Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **2**, 1–19; https://doi.org/10.14232/ejqtde.2024.1.2 www.math.u-szeged.hu/ejqtde/

Exponential decay for a Klein–Gordon–Schrödinger system with locally distributed damping

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Received 11 December 2022, appeared 3 January 2024
Communicated by Vilmos Komornik

Abstract. A coupled damped Klein–Gordon–Schrödinger equations are considered where Ω is a bounded domain of $\mathbb{R}^2$, with smooth boundary Γ and ω is a neighbourhood of $\partial\Omega$ satisfying the geometric control condition. The aim of the paper is to prove the existence, uniqueness and uniform decay for the solutions.

Keywords: Klein–Gordon–Schrödinger, localized damping, exponential stability, asymptotic behavior, existence and uniqueness.

2020 Mathematics Subject Classification: 35B40, 35B45, 35Q55.

1 Introduction

The aim of this paper is to study the following KGS system defined in Ω which is a bounded domain in $\mathbb{R}^2$

$$\begin{aligned} i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi &= \phi\psi\chi(\omega) \in \Omega \times (0,+\infty) \\ \phi_{tt} - \Delta\phi + \phi + \lambda(x)\phi_t &= -\operatorname{Re}\nabla\psi\chi(\omega) \in \Omega \times (0,+\infty) \\ \psi = \phi &= 0, \quad \text{on } \Gamma \times (0,+\infty) \end{aligned} \tag{1.1}$$

with locally distributed damping and where Γ is a smooth boundary and ω is an open subset of Ω such that $\operatorname{meas}(\omega) > 0$ and satisfying the geometric control condition. Let $\alpha > 0$ and $\chi(\omega)$ to represent the characteristic function, that is $\chi = 1$ in ω and $\chi = 0$ in $\Omega \setminus \omega$. We also consider $b, \lambda \in L^\infty(\Omega)$ to be nonnegative functions such that

$$b(x) \geq b_0 > 0 \quad \text{a.e. in } \omega \quad \text{and} \quad \lambda(x) \geq \lambda_0 > 0 \quad \text{a.e. in } \omega,$$

in order for the nonlinearity ψ to exist where the damping terms

$$i\alpha b(x)\psi, \qquad \lambda(x)\phi_t$$

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are effective and reciprocally. If the damping is effective in the whole domain, i.e. $b(x) \geq b_0 > 0$ a.e. in Ω and $\lambda(x) \geq \lambda_0 > 0$ a.e. in Ω we can consider $\chi_\omega \equiv 1$ a.e. in Ω. The variable (complex) ψ stands for the dimensionless low frequency electron field, whereas (real) ϕ denotes the dimensionless low frequency density. This system describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH (Upper Hybrid Heating) plasma heating scheme.

UHH is the dominant branch of the general Electron Cyclotron Resonance Heating (ECRH) scheme, which, for tokamaks and stellarators, constitutes a basic method of plasma build-up and heating. Moreover, ECRH is an attractive method to study transport mechanisms, since it allows for a very localised power deposition, thus influencing temperature and current profiles. The UHH scheme consists in injecting electromagnetic waves in the range $100 - 200GHz$, from the high field side towards the core of the device. Within this frequency range, the incident wave takes on the character of a longitudinal oscillation for the resonant electrons, which become highly energetic. With respect to the physical mechanism involved in the energy damping of the waves, UHH comprises of two stages:

1. Collisionless damping. The energy of the waves is transferred to the resonant electrons, through collisionless mechanisms, e.g. Landau damping. Subsequently, the electrons gain excessive kinetic energy, thus heated.

2. Collisional damping. The excessive electron energy is distributed over electrons and non-resonant ions, through Coulomb collisions, producing bulk heating of the plasma (equipartition).

Collisional damping is very crucial for the success of UHH. If collisions are infrequent, non-thermal distributions will occur, which may result in a reduction in the power delivered to the plasma. Therefore, it is important to determine the operational conditions for the device, under which UHH becomes effective, namely the collisions manage to distribute the excessive electron energy over the species at an exponential rate. The term $Re\nabla\psi$ is a consequence of the different low frequency coupling that was considered, i.e. the polarization drift instead of the ponderomotive force. The system focuses on the vital role of collisions by considering the non-homogeneous polarization drift for the low frequency coupling (see [12]).

By setting $\theta = \phi_t + \epsilon\phi$ where ϵ is a real positive constant to be specified later, the system (1.1) becomes

$$i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi = \phi\psi\chi(\omega), \tag{1.2}$$

$$\phi_t + \epsilon\phi = \theta, \tag{1.3}$$

$$\theta_t + (\lambda(x) - \epsilon)\theta - \Delta\phi + (1 - \epsilon(\lambda(x) - \epsilon))\phi = -\operatorname{Re}\nabla\psi\chi(\omega) \tag{1.4}$$

satisfying the following initial conditions

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \theta(x,0) = \theta_0(x). \tag{1.5}$$

Therefore, one may set the energy equation of the problem by

$$E(t) := \frac{1}{2}\left\{ \|\psi\|^2_{L^2(\Omega)} + \kappa\|\nabla\psi\|^2_{L^2(\Omega)} + \int_\omega \phi|\psi|^2 + \|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)} \right\}. \tag{1.6}$$

Assumption 1.1. *We denote by ω the intersection of Ω with a neighborhood of $\partial\Omega$ in $\mathbb{R}^2$ and we will call it a neighborhood of the boundary of Ω. We assume that $b, \lambda \in L^\infty(\Omega)$ are nonnegative functions such that*

$$b(x) \geq b_0 > 0, \quad \text{a.e. in } \omega, \qquad \lambda(x) \geq \lambda_0 > 0, \quad \text{a.e. in } \omega.$$

In addition, if $b(x) \geq b_0 > 0$ a.e. in Ω then we can consider $\chi_\omega \equiv 1$ in Ω, and if $\lambda(x) \geq \lambda_0 > 0$ a.e. in Ω, then we can consider $\chi_\omega \equiv 1$ in Ω.

Definition 1.2 (Geometric control condition)**.** Let ω geometrically control Ω, i.e there exists $T_0 > 0$, such that every geodesic of Ω travelling with speed 1 and issued at $t = 0$, which enters the set ω in a time $t < T_0$. So, the couple (ω, T_0) satisfies the geometric control condition (GCC, in short) if every geodesic of Ω, traveling with speed 1 and issued at $t = 0$ enters the open set ω before the time T_0.

Assumption 1.3. *We assume that ω satisfies the geometric control condition. The standard example is when ω is a neighbourhood of $\overline{\Gamma(x_0)}$ where*

$$\Gamma(x_0) := \{x \in \Gamma; (x - x_0) \cdot \nu(x) > 0\}$$

and $\nu(x)$ is the unit outward normal at $x \in \Gamma$.

As a consequence of the previous assumption it follows that there exists a couple (ω, T_0), with $T_0 > 0$, such that the following observability inequalities occur:

$$\|\psi_0\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega |\psi(x,t)|^2 dxdt \tag{1.7}$$

for the following problem

$$\begin{cases} i\psi_t + \Delta\psi = 0 \in \Omega \times (0,T), \\ \psi = 0 \quad \text{on } \Gamma \times (0,T), \\ \psi(0) = \psi_0 \in L^2(\Omega) \end{cases} \tag{1.8}$$

and

$$\|\phi_1\|^2_{L^2(\Omega)} + \|\nabla\phi_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\phi_t(x,t)|^2 dxdt \tag{1.9}$$

with regards to the following problem

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 \in \Omega \times (0,T), \\ \phi = 0 \quad \text{on } \Gamma \times (0,T), \\ \phi(0) = \phi_0 \in H^1_0(\Omega), \\ \phi_t(0) = \phi_1 \in L^2(\Omega) \end{cases} \tag{1.10}$$

for some positive constant $C = C(\omega, T_0)$ and for all $T > T_0$. The proof of (1.8) can be found in [13] and [18] while the proof of (1.10) is established in [3] and [15].

The aim of this work is to generalize the previous results of [21] by considering the damped structure $i\alpha b(x)\psi$ instead of $i\alpha\psi$ for the Schrödinger equation following the ideas of [1, 2]. Due to the right-hand side of the wave equation, i.e. $-\operatorname{Re}\nabla\psi\chi(\omega)$ the energy functional of the system depends upon the integral $\int \phi|\psi|^2$ which introduces a time that is required by the damping to smooth out the differences between the kinetic energies of the resonant electrons and non resonant ions. The presence of the damping terms in both equations of the system does not necessary guarantee that the energy $E(t)$ associated to the system is a non increasing function of the parameter t. Indeed in [12] where $b(x), \lambda(x)$ are effective in the whole of Ω and in [21] where $\lambda(x)$ is effective in ω the energy exponential rate depends upon the parameters of the system and t^*.

Our main task is to investigate the parametric energy decay for the system. Specifically, we seek necessary conditions, dependent on the parameters of the system b_0, λ_0, so that energy decay occurs at an exponential rate and therefore improve previous results by focusing on the ω. This ensures that, under specific plasma conditions, the energy of the coupled ion-electron wave is effectively dissipated to the plasma. In fact in Section 3 we will prove that the energy ia a non increasing function. For this purpose, we make use of the observability inequality for both, the wave and the Schrödinger equations. It is important to mention that the use of the observability inequality instead of the multiplier technique allows us to consider sharp regions ω satisfying the geometric control condition. Indeed, the inequalities given in (1.7) and (1.9) are proved by means of microlocal analysis and produce sharp regions when compared with the multiplier method. The main results of this paper are the following:

Theorem 1.4. *Let* $(\psi_0, \phi_0, \theta_0) \in \{H_0^1(\Omega) \cap H^2(\Omega)\}^2 \times H_0^1(\Omega)$ *and assuming that* $(\lambda_0 - \epsilon) > \frac{2}{3\alpha\kappa b_0}$, $(\lambda_0 - \epsilon), (1 - \epsilon(\lambda_0 - \epsilon)) > 0$ *hold then there exists a unique regular solution of* (1.2)–(1.4) *such that*

$$\begin{aligned}
&\psi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \qquad && \psi_t \in L^\infty(0, \infty; L^2(\Omega)),\\
&\phi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \qquad && \phi_t \in L^\infty(0, \infty; H_0^1(\Omega)),\\
&\phi_{tt} \in L^\infty(0, \infty; L^2(\Omega)).
\end{aligned}$$

Theorem 1.5. *Let* $(\psi_0, \phi_0, \theta_0) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ *and the assumptions of Theorem 1.4 hold, then there exists positive constant* C, ν, μ *such that the following decay rate holds*

$$E_\mu(t) \leq Ce^{-\nu t} E(0), \qquad \forall\, t \geq 0$$

for every regular solution of the problem (1.1).

Let us recall the following known results. From Poincaré's inequality we have

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}, \quad \text{for all } u \in H_0^1(\Omega),$$

and the Gagliardo–Nirenberg inequality for dimension $n = 2$

$$\|u\|_{L^4(\Omega)} \leq c\|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega). \tag{1.11}$$

Notation: Denote by $H^s(\Omega)$ both the standard real and complex Sobolev spaces on Ω. For *simplicity reasons* sometimes we use H^s, L^s for $H^s(\Omega)$, $L^s(\Omega)$, and $\|\cdot\|$, $(\cdot,\cdot)$ for the norm and the inner product of $L^2(\Omega)$ and $\|\cdot\|_\omega$, $(\cdot,\cdot)_\omega$ for the norm and the inner product of $L^2(\omega)$ respectively as well as $\int dx$ denotes the integration over the domain Ω. Finally, C is a general symbol for any positive constant.

2 Existence and uniqueness

In this section we derive a priori estimates for the solutions of the system (1.1). Let $\{\omega_\nu\}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ formed by the real eigenfunctions of Δ such that the sequence

$\{\omega_\nu\}$ gives a Hilbert basis of L^2 (i.e. an orthonormal basis of L^2) and let V_m be a subset of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by the first m vectors. Then, let $g_{im} \in \mathbb{C}$ and $h_{im}, k_{im} \in \mathbb{R}$ with

$$\psi_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i, \qquad \phi_m(t) = \sum_{i=1}^m h_{im}(t)\omega_i, \qquad \theta_m(t) = \sum_{i=1}^m k_{im}(t)\omega_i$$

such that $\{(\psi_m(t), \phi_m(t), \theta_m(t))\}$ is the solution to the following Cauchy problem:

$$\begin{cases} i(\psi_{t,m}, u) + \kappa(\Delta\psi_m, u) + i\alpha(b(x)\psi_m, u) = (\phi_m\psi_m\chi(\omega), u), \ \forall\, u \in V_m, \\ (\phi_{t,m}, z) = (\theta_m, z) - \epsilon(\phi_m, z), \ \forall\, z \in V_m, \\ (\theta_{t,m}, v) + ((\lambda(x) - \epsilon)\theta_m, v) - (\Delta\phi_m, v) + ((1 - \epsilon(\lambda(x) - \epsilon))\phi_m, v) \\ \qquad = -\operatorname{Re}(\nabla\psi_m\chi(\omega), v) \ \forall\, v \in V_m, \\ \psi_m(0) = \psi_{0m} \to \psi_0, \quad \phi_m(0) = \phi_{0m} \to \phi_0 \in H_0^1(\Omega) \cap H^2(\Omega), \\ \theta(0) = \theta_{0m} \to \theta_0 \in H_0^1(\Omega). \end{cases} \tag{2.1}$$

Let $Y = (\psi_m, \phi_m, \theta_m)$ then (2.1) also reads

$$\begin{cases} (\psi_{t,m}, u) = i\kappa(\Delta\psi_m, u) + \alpha(b(x)\psi_m, u) - i(\phi_m\psi_m\chi(\omega), u), \ \forall\, u \in V_m, \\ (\phi_{t,m}, z) = (\theta_m, z) - \epsilon(\phi_m, z), \forall\, z \in V_m, \\ (\theta_{t,m}, v) = -((\lambda(x) - \epsilon)\theta_m, v) + (\Delta\phi_m, v) - ((1 - \epsilon(\lambda(x) - \epsilon))\phi_m, v) \\ \qquad - \operatorname{Re}(\nabla\psi_m\chi(\omega), v) \ \forall\, v \in V_m, \\ \psi_m(0) = \psi_{0m} \to \psi_0, \quad \phi_m(0) = \phi_{0m} \to \phi_0 \in H_0^1(\Omega) \cap H^2(\Omega), \\ \theta(0) = \theta_{0m} \to \theta_0 \in H_0^1(\Omega). \end{cases} \tag{2.2}$$

Then the considered matrix is the identity and therefore one may write $Y_t = F(Y)$ with smooth F. Hence the Cauchy–Lipschitz theorem applies straightforward. Since, the approximate system (2.1) is a finite system of ordinary differential equations which has a solution in $[0, t_m[$ the extension of the solution to the whole interval $[0, T]$, for all $T > 0$, is a consequence of the first estimate we are going to obtain. Let us consider $u = \overline{\psi_m}$ in the first equation of (2.1). Then by taking the imaginary part we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_m\|^2 + \alpha\int b(x)|\psi_m|^2 = 0 \tag{2.3}$$

and because

$$\int b(x)|\psi_m|^2 \geq \int_\omega b(x)|\psi_m|^2 \geq b_0\int_\omega |\psi_m|^2 \tag{2.4}$$

almost everywhere in ω we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_m\|^2 + \alpha b_0\|\psi_m\|_\omega^2 \leq 0. \tag{2.5}$$

Finally, multiplying by 2 and integrating over $(0, t)$ for $t \in [0, t_m)$ concludes in

$$\|\psi_m\|^2 + 2\alpha b_0\int_0^t \|\psi_m(s)\|_\omega^2 ds \leq \|\psi_{m0}\|^2. \tag{2.6}$$

Then, since $\psi_{m0} \to \psi_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$(\psi_m) \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \tag{2.7}$$

and for $C_1 = C(\|\psi_0\|) > 0$ we also have

$$\int_0^\infty \|\psi_m(s)\|_\omega^2 ds \leq C_1 = C(\|\psi_0\|). \tag{2.8}$$

Next, taking $u = -\overline{\psi_{t,m}}$ in the first equation of (2.1) and considering the real part produces

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha Im \int b(x)\psi_m\overline{\psi_{t,m}} = -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}} \tag{2.9}$$

and similarly with (2.4) we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha b_0 Im \int_\omega \psi_m\overline{\psi_{t,m}} \leq -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}}. \tag{2.10}$$

Now, substituting $u = \alpha b_0\psi_m$ in the first equation of (2.1), integrating over ω and taking the real part we have

$$\alpha b_0 \operatorname{Im}\int_\omega \psi_m\overline{\psi_{t,m}} = \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2$$

and substituting the expression into (2.10) we obtain

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha\kappa b_0\|\nabla\psi_m 0_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 \leq -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}}. \tag{2.11}$$

Therefore, by taking into consideration that

$$\frac{d}{dt}\int_\omega \phi_m|\psi_m|^2 = \int_\omega \phi_{t,m}|\psi_m|^2 + 2\int_\omega \phi_m\psi_m\overline{\psi}_{t,m}$$

equation (2.11) becomes

$$\frac{1}{2}\frac{d}{dt}\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2\} + \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 \leq \frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2. \tag{2.12}$$

Continuing with the second equation of the system (2.1), let $v = \theta_m$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\} + (\lambda_0-\epsilon)\|\theta_m\|_\omega^2 + \epsilon\|\nabla\phi_m\|^2 \\ + \epsilon(1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \leq -\operatorname{Re}\int_\omega \nabla\psi_m\theta_m.\end{aligned} \tag{2.13}$$

Then, adding equations (2.12) and (2.13) produces

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\} \\ + \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 + (\lambda_0-\epsilon)\|\theta_m\|_\omega^2 + \epsilon\|\nabla\phi_m\|^2 \\ + \epsilon(1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \leq -\operatorname{Re}\int_\omega \nabla\psi_m\theta_m + \frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2,\end{aligned} \tag{2.14}$$

where

$$\frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2 = \frac{1}{2}\operatorname{Re}\int_\omega \theta_m|\psi_m|^2 - \frac{\epsilon}{2}\operatorname{Re}\int_\omega \phi_m|\psi_m|^2$$

and with the use of $\|u\|_4 \le c\|u\|^{1/2}\ \|\nabla u\|^{1/2}$ we have

$$\left|\int_\omega \theta_m \nabla\psi_m\right| \le \frac{\alpha\kappa b_0}{2}\|\nabla\psi_m\|_\omega^2 + \frac{1}{2\alpha\kappa b_0}\|\theta_m\|_\omega^2$$

$$\left|\frac{1}{2}\int_\omega \theta_m|\psi_m|^2\right| \le \|\theta_m\|_\omega\ \|\psi_m\|_{4,\omega}^2 \le \frac{(\lambda_0-\epsilon)}{4}\|\theta_m\|_\omega^2 + \frac{\alpha\kappa b_0}{4}\|\nabla\psi_m\|_\omega^2 + C.$$

Therefore, equation (2.14) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\right\}\\ &\quad+\frac{3\alpha\kappa b_0}{4}\|\nabla\psi_m\|_\omega^2 + (\alpha b_0+\epsilon)\int_\omega \phi_m|\psi_m|^2 + \left(\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0}\right)\|\theta_m\|_\omega^2\\ &\quad+\epsilon\|\nabla\phi_m\|^2 + \epsilon(1+\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \le C\end{aligned} \tag{2.15}$$

for $\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0} > 0$. Set $\beta_0 = \min\{\frac{3\alpha\kappa b_0}{4}, (\alpha b_0+\epsilon), (\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0}), \epsilon, (1-\epsilon(\lambda_0-\epsilon))\}$, with $\beta_0 > 0$ and

$$H_0(t) = \kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2.$$

Hence we have

$$\frac{d}{dt}H_0(t) + \beta_0 H_0(t) \le C. \tag{2.16}$$

Using Gronwall's Lemma we obtain

$$H_0(t) \le \frac{C}{\beta_0}(1-e^{-\beta_0 t}) + H_0(t)e^{-\beta_0 t}$$

and

$$H_0(t) \le H_0(t)e^{-\beta_0 t} + \frac{C}{\beta_0}.$$

Finally, using (1.11) we estimate the following integral

$$\int_\omega \phi_m|\psi_m|^2 \le \frac{\kappa}{2}\|\nabla\psi_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + C$$

then

$$H_0(t) \ge \frac{\kappa}{2}\|\nabla\psi_m\|^2 + \|\theta_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 - C$$

and finally gives

$$\frac{\kappa}{2}\|\nabla\psi_m\|^2 + \|\theta_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \le C.$$

Therefore, we have

$$\begin{aligned}&(\psi_m) \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\ &(\theta_m) \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)),\\ &(\phi_m) \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)).\end{aligned} \tag{2.17}$$

Moving to the next estimate we take the time derivative of the first equation of (2.1) and by choosing $u = \overline{\psi_{t,m}}$ we obtain

$$i(\psi_{tt,m}, \overline{\psi_{t,m}}) + \kappa(\Delta\psi_{t,m}, \overline{\psi_{t,m}}) + i\alpha(b(x)\psi_{t,m}, \overline{\psi_{t,m}}) = (\phi_{t,m}\psi_m\chi(\omega), \overline{\psi_{t,m}}) + (\phi_m\psi_{t,m}\chi(\omega), \overline{\psi_{t,m}}).$$

Taking into consideration the imaginary part we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_{t,m}\|^2 + \alpha\int b(x)|\psi_{t,m}|^2 \leq \int \phi_{t,m}\psi_m\overline{\psi_{t,m}}$$

where since $\|\psi_m\|_\infty \leq c\|\Delta\psi_m\|^{1/2}\|\psi_m\|^{1/2}$ we obtain

$$\left|\int_\omega \phi_{t,m}\psi_m\overline{\psi_{t,m}}\right| \leq \|\phi_{t,m}\|\ \|\psi_m\|_\infty\|\psi_{t,m}\|_\omega \leq \frac{\alpha b_0}{2}\|\psi_{t,m}\|_\omega^2 + \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + C(\|\phi_{t,m}\|, \|\psi_m\|).$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_{t,m}\|^2 + \frac{\alpha b_0}{2}\|\psi_{t,m}\|_\omega^2 \leq \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + C(\|\phi_{t,m}\|, \|\psi_m\|). \tag{2.18}$$

Moving to the next energy estimate by choosing $u = \overline{\Delta\psi_{t,m}} + \epsilon\overline{\Delta\psi_m}$ for the first equation of (2.1) and taking the real part we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\kappa\|\Delta\psi_m\|^2 + 2\alpha\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m}\right\} + \kappa\epsilon\|\Delta\psi_m\|^2\\
&+ 2\alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\epsilon\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m} = \alpha\,\mathrm{Im}\int b(x)\psi_{t,m}\overline{\Delta\psi_m}\\
&- \mathrm{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} + \alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - \epsilon\,\mathrm{Re}\int \phi_m\psi_m\overline{\Delta\psi_m}.
\end{aligned} \tag{2.19}$$

Next, choosing $v = -\Delta\theta$ in the second equation of (2.1) produces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\Delta\phi_m\|^2 + (1+\epsilon^2)\|\nabla\phi_m\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m\right\}\\
&\quad + \epsilon\left\{\|\Delta\phi\|^2 + (1+\epsilon^2)\|\nabla\phi\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m\right\}\\
&\leq -\mathrm{Re}\int_\omega \Delta\psi_m\nabla\theta_m - \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m.
\end{aligned} \tag{2.20}$$

Adding (2.18) with (2.19) and (2.20) produces

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}H_1(t) + \epsilon H_1 = \alpha\,\mathrm{Im}&\int b(x)\psi_{t,m}\overline{\Delta\psi_m}\\
&- \mathrm{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} + \alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m}\\
&- \epsilon\,\mathrm{Re}\int \phi_m\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \Delta\psi_m\nabla\theta_m - \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m\\
&+ \left(\epsilon - \frac{\alpha b_0}{2}\right)\|\psi_{t,m}\|^2 + C\|\nabla\phi_m\|^2\|\nabla\psi_m\|^2
\end{aligned}$$

where

$$\begin{aligned}
H_1(t) = \|\psi_{t,m}\|^2 + \kappa\|\Delta\psi_m\|^2 + 2\alpha\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m} + \|\Delta\phi_m\|^2\\
+ (1+\epsilon^2)\|\nabla\phi_m\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m.
\end{aligned}$$

Set

$$\begin{aligned} F_1(t) = &\alpha \operatorname{Im}\int b(x)\psi_{t,m}\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} \\ &+ \alpha\epsilon \operatorname{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - \epsilon\operatorname{Re}\int \phi_m\psi_m\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \Delta\psi_m\nabla\theta_m \\ &- \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m + \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + \left(\epsilon - \frac{\alpha b_0}{2}\right)\|\psi_{t,m}\|^2 \\ &+ C(c_0, R, \epsilon, \kappa, \alpha, b_0, \|\theta_m\|, |\nabla\phi\|). \end{aligned} \tag{2.21}$$

Evaluating the terms in H_1 and F_1 we have

$$\begin{aligned} \left|\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m}\right| &\le \|\psi_m\|_\infty\|\phi_{t,m}\|\ \|\Delta\psi_m\| \le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + C(\kappa,\epsilon,\|\psi_m\|,\|\phi_{t,m}\|), \\ \left|\int_\omega \Delta\psi_m\nabla\theta_m\right| &\le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + \frac{2}{\kappa\epsilon}\|\nabla\theta_m\|^2, \\ \left|\int b(x)\psi_m\overline{\Delta\psi_m}\right| &\le \|b(x)\|_\infty\|\psi_m\|\ \|\Delta\psi_m\| \le \epsilon_1\|\Delta\psi_m\|^2 + C(\epsilon_1)(\|\psi_m\|,\|b(x)\|_\infty), \\ \left|\int b(x)\psi_{t,m}\overline{\Delta\psi_m}\right| &\le \|b(x)\|_\infty\|\psi_{t,m}\|\ \|\Delta\psi_m\| \le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + C(\kappa,\epsilon,\|b(x)\|_\infty)\|\psi_{t,m}\|^2 \\ \left|\int \phi_m\psi_m\overline{\Delta\psi_m}\right| &\le \|\phi_m\|_4\|\psi_m\|_4\|\Delta\psi_m\| \le \epsilon_2\|\Delta\psi_m\|^2 + C(\epsilon_2)(\|\psi_m\|,\|\phi_m\|,\|\nabla\psi_m\|,\|\nabla\phi_m\|), \\ \left|\int \phi_m\psi_{t,m}\overline{\Delta\psi_m}\right| &\le \|\phi_m\|_\infty\|\psi_{t,m}\|\ \|\Delta\psi_m\| \\ &\le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + \frac{\epsilon}{2}\|\Delta\phi_m\|^2 + \frac{\alpha b_0}{4}\|\psi_{t,m}\|^2 + C(\kappa,\epsilon,\alpha,b_0,c,\|\phi_m\|), \\ \left|\int \lambda(x)\phi_t\Delta\phi_m\right| &\le \|\lambda(x)\|_\infty\|\phi_{t,m}\|\ \|\Delta\phi_m\| \le \epsilon_3\|\Delta\phi_m\|^2 + C(\epsilon_3)(\|\lambda(x)\|_\infty,\|\phi_{t,m}\|). \end{aligned}$$

Therefore there exists a constant $\beta_1 > 0$ such that

$$\beta_1 H_1(t) \le F_1 + C(\kappa,\epsilon,\alpha,b_0,\epsilon_1,\epsilon_2,\epsilon_3,\|\lambda(x)\|_\infty,\|b(x)\|_\infty,\|\phi_m\|,\|\psi_m\|,\|\nabla\psi_m\|,\|\nabla\phi_m\|,\|\phi_{t,m}\|)$$

and

$$\frac{d}{dt}H_1(t) + \beta_1 H_1(t) \le C. \tag{2.22}$$

Employing Gronwall's Lemma we finally obtain

$$\|\psi_{t,m}\|^2 + \|\Delta\psi_m\|^2 + \|\Delta\phi_m\|^2 + \|\nabla\theta_m\|^2 \le C. \tag{2.23}$$

Hence,

$$\begin{aligned} (\psi_m) &\text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)\cap H_0^2(\Omega)), \\ (\theta_m) &\text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)), \\ (\phi_m) &\text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)\cap H_0^2(\Omega)) \\ (\psi_{t,m}) &\text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)). \end{aligned} \tag{2.24}$$

Therefore we may extract subsequences $\{\psi_\nu\} \subset \{\psi_m\}$, $\{\phi_\nu\} \subset \{\phi_m\}$ and $\{\theta_\nu\} \subset \{\theta_m\}$ such that

$$\begin{aligned} \psi_\nu \rightharpoonup \psi \quad & \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega) \cap H_0^2(\Omega)), \\ \theta_\nu \rightharpoonup \theta \quad & \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega)), \\ \phi_\nu \rightharpoonup \phi \quad & \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega) \cap H_0^2(\Omega)) \\ \psi_{t,\nu} \rightharpoonup \psi_t \quad & \text{for the weak star topology of } L^\infty(0,\infty; L^2(\Omega)). \end{aligned} \tag{2.25}$$

These convergencies are sufficient to pass to the limit (on a standard manner) in (2.1) which results in

$$\begin{aligned} i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi = \phi\psi\chi(\omega) \qquad & \text{in } L^\infty(0,\infty; L^2(\Omega)), \\ \phi_{tt} - \Delta\phi + \phi + \lambda(x)\phi_t = -\operatorname{Re}\nabla\psi\chi(\omega) \quad & \text{in } L^\infty(0,\infty; L^2(\Omega)). \end{aligned} \tag{2.26}$$

From [22, Lemma 4.1, Chapter II] we may derive that

$$\phi \in C(0,\infty; H_0^1(\Omega) \cap H_0^2(\Omega)) \quad \text{and} \quad \phi_t \in C(0,\infty; L^2(\Omega))$$

and since $\psi_t = \frac{1}{i}(-\kappa\Delta\psi - i\alpha b(x)\psi + \phi\psi\chi(\omega)) \in L^\infty(0,\infty; L^2(\Omega)$ using results in [16] we then obtain that

$$\psi \in C(0,\infty; H_0^1(\Omega) \cap H_0^2(\Omega)).$$

Let (ψ_1, ϕ_1) and (ψ_2, ϕ_2) be two solutions of the problem. Then by setting $z = \psi_1 - \psi_2$ and $w = \phi_1 - \phi_2$ the uniqueness of the solutions follows using the same above analysis.

This concludes the proof of Theorem 1.4. □

3 Uniform decay rates

In order to prove the energy decay of the system we derive some useful estimates.

Theorem 3.1. *Assume that Theorem 1.4 holds and let $C^* > 0$ denote a constant such that $|E(0)| \leq C^*$. Then there exists a $t^* > 0$ such that for every $t \geq t^*$, $E(0) > 0$.*

Proof. Taking into consideration the assumptions of Theorem 1.4 and the result $\|\psi\| \leq \epsilon^*$ for all $t \geq t^* > 0$ we evaluate the integral of the energy functional, that is

$$\left| \int_\omega \phi|\psi|^2 dx \right| \leq c\|\phi\|\|\psi\|_4^2 \leq \frac{1}{2}\|\phi\|^2 + \frac{c^2(\epsilon^*)^2}{2}\|\nabla\psi\|^2.$$

Therefore we have

$$E(t) \geq \frac{1}{2}\left[\|\psi\|^2 + \left(\kappa - \frac{c^2(\epsilon^*)^2}{2}\right)\|\nabla\psi\|^2 + \|\nabla\phi\|^2 + \|\phi_t\|^2\right], \quad \text{for } t \geq t^* \tag{3.1}$$

which completes the proof by choosing $\kappa > \frac{c^2(\epsilon^*)^2}{2}$. □

Proceeding with the analysis we take the inner product of (1.2) with $\overline{\psi_t + \alpha\psi}$, adding equation (2.3) and following similar steps as the ones for the a priori estimates we have

$$\frac{1}{2}\frac{d}{dt}\left\{\|\psi\|^2 + \kappa\|\nabla\psi\|^2 + \int_\omega \phi|\psi|^2\right\} + \kappa\alpha b_0\|\nabla\psi\|_\omega^2 + \alpha\int b(x)|\psi|^2 + \alpha b_0\int_\omega \phi|\psi|^2 = \frac{1}{2}\int_\omega \phi_t|\psi|^2. \tag{3.2}$$

Next, taking the inner product of (1.1) with ϕ_t gives

$$\frac{1}{2}\frac{d}{dt}\{\|\phi_t\|^2 + \|\nabla\phi\|^2 + \|\phi\|^2\} + \int \lambda(x)|\phi_t|^2 = \text{Re}\int_\omega \nabla\psi\phi_t. \tag{3.3}$$

Adding equations (2.5), (3.2) and (3.3) results in

$$\begin{aligned} E_t(t) + ab_0\|\psi\|_\omega^2 + \kappa\alpha b_0\|\nabla\psi\|^2 + \alpha\int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2 \\ + \alpha b_0\int_\omega \phi|\psi|^2 = \frac{1}{2}\int_\omega \phi_t|\psi|^2 + \text{Re}\int_\omega \nabla\psi\phi_t. \end{aligned} \tag{3.4}$$

From equation (2.3) we have

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_\omega^2 + \alpha\|\psi\|_\omega^2 \leq 0$$

from which we get

$$\|\psi\|_\omega \leq \|\psi(0)\|_\omega e^{-\alpha t} = \epsilon^* \tag{3.5}$$

and therefore since

$$\limsup_{t\to\infty} \|\psi\|_\omega = 0.$$

Next, evaluating the integrals

$$\begin{aligned} \left|\alpha b_0\int_\omega \phi|\psi|^2\right| &\leq \frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|_\omega^2 + \frac{\kappa\alpha b_0}{2}\|\nabla\psi\|_\omega^2, \\ \left|\text{Re}\int_\omega \nabla\psi\phi_t\right| &\leq \frac{1}{2\epsilon}\int \lambda(x)|\phi_t|^2 + \frac{\epsilon}{2\lambda_0}\|\nabla\psi\|_\omega^2, \\ \left|\frac{1}{2}\int_\omega \phi_t|\psi|^2\right| &\leq \frac{\epsilon^*}{8\epsilon}\int \lambda(x)|\phi_t|^2 + \frac{\epsilon}{2\lambda_0}\|\nabla\psi\|_\omega^2. \end{aligned}$$

Therefore

$$\begin{aligned} E_t(t) \leq & -\left(\frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0}\right)\|\nabla\psi\|_\omega^2 - \alpha\int b(x)|\psi|^2 - \left(1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon}\right)\int \lambda(x)|\phi_t|^2 \\ & + \frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|_\omega^2 - \alpha b_0\|\psi\|_\omega^2. \end{aligned} \tag{3.6}$$

For $\mu > 0$ let us define the perturbed energy

$$E_\mu(t) = E(t) + \mu p(t) \tag{3.7}$$

where

$$p(t) = (\psi(t), \psi(t)) + (\phi_t(t), \phi(t))_\omega. \tag{3.8}$$

Lemma 3.2. *For $\mu, C > 0$ we have that*

$$|E_\mu(t) - E(t)| \leq \mu C E(t), \quad \textit{for all } t \geq t^*.$$

Proof. We have

$$|p(t)| \leq \|\psi\|^2 + \frac{1}{2}\|\phi_t\|^2 + \frac{c_1}{2}\|\nabla\phi\|^2 \leq C^* E(t)$$

which completes the proof. □

Next, by taking the time derivative of $p(t)$ we obtain

$$\begin{aligned} p_t(t) &= 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi)_\omega + (\phi_t,\phi_t)_\omega \\ &\le 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi)_\omega + \frac{1}{\lambda_0}\int_\omega \lambda(x)|\phi_t|^2 \\ &\le 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi) + \frac{1}{\lambda_0}\int \lambda(x)|\phi_t|^2 \end{aligned}$$

which with the help of (1.1) we can deduce that

$$p_t(t) \le -2\alpha\int b(x)|\psi|^2 - \|\nabla\phi\|^2 - \|\phi\|^2 + \frac{1}{\lambda_0}\int \lambda(x)|\phi_t|^2 - \int \lambda(x)\phi_t\phi - \mathrm{Re}\int_\omega \nabla\psi\phi. \quad (3.9)$$

Adding equations (3.6)–(3.9) gives

$$\begin{aligned} E_{t,\mu} &= E_t(t) + \mu p_t(t) \\ &\le -\left(\frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0}\right)\|\nabla\psi\|_\omega^2 - \alpha(2\mu+1)\int b(x)|\psi|^2 \\ &\quad - \left(1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0}\right)\int \lambda(x)|\phi_t|^2 \\ &\quad + \frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|_\omega^2 - \mu\|\nabla\phi\|^2 - \mu\|\phi\|^2 - \mu\int \lambda(x)\phi_t\phi - \mathrm{Re}\,\mu\int_\omega \nabla\psi\phi - \alpha b_0\|\psi\|_\omega^2, \end{aligned} \quad (3.10)$$

where

$$\left|\mu\int_\omega \nabla\psi\phi\right| \le \frac{c\mu}{2}\|\nabla\psi\|_\omega^2 + \frac{\mu}{2}\|\nabla\phi\|^2,$$
$$\left|\mu\int \lambda(x)\phi_t\phi\right| \le \frac{c\mu\|\lambda\|_\infty}{2}\int \lambda(x)|\phi_t|^2 + \frac{\mu}{2}\|\nabla\phi\|^2$$

which concludes in

$$\begin{aligned} E_{t,\mu} &= E_t(t) + \mu p_t(t) \\ &\le -\left(\frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0} - \frac{c\mu}{2}\right)\|\nabla\psi\|_\omega^2 - \alpha(2\mu+1)\int b(x)|\psi|^2 \\ &\quad - \left(1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0} - \frac{c\mu\|\lambda\|_\infty}{2}\right)\int \lambda(x)|\phi_t|^2 - \mu\left(1 - \frac{\epsilon^* c\alpha b_0}{2\kappa\mu}\right)\|\phi\|^2. \end{aligned}$$

Therefore we will require the following expressions to be nonnegative

$$\begin{cases} \frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0} - \frac{c\mu}{2} > 0, \\ 1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0} - \frac{c\mu\|\lambda\|_\infty}{2} > 0, \\ 1 - \frac{\epsilon^* c\alpha b_0}{2\kappa\mu} > 0. \end{cases}$$

Therefore, choosing κ sufficiently large enough such that the following inequality holds

$$2\kappa\lambda_0 > \epsilon^* acb_0(2 + \lambda_0\|\lambda\|_\infty)$$

we may deduce that there exists a k such that

$$E_{t,\mu}(t) \le -k\left[\int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2\right] \quad (3.11)$$

and hence $E_\mu(t)$ would be a non increasing function.

Remark 3.3. The time t^* introduced in the energy decay analysis which is present through the constant ϵ^* has a specific physical meaning. This is the time so that the non-collisional integral $\int \phi|\psi|^2$ is absorbed by the collisional terms (see (3.1)). Therefore, t^* roughly signifies the time required by the collisional damping to smooth out the excessive difference of the kinetic energies of the resonant electrons and the non-resonant ions (equipartition). It is important to note that equation (3.6) is a non increasing function due to the positive term $\|\phi\|_\omega$ which depends heavily on the t^*.

Lemma 3.4. *For all $T > T_0$ there exists a positive constant $C = C(t)$ such that if (ψ, ϕ) is the regular solution of the system (1.1) where $(\psi_0, \phi_0, \phi_1) \in \{H_0^1(\Omega) \cap H^2(\Omega)\}^2 \times H_0^1(\Omega)$ we have*

$$E_\mu(0) \leq C \int_0^T \left[\int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2 \right] dt. \tag{3.12}$$

Proof. We will prove this lemma by contradiction. Assume (3.12) is not true and let $(\psi_k(0), \phi_k(0), \phi_{t,k}(0))$ be a sequence of initial data where the corresponding solutions $(\psi_k, \phi_k, \phi_{t,k})$ with $E_{\mu,k}(0)$, uniformly bounded in k satisfy

$$\lim_{k\to+\infty} \frac{E_{\mu,k}(0)}{\int_0^T \left[\int b(x)|\psi_k|^2 + \int \lambda(x)|\phi_{t,k}|^2 \right] dt} = +\infty. \tag{3.13}$$

Since $E_{\mu,k}(t)$ is non increasing and $E_{\mu,k}(0)$ remains bounded we may obtain a subsequence, denoted again as (ψ_k, ϕ_k) for which we have

$$\begin{cases} \psi_k \rightharpoonup \psi \text{ weak star in } L^\infty(0, T; H_0^1(\Omega)), \\ \phi_k \rightharpoonup \phi \text{ weak star in } L^\infty(0, T; H_0^1(\Omega)), \\ \phi_{t,k} \rightharpoonup \phi_t \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \\ \psi_{t,k} \rightharpoonup \psi_t \text{ weak star in } L^\infty(0, T; L^2(\Omega)). \end{cases} \tag{3.14}$$

By compactness results, see [14] we get

$$\begin{aligned} \psi_k \to \psi_k \text{ strongly in } L^\infty(0, T; L^2(\Omega)), \\ \phi_k \to \phi_k \text{ strongly in } L^\infty(0, T; L^2(\Omega)). \end{aligned} \tag{3.15}$$

Now, taking into consideration (3.13) and (3.14) we obtain

$$\begin{aligned} \lim_{k\to+\infty} \int_0^T \int b(x)|\psi_k|^2 dx dt = 0, \\ \lim_{k\to+\infty} \int_0^T \int \lambda(x)|\phi_{t,k}|^2 dx dt = 0. \end{aligned} \tag{3.16}$$

Setting

$$c_k := [E_{\mu,k}(0)]^{1/2} \quad \text{and} \quad \hat{\phi}_k = \frac{1}{c_k}\phi_k, \quad \hat{\psi}_k = \frac{1}{c_k}\psi_k$$

we infer that

$$\hat{E}_{\mu,k}(t) := \frac{E_{\mu,k}(t)}{c_k^2}$$

for which we have

$$\hat{E}_k(0) = 1. \tag{3.17}$$

Taking into consideration the following system

$$\begin{cases} i\hat{\psi}_{t,k} + \kappa\Delta\hat{\psi}_k + i\alpha b(x)\hat{\psi}_k = \hat{\phi}_k\psi_k\chi(\omega), \\ \hat{\phi}_{tt,k} - \Delta\hat{\phi}_k + \hat{\phi}_k + \lambda(x)\hat{\phi}_{t,k} = -\operatorname{Re}\nabla\hat{\psi}_k\chi(\omega), \\ \hat{\psi}_k = \hat{\phi}_k = 0 \in \Gamma\times(0,T), \\ \hat{\psi}_k(0) = \hat{\psi}_{0k},\ \hat{\phi}_k(0) = \hat{\phi}_{0k},\ \hat{\phi}_{t,k}(0) = \hat{\phi}_{1k} \text{ in } \Omega, \\ \hat{\phi}_{t,k} \to 0 \in L^2(0,T;L^2(\omega)) \end{cases} \tag{3.18}$$

and since $E_{\mu,k}(0) = 1$ we may deduce that for a subsequence $(\hat{\psi}_k, \hat{\phi}_k)$ it is true that

$$\begin{cases} \hat{\psi}_k \rightharpoonup \hat{\psi} \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \hat{\psi}_k \to \hat{\psi} \text{ strongly in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\psi}_{t,k} \to \hat{\psi}_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\phi}_k \rightharpoonup \hat{\phi} \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \hat{\phi}_{t,k} \rightharpoonup \hat{\phi}_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\phi}_k \to \hat{\phi} \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.19}$$

From the (3.19), we obtain

$$\begin{aligned} \lim_{k\to+\infty}\int_0^T\int b(x)|\hat{\psi}_k|^2 dxdt = 0, \\ \lim_{k\to+\infty}\int_0^T\int \lambda(x)|\hat{\phi}_{t,k}|^2 dxdt = 0, \end{aligned} \tag{3.20}$$

and therefore by (3.20) and by the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$

$$\begin{aligned} \lim_{k\to+\infty}\int_0^T\int_\omega |\nabla\hat{\psi}_k|^2 dxdt = 0, \\ \lim_{k\to+\infty}\int_0^T\int_\omega |\hat{\phi}_k\psi_k|^2 dxdt = 0. \end{aligned} \tag{3.21}$$

Taking into consideration (3.20) and letting the limit $k \to +\infty$ for the system (3.18) we get for the wave equation

$$\begin{cases} \hat{\phi}_{tt} - \Delta\hat{\phi} + \hat{\phi} = 0 \text{ in } \Omega\ \times(0,T), \\ \hat{\phi} = 0 \in \Gamma\times(0,T), \\ \phi_t = 0 \text{ a.e.} \in \omega\times(0,T) \end{cases} \tag{3.22}$$

and for the Schrödinger equation

$$\begin{cases} i\hat{\psi}_t + \kappa\Delta\hat{\psi} = 0, \text{in } \Omega\ \times(0,T), \\ \hat{\psi} = 0 \text{ on } \Gamma\times(0,T). \end{cases} \tag{3.23}$$

Setting $\hat{\phi}_t = v$ equation (3.22) in the distributional sense becomes

$$\begin{cases} v_{tt} - \Delta v = 0 \in D'(\Omega\ \times(0,T)), \\ v = 0 \in \Gamma\times(0,T), \\ v = 0 \text{ a.e.} \in \omega\times(0,T). \end{cases} \tag{3.24}$$

From standard uniqueness results from equation (3.24) we may conclude that $v \equiv 0$, that is $\hat{\phi}_t \equiv 0$. Therefore for a.e. $t \in (0,T)$

$$\begin{cases} -\Delta\hat{\phi} = 0 \in \Omega, \\ \hat{\phi} = 0 \in \Gamma \end{cases} \tag{3.25}$$

which multiplying by $\hat{\phi}$ implies that $\hat{\phi} \equiv 0$. Following a similar procedure for the Schrödinger equation the uniqueness theorem concludes that $\hat{\psi} = 0$ a.e. $\in \Omega$.

In order to achieve a contradiction it is enough to prove that $\hat{E}_{\mu,k}(0) \to 0$ as $k \to +\infty$.

$$\begin{aligned} \hat{E}_{\mu,k}(0) = & \frac{1}{2}\Big\{\int((2\mu+1)|\hat{\psi}(x,0)|^2 + \kappa|\nabla\hat{\psi}(x,0)|^2 + |\hat{\phi}(x,0)|^2 + |\nabla\hat{\phi}(x,0)|^2 + |\hat{\phi}_t(x,0)|^2) \\ & + \int_\omega \phi(x,0)|\hat{\psi}(x,0)|^2 + 2\mu\int_\omega \hat{\phi}_t(x,0)\hat{\phi}(x,0)\Big\} \\ \le & \frac{1}{2}\Big\{\int((2\mu+1)|\hat{\psi}(x,0)|^2 + \left(\kappa + \frac{c}{2}\right)|\nabla\hat{\psi}(x,0)|^2 + \frac{3}{2}|\hat{\phi}(x,0)|^2 \\ & + (\mu c+1)|\nabla\hat{\phi}(x,0)|^2 + (\mu+1)|\hat{\phi}_t(x,0)|^2)\Big\} = E_{\mu,\hat{\psi}_k}(0) + E_{\mu,\hat{\phi}_k}(0). \end{aligned} \tag{3.26}$$

Our aim is to prove that $E_{\mu,\hat{\psi}_k}(0) \to 0$ and $E_{\mu,\hat{\phi}_k}(0) \to 0$ with the help of (1.7) and (1.9). For this purpose let $\hat{\psi}_k = v_k + w_k$ where $\hat{\psi}_k$ is the solution of the system (3.18) and v_k, w_k are the solutions of the following systems respectively,

$$\begin{cases} iv_{t,k} + \kappa\Delta v_k = 0 \in \Omega \times (0,T), \\ v_k = 0 \in \Gamma \times (0,T), \\ v_k(0) = \hat{\psi}_{0,k} \in \Omega \end{cases} \tag{3.27}$$

and

$$\begin{cases} iw_{t,k} + \kappa\Delta w_k = -i\alpha b(x)\hat{\psi}_k + \hat{\phi}_k\psi_k\chi(\omega), \in \Omega \times (0,T), \\ w_k = 0 \in \Gamma \times (0,T), \\ w_k(0) = 0 \in \Omega. \end{cases} \tag{3.28}$$

Similarly for the wave equation we obtain $\hat{\phi}_k = z_k + u_k$ which produces the following problems

$$\begin{cases} z_{tt,k} + \Delta z_k = 0 \in \Omega \times (0,T), \\ z_k = \hat{\phi}_{0k} \in \Gamma \times (0,T), \\ z_k(0) = \hat{\phi}_{0,k} \in \Omega, \\ z_{t,k}(0) = \hat{\phi}_{1k} \in \Omega \end{cases} \tag{3.29}$$

and

$$\begin{cases} u_{tt,k} + \Delta u_k = -\lambda(x)\hat{\phi}_{t,k} - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) \in \Omega \times (0,T), \\ u_k = 0 \in \Gamma \times (0,T), \\ u_k(0) = 0 \in \Omega, \\ u_{t,k}(0) = 0 \in \Omega. \end{cases} \tag{3.30}$$

Therefore, it follows that

$$\begin{aligned}\hat{E}_{\mu,k} &\leq E_{\mu,\hat{\psi}_k}(0) + E_{\mu,\hat{\phi}_k}(0) = E_{\mu,v_k}(0) + E_{\mu,z_k}(0) \leq c_1 \int_0^T \int_\omega |v_k|^2 + c_2 \int_0^T \int_\omega |z_{t,k}|^2 \\ &\leq c_1\Big(\int_0^T \int b(x)|\hat{\psi}_k|^2 + \int_0^T \int_\omega |w_k|^2\Big) + c_2\Big(\int_0^T \int \lambda(x)|\hat{\phi}_{t,k}|^2 + \int_0^T \int_\omega |x_k|^2\Big).\end{aligned} \tag{3.31}$$

From equation (3.28) we have the following integral form

$$\hat{w}_k(t) = S(t)\hat{w}_k(0) + \int_0^T S(t-\tau)F(\tau)d\tau, \tag{3.32}$$

where $S(t)$ is the semigroup generated by

$$\begin{aligned} A : D(A) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) &\to L^2(\Omega), \\ y \to Ay &= -i\Delta y \end{aligned}$$

and $F(t) = \hat{\phi}_k(t)\psi_k(t)\chi(\omega) - i\alpha b(x)\hat{\psi}_k(t)$. Thus, taking into consideration that $\|S(t)\|_{\mathcal{L}(L^2(\Omega))} \leq C$ we have

$$\|w_k\|^2 \leq c_1\|w_{0,k}\|^2 + c_2\Big(\int_0^T \|F(\tau)\|d\tau\Big)^2 \leq C(\|w_{0k}\|^2 + \|F\|^2_{L^1(0,T;L^2(\Omega))})$$

which with the help of the embedding $L^\infty(0,T;L^2(\Omega)) \to L^1(0,T;L^2(\Omega))$ and $w_k(0) = 0$ produces

$$\begin{aligned}\int_0^T \int_\omega |w_k|^2 &\leq \|w_k\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\|F\|^2_{L^1(0,T;L^2(\Omega))} \leq C\int_0^T \int |\hat{\phi}_k\psi_k\chi(\omega) - i\alpha b(x)\hat{\psi}_k|^2.\end{aligned} \tag{3.33}$$

Moving onto the wave equation we have the following integral form expression for the system (3.29)

$$U_k(t) = S(t)U_{0k} + \int_0^T S(t-\tau)F(\tau)d\tau$$

where

$$U_k = \begin{pmatrix} u_k \\ u_{t,k} \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ -\lambda(x)\hat{\phi}_k - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) \end{pmatrix}.$$

Evaluating the following integral

$$\begin{aligned}\int_0^T \int_\omega |u_{t,k}|^2 &\leq \|u_{t,k}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\|F\|^2_{L^1(0,T;L^2(\Omega))} \leq C\int_0^T \int |-\lambda(x)\hat{\phi}_k - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega)|^2.\end{aligned} \tag{3.34}$$

Therefore, from equation (3.31) we obtain

$$\begin{aligned}\hat{E}_{\mu,k}(t) \leq C\Big(&\int_0^T \int b(x)|\hat{\psi}_k|^2 + \int_0^T \int |\hat{\phi}_k\psi_k\chi(\omega) - i\alpha b(x)\hat{\psi}_k|^2 \\ &+ \int_0^T \int \lambda(x)|\hat{\phi}_{t,k}|^2 + \int_0^T \int |-\operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) - \lambda(x)\hat{\phi}_{t,k}|^2\Big),\end{aligned} \tag{3.35}$$

which taking into consideration (3.20) and (3.21) produces $\hat{E}_{\mu,k}(0) \to 0$ as $k \to +\infty$ and therefore contradicts the expression (3.17). □

Proof of Theorem 1.5. Continuing with the proof of Theorem 1.5 and by taking $T_0 > 0$ large enough from (3.11) we may deduce that

$$E_\mu(T_0) - E_\mu(0) \leq -k\left[\int_0^{T_0} b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2\right] \tag{3.36}$$

and from Lemma 3.4 we also have

$$E_\mu(0) \leq C\int_0^{T_0} D(t)$$

where

$$D(t) := \int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2.$$

Therefore, we get

$$E_\mu(T_0) \leq E_\mu(0) \leq C\int_0^{T_0} D(t) \leq -\frac{C}{k}E_\mu(T_0) + \frac{C}{k}E_\mu(0),$$

so

$$\left(1+\frac{C}{k}\right)E_\mu(T_0) \leq \frac{C}{k}E_\mu(0).$$

Hence,

$$E_\mu(T_0) \leq \nu E_\mu(0), \qquad 0 < \nu < 1.$$

Proceeding in a similar way from T to $2T$ and eventually to nT we have

$$E_\mu(nT) \leq \nu^n E_\mu(0), \qquad \forall\, T > T_0.$$

Finally, let $t > T_0$ then $t = nT_0 + r$ for $0 \leq r \leq T_0$ and

$$E_\mu(t) \leq E_\mu(t-r) = E_\mu(nT_0) \leq \nu^n E_\mu(0) = \nu^{\frac{t-r}{T_0}} E_\mu(0) = e^{\frac{t-r}{T_0}\ln\nu} E_\mu(0).$$

Moreover, by Lemma 3.2 we have

$$E_\mu(t) \leq 2E(t) \quad \text{for } t \geq 0$$

therefore

$$E_\mu(t) \leq 2E(0)e^{\frac{t-r}{T_0}\ln\nu} \quad \text{for } t \geq 0$$

which completes the proof of Theorem 1.5. □

Acknowledgements

The authors would like to thank the referee for his comments that have improved the results of this work. The publication of this paper has been partly supported by the University of Piraeus Research Center

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