be deduced from (H3) that

$$\begin{aligned}
\left|\sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{y^i(t_j)} I_{ij}(s)\mathrm{d}s\right| &\leq \sum_{j=1}^{pl}\sum_{i=1}^{N}\frac{1}{2}d_{ij}\left|y\left(t_j\right)\right|^2 \\
&\leq \sum_{j=1}^{pl}\sum_{i=1}^{N} d_{ij}\left(\frac{\|y\|_{L^2}^2}{pT} + pT\|\dot{y}\|_{L^2}^2\right) \\
&\leq \frac{\rho}{T}\|y\|_{L^2}^2 + \rho p^2 T\|\dot{y}\|_{L^2}^2, \qquad y \in H_{pT}^1.
\end{aligned} \tag{2.14}$$

To the best of our knowledge, the formula (2.14) was first proved in [2]. For more details, please refer to (2.9) in [2]. From (H4), (2.9), (2.12), (2.13) and (2.14), we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{2}\int_0^{pT}|\dot{x}|^2\mathrm{d}t - \int_0^{pT}V(t,x)\mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{x^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\geq \left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}|\dot{x}|^2\mathrm{d}t - \frac{\rho}{T}\int_0^{pT}|x|^2\mathrm{d}t - \int_0^{pT}V(t,x)\mathrm{d}t \\
&= \left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}\left(|\dot{x}_1|^2+|\dot{x}_2|^2-\frac{\omega^2}{p^2}|x_1|^2-\frac{\omega^2}{p^2}|x_2|^2\right)\mathrm{d}t \\
&\quad + \int_0^{pT}\left[\left(\frac{\omega^2}{2p^2}-\rho\omega^2 T-\frac{\rho}{T}\right)|x|^2 - V(t,x)\right]\mathrm{d}t \\
&\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}|\dot{x}_2|^2\,\mathrm{d}t + \int_0^{pT}\left[G(x)+\gamma(t)\right]\mathrm{d}t \\
&\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\|\dot{x}_2\|_{L^2}^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - \int_0^{pT}G(-x_2)\,\mathrm{d}t + \int_0^{pT}\gamma(t)\mathrm{d}t \\
&\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\|\dot{x}_2\|_{L_2}^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - pT(1+\|x_2\|_\infty+\|\gamma\|_\infty) \\
&\geq \frac{8\omega^2}{9\omega^2+p^2}\left(\frac{1}{2}-\rho p^2 T\right)\|x_2\|^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - C_1\|x_2\| - C_2,
\end{aligned} \tag{2.15}$$

where C_1 and C_2 are positive constants. With $\int_0^{pT}G(x_1)\mathrm{d}t$ being coercive on X_1 and

$$\|x\|\to\infty \quad \text{if and only if} \quad (\|x_1\|^2+\|x_2\|^2)^{\frac{1}{2}}\to\infty,$$

it follows from (2.15) that $\varphi(x)$ is coercive on X_1.

Through replacing X_1^1 with

$$\begin{aligned}
X_2^1 &= \left\{ b\sin\frac{2\omega t}{p} \mid b\in\mathbb{R}^N \right\}, \\
Y_1^1 &= \left\{ c\cos\frac{\omega t}{p} \mid c\in\mathbb{R}^N \right\},
\end{aligned}$$

repeating the above arguments with a small modification, one can prove the coercivity of φ on X_2 and Y_1. □

Proof of Theorem 1.1. According to Lemma 2.2, Lemma 2.3 and Lemma 2.5, there exists $x_1^*\in X_1$ such that

$$\langle\varphi'(x_1^*),y\rangle = 0, \qquad \forall y\in H_{pT}^1. \tag{2.16}$$

In what follows, we show that the minimal period of x_1^* is pT by contradiction. Suppose that x_1^* has minimal period $\frac{pT}{q_1}$, where $q_1>1$ is an integer. From Lemma 2.3, the minimal period of x_1^* is multiple of $\frac{T}{2}$, which means that $q_1=2$ or $q_1\geq s_p$.

If $q_1=2$, by Fourier expansion,

$$x_1^* = \sum_{k=1}^{+\infty} a_k^*\sin\frac{2k\omega t}{p}, \qquad a_k^*\in\mathbb{R}^N.$$

However, for every $x \in X_1$, we have

$$x = \sum_{k=1}^{+\infty} a_k \sin \frac{(2k-1)\omega t}{p}, \qquad a_k \in \mathbb{R}^N,$$

which implies $x_1^* = 0$. It contradicts that x_1^* has minimal period $\frac{pT}{2}$. So we get $q_1 \geq s_p$ and

$$x_1^* = \sum_{k=1}^{+\infty} a_k^* \sin \frac{kq_1\omega t}{p}, \qquad a_k^* \in \mathbb{R}^N. \tag{2.17}$$

It can be deduced from Parseval's equality and (2.17) that

$$\|\dot{x}_1^*\|_{L^2} \geq \frac{q_1\omega}{p}\|x_1^*\|_{L^2}. \tag{2.18}$$

Now, from (H2), (H4), (2.14) and (2.18), we have

$$\begin{aligned}
\varphi(x_1^*) &= \frac{1}{2}\int_0^{pT} |\dot{x}_1^*|^2 \mathrm{d}t - \int_0^{pT} V(t, x_1^*)\mathrm{d}t - \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(x_1^*)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\geq \frac{1}{2}\|\dot{x}_1^*\|_{L^2}^2 - \int_0^{pT} [V(t,x_1^*) - (\nabla V(t,0), x_1^*)]\,\mathrm{d}t - \int_0^{pT} (\nabla V(t,0), x_1^*)\,\mathrm{d}t \\
&\quad - \frac{\rho}{T}\|x_1^*\|_{L^2}^2 - \rho p^2 T\|\dot{x}_1^*\|_{L^2}^2 \\
&\geq \left(\frac{1}{2} - \rho p^2 T\right)\|\dot{x}_1^*\|_{L^2}^2 - \left(\frac{A}{2} + \frac{\rho}{T}\right)\|x_1^*\|_{L^2}^2 - \|\nabla V(t,0)\|_{L^2}\,\|x_1^*\|_{L^2} \\
&\geq \left(\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T}\right)\|x_1^*\|_{L^2}^2 - \|\nabla V(t,0)\|_{L^2}\,\|x_1^*\|_{L^2}.
\end{aligned} \tag{2.19}$$

It follows from (H4) and $q_1 \geq s_p$ that

$$\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T} > 0,$$

which combined with (2.19) yields to

$$\varphi(x_1^*) \geq -\frac{1}{4}\left(\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T}\right)^{-1} \|\nabla V(t,0)\|_{L^2}^2. \tag{2.20}$$

Choosing $\bar{x}_1(t) = (\delta \sin\frac{\omega t}{p}, 0, \ldots, 0) \in X_1$, where δ defined in (H2), the minimal period of $\bar{x}_1(t)$ is pT. According to mean value theorem, the Cauchy–Schwarz inequality and (H3), we have

$$\sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(\bar{x}_1)^i(t_j)} I_{ij}(s)\mathrm{d}s \leq \sum_{j=1}^{pl}\sum_{i=1}^{N} d_{ij}|\theta|\left|\delta \sin\frac{\omega t_j}{p}\right| \leq p\delta^2 \sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}}, \tag{2.21}$$

where $\theta \in \left(0, \delta \sin \frac{\omega t_j}{p}\right)$. In view of (2.21) and (H2), we get

$$\begin{aligned}
\varphi(\bar{x}_1) &= \frac{1}{2}\int_0^{pT} |\dot{\bar{x}}_1|^2 \mathrm{d}t - \int_0^{pT} V(t,\bar{x}_1)\mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(\bar{x}_1)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\le \frac{1}{2}\int_0^{pT} \delta^2 \frac{\omega^2}{p^2}\cos^2\frac{\omega t}{p}\mathrm{d}t - \int_0^{pT} V\left(t, \delta \sin\frac{\omega t}{p}\right)\mathrm{d}t + p\delta^2 \sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{4}\delta^2\frac{\omega^2}{p}T - \int_0^{pT}\frac{\bar{A}}{2}\delta^2 \sin^2\frac{\omega t}{p}\mathrm{d}t + p\delta^2 \sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \\
&= \frac{1}{4}\delta^2\frac{\omega^2}{p}T - \frac{\bar{A}}{4}\delta^2 pT + p\delta^2 \sum_{j=1}^{l}\left(\sum_{j=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \\
&= -\frac{1}{4}\left[\delta^2 pT\left(\bar{A} - \frac{\omega^2}{p^2}\right) - 4p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N} {d_{ij}}^2\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{2.22}$$

By (H4), (H5), (2.16), (2.20) and (2.22), we find

$$\inf_{x\in X_1}\varphi(x) = \varphi(x_1^*) > \varphi(\bar{x}_1).$$

That is a contradiction. Hence, there exists a critical point $x_1^* \in X_1$ of φ with minimal period pT.

Similarly, we can find $x_2^* \in X_2$ such that $\langle \varphi'(x_2^*), y\rangle = 0$, for every $y \in H^1_{pT}$. If the minimal period of x_2^* is not equal to $\frac{pT}{2}$, then there exists $q_2 > 1$ such that x_2^* has minimal period $\frac{pT}{2q_2}$. Lemma 2.3 implies that $q_2 \ge s_p$. Additionally, we have

$$x_2^* = \sum_{k=1}^{+\infty} b_k^* \sin\frac{2kq_2\omega t}{p}, \qquad b_k^* \in \mathbb{R}^N$$

and

$$\|\dot{x}_2^*\|_{L^2} \ge \frac{2q_2\omega}{p}\|x_2^*\|_{L^2}. \tag{2.23}$$

It follows from (H2), (H4), (2.14) and (2.23) that

$$\begin{aligned}
\varphi\left(x_2^*\right) &= \frac{1}{2}\int_0^{pT}|\dot{x}_2^*|^2\mathrm{d}t - \int_0^{pT} V(t,x_2^*)\mathrm{d}t - \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(x_2^*)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\ge -\frac{1}{4}\left(\frac{2\omega^2q^2}{p^2} - 4\rho T\omega^2 q^2 - \frac{A}{2} - \frac{\rho}{T}\right)^{-1} \|\nabla V\left(t,0\right)\|^2_{L^2}.
\end{aligned} \tag{2.24}$$

Let $\bar{x}_2(t) = \left(\delta \sin\frac{2\omega t}{p}, 0, \ldots, 0\right) \in X_2$, then $\bar{x}_2(t)$ has minimal period $\frac{pT}{2}$. After a computation like (2.22), we get

$$\varphi(\bar{x}_2) \le -\frac{1}{4}\left[\delta^2 pT\left(\bar{A} - \frac{4\omega^2}{p^2}\right) - 4p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N}{d_{ij}}^2\right)^{\frac{1}{2}}\right]. \tag{2.25}$$

Taking (H4), (H5), (2.24) and (2.25) into consideration, we obtain

$$\inf_{x\in X_2}\varphi(x) = \varphi(x_2^*) > \varphi(\bar{x}_2).$$

This contradiction leads to the fact that the minimal period of x_2^* is $\frac{pT}{2}$.

Using a similar argument, a critical points y_1^* of φ with minimal period pT can be found on Y_1. It is clear that x_1^*, x_2^* and y_1^* are nonzero, which combines with $H^1_{pT} = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$ to give that three points are different. □

3 Example

To show how our theorem applies in practice, we give the following example.

Example 3.1. Consider the impulsive system (1.1) with $T = 1, N = l = p = 3$,

$$I_{ij}(s) = \frac{|s|}{3240}, \qquad i, j \in \{1, 2, 3\},$$

and

$$V(t,x) = \begin{cases} \frac{7-\cos 4\pi t}{18}\omega^2|x|^2, & |x| \le 1, \\ \left(\frac{68\pi^2-1}{162} - \frac{7-\cos 4\pi t}{18}\omega^2 - \frac{1}{2}\right)|x|^2 & \\ \quad + \left(\frac{7-\cos 4\pi t}{9}\omega^2 - \frac{68\pi^2-1}{162} + \frac{1}{2}\right)(2|x|-1), & 1 < |x| \le 2, \\ \frac{68\pi^2-1}{324}|x|^2 - \ln|x|^2 + \frac{7-\cos 4\pi t}{9}\omega^2 - \frac{68\pi^2-1}{162} + 2\ln 2 - \frac{1}{2}, & |x| > 2. \end{cases}$$

We can take

$$\omega = 2\pi, \qquad \rho = \frac{1}{4p^2 s_p^2} = \frac{1}{324},$$

then

$$(1 - 2\rho p^2 T)\frac{\omega^2}{2p^2} - \frac{\rho}{T} = \frac{68\pi^2 - 1}{324}.$$

It is easy to verify that $V(t,x)$ satisfies (H1). Let

$$d_{ij} = \frac{\rho}{9}, \qquad i, j \in \{1, 2, 3\},$$

and

$$A = \frac{8}{9}\omega^2, \qquad \bar{A} = \frac{2}{3}\omega^2, \qquad \delta = 1,$$

then $D = \frac{\rho}{3}$. One can easily check that (H3) is true and $V(t,x)$ satisfies (H2), (H4) and (H5). By Theorem 1.1, Example 3.1 possesses at least three periodic solutions. Two of them have minimal period 3 and the other one has minimal period 1.5.

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