Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **62**, 1–17; https://doi.org/10.14232/ejqtde.2022.1.62 www.math.u-szeged.hu/ejqtde/

Uniqueness and Liouville type results for radial solutions of some classes of k-Hessian equations

Mohamed Ben Chrouda ✉

Université de Monastir, Institut Supérieur d'Informatique et de Mathématiques de Monastir, Tunisie
Université de Sousse, Laboratoire de Mathématiques: Modélisation Déterministe et Aléatoire, Tunisie

Received 19 July 2022, appeared 18 December 2022
Communicated by Petru Jebelean

Abstract. We establish a uniqueness theorem and a Liouville type result for positive radial solutions of some classes of nonlinear autonomous equation with the k-Hessian operator. We also give some interesting qualitative properties of solutions. We provide an approach, based upon a Pohozaev-type identity, that unifies all our results.

Keywords: uniqueness, Liouville-type theorem, radial solutions, k-Hessian operators.

2020 Mathematics Subject Classification: 35A02, 35B08, 35B09, 35B40, 35B53.

1 Introduction

Let $\Omega = \mathbb{R}^n$ or $\Omega = B$ a finite ball about the origin. For $1 \leq k \leq n$ and $u \in C^2(\Omega)$, let $S_k(D^2u)$ denote the k-Hessian operator of u which is defined by

$$S_k(D^2u) = \sigma_k\left(\lambda[D^2u]\right) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k},$$

where $\lambda[D^2u] = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix D^2u of u and σ_k is the k^{th} elementary symmetric polynomial in n variables. This family of partial differential operators includes the Laplace and Monge–Ampère operator, respectively, when $k = 1$ and $k = n$. The study of general k-Hessian operators have many applications in geometry, optimization theory and other related fields. It began with the work of Krylov [15] and Caffarelli, Nirenberg, and Spruck [3]; and was continued by Jacobson [13], Trudinger and Wang [26,27], Tso [28] and Wang [29], among others. In this paper, we consider the Dirichlet boundary value problem

$$\begin{aligned} S_k(D^2(-u)) &= f(u) \quad \text{on } \Omega \\ u &> 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

✉ Email: Mohamed.BenChrouda@isimm.rnu.tn

For $\Omega = \mathbb{R}^n$, the Dirichlet boundary condition is understood to mean that

$$\lim_{|x|\to\infty} u(x) = 0.$$

When $k = 1$, problem (1.1) is reduced to

$$\begin{aligned} -\Delta u &= f(u) \quad \text{on } \Omega \\ u &> 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The interest in radial solutions is sparked by the well known results of Gidas, Ni, and Nirenberg [11, 12]. The authors showed that any solution of problem (1.2) is necessarily radially symmetric. The uniqueness of ground state solution (radial solution in $\Omega = \mathbb{R}^n$) plays an important role in physics. This importance was mentioned, for instance, by Troy [25] and the references therein for the logarithmic Schrödinger equation $-\Delta u = u \ln u$. Over the past half century, the question of uniqueness of radial solution of problem (1.2) has been explored under a variety of conditions on the non-linearity $f(u)$. For relevant references, see [1,4,6–9,14,16–19,21–25,31]. In general, two types of datum $f(u)$ are considered: $f(u) > 0$ on the whole of the interval $(0,\infty)$, or $f(u) < 0$ on $(0,\gamma)$ and $f(u) > 0$ on (γ,∞) for some $\gamma > 0$. The fundamental examples correspond to $f(u) = u^p$ and $f(u) = u^p - u$, $p > 1$. A natural question to ask is whether uniqueness of radial solutions of problem (1.1) continues to hold for general k-Hessian operators. This question seems to have received almost no attention in the literature and little is known when $2 \le k \le n$. Clément, Figueiredo and Mitidieri [5] studied problem (1.1) in $\Omega = B$ and $f(u) = \lambda e^u$, $\lambda > 0$, which is known as Liouville–Gelfand problem in the literature. The authors proved the existence of $\lambda^* > 0$ such that the Liouville–Gelfand problem has exactly two radial solutions if $0 < \lambda < \lambda^*$, a unique radial solution if $\lambda = \lambda^*$, and no radial solutions if $\lambda > \lambda^*$. The question of uniqueness has been explored also by Wei [30] and Zhang [32] over the last few years. Wei apply the argument of Erbe and Tang [8, 9] to prove the uniqueness of radial solutions to problem (1.1) when $1 \le k < n/2$ and $f(u)$ satisfies some convexity conditions. The argument is based upon a Pohozaev-type identity and the monotone separation techniques. Zhang proved existence and uniqueness of radial solution of problem (1.1) where $f(u)$ is replaced by $\lambda f(u)$ a positive continuous function which satisfies some growth conditions at ∞ and 0, and λ is a large parameter. We also note that a characterization of semi-stable radial solutions of some class of autonomous k-Hessian equation in the unit ball have been studied recently in [20]. We mention here that all the aforementioned authors investigated problem (1.1) in $\Omega = B$ and a non-linearity $f(u)$ which is always assumed to be positive on the whole of $(0,\infty)$.

In the present work, we are concerned with radial solutions of problem (1.1) in $\Omega = B$ or $\Omega = \mathbb{R}^n$ and datum $f(u)$ of the form:

(a) $f_1(u) = u^p - u^k$, $k < p$.

(b) $f_2(u) = u^k - u^p$, $0 < p < k$.

(c) $f_3(u) = u^k(\ln u + \beta)$, $\beta \in \mathbb{R}$.

Here $k \in \{1,\dots,n\}$ is the index of the Hessian operator and p is a parameter. As far as we know this is the first work dealing with non-linearity $f(u)$ which change sign on $(0,\infty)$. Our main results are the following:

Theorem 1.1 (Liouville-type results)**.** *Let $n \geq 1$ and $k \in \{1, \dots, n\}$.*

1. *If $f(u) = u^p - u^k$, $p > k$ and*
$$p(n-2k) \geq k(n+2), \qquad n > 2k,$$
then problem (1.1) has no radial solutions in $\Omega = B$.

2. *If $f(u) = u^k - u^p$, $0 < p < k$ then problem (1.1) has no radial solutions in $\Omega = \mathbb{R}^n$.*

We mention that similar nonexistence result of radial solutions of problem (1.1) in $\Omega = B$ was established in [5, 28] when $f(u) = u^p$, $p > 1$. In our second main result, we give some interesting qualitative properties of radial solutions of problem (1.1).

Theorem 1.2. *Let $n \geq 1$ and $k \in \{1, \dots, n\}$. Let $f(u) = f_i(u)$, $i = 1, 2$ or 3.*

1. *Suppose that problem (1.1) admits a radial solution u_j in $\Omega = B_j$ a finite ball of radius b_j, $j = 1, 2$. If $u_1(0) < u_2(0)$ then $b_2 < b_1$, and $u_1(r)$ and $u_2(r)$ intersect exactly once in $(0, b_2)$.*

2. *If problem (1.1) admits a radial solution u in $\Omega = B$ for some ball B then there is no radial solution v in $\Omega = \mathbb{R}^n$ such that $u(0) < v(0)$.*

As an immediate consequence, we have the following uniqueness result in balls.

Corollary 1.3. *Problem (1.1) has at most one radial solution in $\Omega = B$.*

Theorem 1.4 (Uniqueness results in $\Omega = \mathbb{R}^n$)**.** *Let $n \geq 1$ and $k \in \{1, \dots, n\}$.*

1. *If $f(u) = u^p - u^k$, $k < p$ and $p(n-2k) < k(n+2)$ then problem (1.1) has at most one radial solution in $\Omega = \mathbb{R}^n$. Furthermore, if such a radial solution u exists then*
$$u(x) \leq C \exp\left[-\frac{k+1}{2k} \left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}} \right)^{\frac{1}{k+1}} |x|^{\frac{2k}{k+1}} \right],$$
where C is a positive constant.

2. *If $f(u) = u^k(\ln u + \beta)$ then the function*
$$u(x) = \exp\left[-\left(\frac{1}{2\, C_{n-1}^{k-1}} \right)^{\frac{1}{k}} \frac{|x|^2}{2} + \frac{n}{2k} - \beta \right]$$
is the unique radial solution of problem (1.1) in $\Omega = \mathbb{R}^n$.

This theorem is an extension of the uniqueness results established in [16, 25] when $k = 1$. We note that the kind of bound of solutions given in the first statement when $k = 1$ has been proved in the celebrated work of Berestycki and Lions [2].

The following result is an immediate consequence of Theorems 1.2 and 1.4.

Corollary 1.5. *For $f(u) = u^k(\ln u + \beta)$, problem (1.1) has no radial solution u such that*
$$u(0) < \exp\left(\frac{n}{2k} - \beta \right)$$
neither in finite balls nor in the whole of $\mathbb{R}^n$.

We provide an approach, based upon a Pohozaev-type identity due to Tso [28], that unifies all our uniqueness and Liouville type results. Our method can be used without any restriction on $k \in \{1, \dots, n\}$, including the Laplace operator when $k = 1$ and also the Monge–Ampère operator when $k = n$.

We finally note that the question of uniqueness for problems of type (1.1) has been also explored with non-local operators. In this context, Frank, Lenzmann, and Silvestre [10] showed the uniqueness of ground state solutions for the non-linear equation

$$(-\Delta)^s u + u - u^p = 0 \quad \text{in } \mathbb{R}^n,$$

where $(-\Delta)^s$ denotes the fractional Laplacian with $s \in (0,1)$ and $p > 1$ a real number. A little is known on the uniqueness of positive radial solutions for the fractional Laplacian.

2 Properties of radial solutions

For radial function $v(x) = v(r)$ with $r = |x|$, we have

$$\begin{aligned} S_k(D^2(-v))(x) &= r^{1-n} \frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k \right)' \\ &= C_{n-1}^{k-1} \left(\frac{-v'}{r} \right)^{k-1} \left(-v'' - \frac{n-k}{k} \frac{v'}{r} \right), \end{aligned}$$

where $C_{n-1}^{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$ and $1 \le k \le n$. Therefore, when referring to a radial solution v of problem (1.1) in Ω, we mean a C^2 function $v(|x|) = v(r)$ satisfies

$$\begin{aligned} \frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k \right)' &= r^{n-1} f(v) \quad \text{on } (0,b) \\ v &> 0 \qquad\qquad\quad \text{on } (0,b) \\ \lim_{r \to b} v(r) &= 0, \end{aligned} \tag{2.1}$$

where b denotes the radius of Ω ($0 < b \le \infty$) and $f = f_i$, $i = 1,2,3$, the function defined by

(a) $f_1(v) = v^p - v^k, \;\; k < p.$

(b) $f_2(v) = v^k - v^p, \;\; 0 < p < k.$

(c) $f_3(v) = v^k \ln v + \beta v^k, \;\; \beta \in \mathbb{R}.$

Denote by γ_f the unique zero of f in $(0, \infty)$. We note that $f < 0$ on $(0, \gamma_f)$, $f > 0$ on (γ_f, ∞) and

$$\gamma_{f_1} = \gamma_{f_2} = 1, \quad \gamma_{f_3} = e^{-\beta}.$$

In the following lemma, we shall focus attention on some basic properties of solutions of problem (2.1).

Lemma 2.1. *If v is a solution of (2.1), then $v'(0) = 0$, $v(0) > \gamma_f$ and $v'(r) < 0$ for $0 < r < b$.*

Proof. The results are well known when $k = 1$, see for instance [24]. So we can assume that $k \ge 2$. We first write equation (2.1) in the following equivalent form

$$C_{n-1}^{k-1} \left(\frac{-v'}{r} \right)^{k-1} \left(-v'' - \frac{n-k}{k} \frac{v'}{r} \right) = f(v). \tag{2.2}$$

If $v'(0) \neq 0$ then by letting r tend to 0 we obtain $\infty = f(v(0))$, a contradiction. Hence $v'(0) = 0$. Suppose that there exists $r_0 \in (0, b)$ such that $v'(r_0) = 0$ and $v' \neq 0$ on $(0, r_0)$. Then integrate the equation in (2.1) from 0 to r_0 to get

$$0 = \frac{C_{n-1}^{k-1}}{k} \left[r^{n-k}(-v')^k \right]_0^{r_0} = \int_0^{r_0} r^{n-1} f(v)\, dr. \tag{2.3}$$

On the other hand, since $v'(r_0) = 0$, it follows from (2.2) that $f(v(r_0)) = 0$, yielding $v(r_0) = \gamma_f$ the unique zero of f other than 0. Since v is strictly monotone on the interval $(0, r_0)$, this implies that $f(v)$ does not change sign on $(0, r_0)$, which contradicts (2.3). Therefore $v' \neq 0$ on $(0, b)$, and hence $v' < 0$ as desired. Suppose that $v(0) \leq \gamma_f$. Since v decreases on $(0, b)$ and $f < 0$ on $(0, \gamma_f)$, this implies that $f(v) < 0$ on $(0, b)$ and thus

$$\int_0^r t^{n-1} f(v) dt < 0.$$

But this is impossible since

$$\int_0^r t^{n-1} f(v)\, dt = \frac{C_{n-1}^{k-1}}{k} r^{n-k}(-v')^k > 0.$$

Hence $v(0) > \gamma_f$. This completes the proof. □

Let

$$F(v) = \int_0^v f(t) dt.$$

One easily checks that

$$F_1(v) = \frac{v^{p+1}}{p+1} - \frac{v^{k+1}}{k+1}, \qquad F_2(v) = \frac{v^{k+1}}{k+1} - \frac{v^{p+1}}{p+1}, \qquad F_3(v) = \frac{v^{k+1}}{k+1} \left(\ln v + \beta - \frac{1}{k+1} \right).$$

We denote by γ_F the unique zero of F in $(0, \infty)$. It can be easily calculated

$$\gamma_{F_1} = \left(\frac{p+1}{k+1} \right)^{\frac{1}{p-k}}, \qquad \gamma_{F_2} = \left(\frac{k+1}{p+1} \right)^{\frac{1}{k-p}}, \qquad \gamma_{F_3} = e^{\frac{1}{k+1} - \beta}.$$

We note that

$$\gamma_{F_i} > \gamma_{f_i}, \qquad i = 1, 2, 3.$$

For a given solution v of problem (2.1), we define

$$E(r, v) := C_{n-1}^{k-1} r^{1-k}(-v')^{k+1} + (k+1)F(v), \qquad 0 \leq r < b. \tag{2.4}$$

Lemma 2.2. *If v is a solution of problem (2.1), then $v(0) > \gamma_F$ and*

$$E(r, v) > 0, \qquad 0 \leq r < b. \tag{2.5}$$

Proof. Let v be a solution of (2.1). By using (2.2), a straightforward computation gives

$$\frac{d}{dr} E(r, v) = -(n + k(n-2)) \frac{C_{n-1}^{k-1}}{k} \frac{(-v')^{k+1}}{r^k}.$$

Since $v' < 0$ on $(0, b)$, this implies that $E(r, v)$ decreases on $(0, b)$, yielding

$$E(r, v) > \lim_{r \to b} E(r, v) = C_{n-1}^{k-1} \lim_{r \to b} r^{1-k}(-v')^{k+1} \geq 0.$$

For $r = 0$, we obtain $0 < E(0, v) = (k+1)F(v(0))$ from which we conclude that $v(0) > \gamma_F$. □

For $i \in \{1,2,3\}$, let

$$G_i(t) = (n-2k)tf_i(t) - n(k+1)F_i(t). \tag{2.6}$$

A straightforward computation shows that

$$G_1(t) = C(n,k,p)t^{p+1} + 2kt^{k+1},$$
$$G_2(t) = -2kt^{k+1} - C(n,k,p)t^{p+1}$$

and

$$G_3(t) = C(n,k,p)t^{k+1}\ln t + \left(\frac{n}{k+1} - 2\beta k\right)t^{k+1},$$

where

$$C(n,k,p) = n - 2k - \frac{n(k+1)}{p+1} \tag{2.7}$$

with the obvious convention $p = k$ when $f(t) = t^k \ln t + \beta t^k$.

Remark 2.3.

1. We note that $C(n,k,p) < 0$ when $f = f_2$ or $f = f_3$, that is, when $0 < p \leq k$. However, the mapping $p \to C(n,k,p)$ can change sign when $p > k$.
2. If $C(n,k,p) \geq 0$ then G_1 is positive on $(0,\infty)$.
3. If $C(n,k,p) < 0$ then G_1 is positive on $(0,\gamma_{G_1})$ and negative on (γ_{G_1},∞), where

$$\gamma_{G_1} = \left(\frac{2k}{-C(n,k,p)}\right)^{\frac{1}{p-k}}.$$

4. For $i = 2,3$, G_i is positive on $(0,\gamma_{G_i})$ and negative on (γ_{G_i},∞), where

$$\gamma_{G_2} = \left(\frac{-C(n,k,p)}{2k}\right)^{\frac{1}{k-p}}, \qquad \gamma_{G_3} = \exp\left(-\beta + \frac{n}{2k(k+1)}\right).$$

5. It is worth noting that

$$\gamma_{F_1} < \gamma_{G_1}, \qquad \gamma_{G_2} < \gamma_{F_2}$$

   and

$$\gamma_{F_3} < \gamma_{G_3} \quad \text{if } n-2k > 0, \qquad \gamma_{G_3} < \gamma_{F_3} \quad \text{if } n-2k < 0.$$

6. A straightforward computation shows

$$\frac{G(t)}{t^{k+1}} - \frac{G(s)}{s^{k+1}} = C(n,k,p)\left(\frac{f(t)}{t^k} - \frac{f(s)}{s^k}\right), \qquad t,s > 0. \tag{2.8}$$

   This identity will be crucial in the proof of uniqueness results.

For a solution v of problem (2.1) in $(0,b)$, let

$$P(r,v) = r^n\left[C_{n-1}^{k-1}r^{1-k}(-v')^{k+1} + (k+1)F(v) - C_{n-1}^{k-1}\frac{n-2k}{k}v\left(\frac{-v'}{r}\right)^k\right].$$

We note that the radial form of the Pohozaev identity for k-Hessian equation established in [28] states that

$$P(r,v) = -\int_0^r t^{n-1}G(v)dt. \tag{2.9}$$

Lemma 2.4. *If v is a solution of problem (2.1) in $(0,b)$, $0<b<\infty$, then $v(0)>\gamma_G$ and*

$$P(r,v)>0, \qquad r\in(0,b). \tag{2.10}$$

Proof. Let v be a solution of problem (2.1) in $(0,b)$. By (2.9),

$$\frac{d}{dr}P(r,v)=-r^{n-1}G(v).$$

By Remarks 2.3, G is positive on $(0,\gamma_G)$ and negative on (γ_G,∞). Suppose that $v(0)\le\gamma_G$. Then $G(v)>0$ on $(0,b)$ which implies that $P(r,v)$ decreases on $(0,b)$. Thus,

$$0=P(0,v)>\lim_{r\to b}P(r,v)=C_{n-1}^{k-1}b^{n+1-k}(-v'(b))^{k+1}\ge 0,$$

a contradiction. Hence $v(0)>\gamma_G$. Since v is decreasing on $(0,b)$, the condition $v(0)>\gamma_G$ implies that $G(v)$ is positive-negative on $(0,b)$, yielding $P(r,v)$ increases-decreases on $(0,b)$. Since $P(0,v)=0$ and $P(b,v)\ge 0$, we immediately deduce that $P(r,v)>0$ for every $r\in(0,b)$. This completes the proof. □

We now provide some interesting estimates of radial solutions of problem (1.1) in $\Omega=\mathbb{R}^n$.

Lemma 2.5.

(a) *Let $f(v)=v^p-v^k,\ \ k<p$. If v is a solution of problem (2.1) in $(0,\infty)$, then there exist two constants $C_1,C_2>0$ such that*

$$v(r)\le C_1\exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}r^{\frac{2k}{k+1}}\right] \tag{2.11}$$

and

$$-rv'(r)\le C_2\exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]. \tag{2.12}$$

(b) *Let $f(v)=v^k\ln v+\beta v^k,\ \ \beta\in\mathbb{R}$. If v is a solution of problem (2.1) in $(0,\infty)$, then there exist two constants $C_1,C_2>0$ such that*

$$v(r)\le C_1\exp\left[-\left(\frac{1}{(k+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}r^{\frac{2k}{k+1}}\right] \tag{2.13}$$

and

$$-rv'(r)\le C_2\exp\left[-\left(\frac{1}{(k+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]. \tag{2.14}$$

Proof. We shall use the same lines of reasoning in the proof of both cases. It is for this reason that we omit the proof of the second statement. Let $f(v)=v^p-v^k,\ \ k<p$. Let v be a solution of problem (2.1) in $(0,\infty)$. We denote by b_1 the unique positive constant such that $v(b_1)=\gamma_f(=1)$. By (2.5), we have $E(r,v)>0$. This means that

$$C_{n-1}^{k-1}r^{1-k}(-v')^{k+1}>-(k+1)F(v)=v^{k+1}\left(1-\frac{k+1}{p+1}v^{p-k}\right).$$

Since v decreases, we then obtain, for every $r \geq b_1$,

$$(-v')^{k+1} \geq \frac{(p-k)}{(p+1)C_{n-1}^{k-1}} r^{k-1} v^{k+1},$$

or equivalently,

$$-\frac{v'}{v} \geq \left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{k-1}{k+1}}.$$

Integrating from b_1 to r gives

$$-\ln v \geq \frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(r^{\frac{2k}{k+1}} - b_1^{\frac{2k}{k+1}}\right).$$

Thus

$$v(r) \leq C \exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{2k}{k+1}}\right], \qquad r \geq b_1,$$

where

$$C = \exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} b_1^{\frac{2k}{k+1}}\right].$$

Since v is continuous on the whole of $\mathbb{R}_+$, this yields the existence of a constant $C_1 > 0$ such that (2.11) holds for all $r \geq 0$. The estimate (2.12) follows from the fact that v is convex on (b_1, ∞) together with (2.11). Indeed, v is convex on (b_1, ∞) since

$$v'' = -\frac{n-k}{k}\frac{v'}{r} - \frac{1}{C_{n-1}^{k-1}}\left(\frac{r}{-v'}\right)^{k-1} f(v) > 0.$$

Thus, for every $b_1 < t < r$, we have

$$\frac{v(r) - v(t)}{r - t} \leq v'(r).$$

For $t = \frac{r}{2}$, we get

$$2v(r) - 2v(\frac{r}{2}) \leq rv'(r).$$

Multiplying by

$$\exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]$$

and then letting r tend to ∞, we conclude using (2.11) that

$$-\infty < \lim_{r\to\infty} rv'(r) \exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right].$$

This completes the proof of the first statement. □

We mention here that this kind of bound when $k = 1$ and $f(u) = u^p - u$ has been proved in the celebrated work [2].

Lemma 2.6. *Assume that $f(v) = f_i(v)$, $i \in \{1,3\}$. If v is a solution of (2.1) in $(0,\infty)$ then $v(0) > \gamma_G$,*

$$\lim_{r\to\infty} P(r,v) = 0 \tag{2.15}$$

and

$$P(r,v) > 0.$$

Proof. The property (2.15) follows from the estimates (2.11) and (2.12) when $f(v) = f_1(v)$, and the estimates (2.13) and (2.14) when $f(v) = f_3(v)$. The rest of the proof is similar to that of Lemma 2.4. □

3 Proof of Theorem 1.1

1. Assume that $f(v) = v^p - v^k$, $k < p$ and $p(n-2k) \geq k(n+2)$. Suppose that problem (1.1) has a radial solution v in $\Omega = B$ a finite ball of radius b. By (2.9), we have

$$\frac{d}{dr}P(r,v) = -r^{n-1}G_1(v).$$

$G_1(v) > 0$ by Remarks 2.3 since $C(n,k,p) \geq 0$ by hypothesis. Thus $P(r,v)$ is decreasing on $(0,b)$, and hence

$$0 = P(0,v) > P(b,v) = C_{n-1}^{k-1} b^{n+1-k}(-v'(b))^{k+1} \geq 0,$$

a contradiction. Therefore, problem (1.1) has no radial solutions in $\Omega = B$.

2. Assume that $f(v) = v^k - v^p$, $0 < p < k$. Striving for a contradiction, suppose that problem (1.1) admits a radial solution v in $\mathbb{R}^n$. By (2.5), we have $E(r,v) > 0$. Thus,

$$C_{n-1}^{k-1} r^{1-k}(-v')^{k+1} \geq -(k+1)F(v) = v^{p+1}\left(\frac{k+1}{p+1} - v^{k-p}\right).$$

Let $b_1 \in (0,\infty)$ so that $v(b_1) = 1$. Since v is decreasing, it follows that, for every $r \geq b_1$,

$$(-v')^{k+1} \geq \frac{(k-p)}{(p+1)C_{n-1}^{k-1}} r^{k-1} v^{p+1},$$

or equivalently,

$$-v'v^{-\frac{p+1}{k+1}} \geq \left(\frac{(k-p)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{k-1}{k+1}}.$$

Integrate from b_1 to r gives

$$1 - v^{\frac{k-p}{k+1}} \geq \frac{k-p}{2k}\left(\frac{(k-p)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(r^{\frac{2k}{k+1}} - b_1^{\frac{2k}{k+1}}\right).$$

Now, let r tend to ∞ we obtain $1 \geq \infty$, a contradiction. This completes the proof. □

4 Proof of Theorem 1.2

Let $k \in \{1, \cdots, n\}$ be the index of the Hessian operator. Let $f(v)$ be a function defined on $[0, \infty[$ which takes one of the following forms:

(a) $f_1(v) = v^p - v^k,\ \ k < p$ and $p(n-2k) < k(n+2)$.

(b) $f_2(v) = v^k - v^p,\ \ 0 < p < k$.

(c) $f_3(v) = v^k \ln v + \beta v^k,\ \ \beta \in \mathbb{R}$.

Assume that v and w are two solutions of problem (2.1) in $(0, b)$ and $(0, c)$ respectively. We shall prove that if $v(0) < w(0)$ and $b < \infty$ then $c < \infty$ and

$$c < b.$$

Furthermore, v and w intersect exactly once in $(0, c)$. The proof will be divided into a sequence of lemmas. Assume that $v(0) < w(0)$ and $b < \infty$. Arguing by contradiction, suppose that

$$c \geq b.$$

For $0 \leq r \leq b$, let

$$Y(r) = vw' - v'w.$$

Lemma 4.1. *Let v and w be two positive solutions of the equation*

$$\frac{C_{n-1}^{k-1}}{k}\left(r^{n-k}(-u')^k\right)' = r^{n-1} f(u).$$

If $v(0) < w(0)$ then $Y(r) < 0$ as long as $v(r) < w(r)$.

Proof. By writing the above equation in the following equivalent form

$$C_{n-1}^{k-1}\left(\frac{-v'}{r}\right)^{k-1}\left(-v'' - \frac{n-k}{k}\frac{v'}{r}\right) = f(v), \tag{4.1}$$

we see that

$$\lim_{r\to 0}\frac{-v'}{r} = \left(\frac{k}{n\, C_{n-1}^{k-1}} f(v(0))\right)^{\frac{1}{k}}.$$

Thus

$$\lim_{r\to 0}\frac{Y(r)}{r} = \left(\frac{k}{n\, C_{n-1}^{k-1}}\right)^{\frac{1}{k}}\left[w\,(f(v))^{\frac{1}{k}} - v\,(f(w))^{\frac{1}{k}}\right](0).$$

Together with the fact that $f(t)/t^k$ increases on $(0, \infty)$ and $v(0) < w(0)$, this implies that $Y < 0$ on some neighbourhood $(0, \varepsilon)$. Arguing by contradiction, suppose that there exists $a \in (0, b)$ such that $Y(a) = 0$, $Y < 0$ on $(0, a)$ and $v(a) < w(a)$. It is obvious that $Y'(a) \geq 0$. On the other hand, using the fact that v and w satisfy equation (4.1), we easily obtain

$$Y'(r) = \frac{r^{k-1}}{C_{n-1}^{k-1}} vw\left(\frac{f(v)}{v(-v')^{k-1}} - \frac{f(w)}{w(-w')^{k-1}}\right) - \frac{n-k}{kr} Y(r).$$

Since $Y(a) = 0$, this implies that

$$Y'(a) = \frac{a^{k-1}}{C_{n-1}^{k-1}} v(a) w(a)\left(\frac{v}{-v'}\right)^{k-1}(a)\left(\frac{f(v)}{v^k} - \frac{f(w)}{w^k}\right)(a) < 0,$$

in contradiction with $Y'(a) \geq 0$. This completes the proof. □

It is clear that Y must vanish on $(0,b]$ since $Y(0)=0$, $Y<0$ near 0 and $Y(b)\geq 0$. Let τ denote the first zero of Y in $(0,b]$. Therefore,

$$Y(\tau)=0 \quad \text{and} \quad Y(r)<0, \qquad 0<r<\tau.$$

Define

$$Z(r)=\frac{C_{n-1}^{k-1}}{k}r^{n-k}\left[(-w')^k-\left(-v'\frac{w}{v}\right)^k\right], \qquad 0\leq r\leq b. \tag{4.2}$$

Lemma 4.2.

(a) For every $r\in(0,b)$, we have

$$Z'(r)+C_{n-1}^{k-1}r^{n-k}Y\frac{w^{k-1}(-v')^k}{v^{k+1}}=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]. \tag{4.3}$$

(b) For every $r\in(0,\tau)$, we have

$$-w'Z+C_{n-1}^{k-1}r^{n-k}Y\frac{(-v'w)^k}{v^{k+1}}>0. \tag{4.4}$$

Proof. (a) A straightforward computation using equation (2.1) gives

$$\begin{aligned}Z'(r)&=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]-C_{n-1}^{k-1}r^{n-k}\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k-1}(-v')^k\\&=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]-C_{n-1}^{k-1}r^{n-k}Y\frac{w^{k-1}(-v')^k}{v^{k+1}}.\end{aligned}$$

(b) For $r\in(0,\tau)$ we have $-vw'>-v'w>0$. Thus,

$$\begin{aligned}Z(r)&=\frac{C_{n-1}^{k-1}}{k}\frac{r^{n-k}}{v^k}\left[(-vw')^k-(-wv')^k\right]\\&\geq -C_{n-1}^{k-1}\frac{r^{n-k}}{v^k}(-wv')^{k-1}Y.\end{aligned}$$

The last inequality follows immediately from the fact that $t^k-s^k\geq k(t-s)s^{k-1}$ for $t>s>0$. Hence,

$$\begin{aligned}-w'Z+C_{n-1}^{k-1}r^{n-k}Y\frac{(-wv')^k}{v^{k+1}}&\geq C_{n-1}^{k-1}r^{n-k}Y\frac{(-wv')^{k-1}}{v^{k+1}}(w'v-wv')\\&=C_{n-1}^{k-1}r^{n-k}\frac{(-wv')^{k-1}}{v^{k+1}}Y^2>0.\end{aligned}$$

□

For $0\leq r\leq b$, we define

$$\Phi(r)=P(r,w)-\left(\frac{w}{v}\right)^{k+1}P(r,v).$$

We now show by two methods that $\Phi(\tau)>0$ and $\Phi(\tau)\leq 0$ which is the required contradiction.

Lemma 4.3. $\Phi(\tau)>0$.

Proof. By differentiation, we have

$$\Phi'(r) = r^{n-1}\left[\left(\frac{w}{v}\right)^{k+1} G(v) - G(w)\right] - (k+1)\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k} P(r,v).$$

Using (2.8) and (4.3), we obtain

$$\begin{aligned}\Phi'(r) &= C(n,k,p)r^{n-1}w\left[\left(\frac{w}{v}\right)^{k} f(v) - f(w)\right] - (k+1)\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k} P(r,v)\\ &= -C(n,k,p)\left[wZ'(r) + C_{n-1}^{k-1}r^{n-k}Y\frac{(-v'w)^k}{v^{k+1}}\right] - (k+1)Y\frac{w^k}{v^{k+2}}P(r,v).\end{aligned}$$

Seeing that $Z(0) = Z(\tau) = 0$, an integration by parts yields

$$\int_0^\tau wZ'dr = [wZ]_0^\tau - \int_0^\tau w'Zdr = -\int_0^\tau w'Zdr.$$

Thus

$$\Phi(\tau) = -C(n,k,p)\int_0^\tau\left[-w'Z + C_{n-1}^{k-1}r^{n-k}Y\frac{(-v'w)^k}{v^{k+1}}\right]dr - (k+1)\int_0^\tau \frac{Yw^k}{v^{k+2}}P(r,v)dr.$$

Now, $\Phi(\tau) > 0$ follows from (2.10) and (4.4) together with the fact that $Y < 0$ on $(0,\tau)$ and $C(n,k,p) < 0$. □

On the other hand, we have

Lemma 4.4. $\Phi(\tau) \leq 0$.

Proof. For $r \in (0,b)$, we have

$$\begin{aligned}\Phi(r) = C_{n-1}^{k-1}r^{n+1-k}&\left[(-w')^{k+1} - \left(-\frac{wv'}{v}\right)^{k+1}\right] - (k+1)r^n w^{k+1}\left[\frac{F(v)}{v^{k+1}} - \frac{F(w)}{w^{k+1}}\right]\\ &- (n-2k)\frac{C_{n-1}^{k-1}}{k}r^{n-k}w\left[(-w')^k - \left(-\frac{wv'}{v}\right)^k\right].\end{aligned}$$

Thus

- If $\tau < b$ then
$$\Phi(\tau) = -(k+1)\tau^n w(\tau)^{k+1}\left[\frac{F(v)}{v^{k+1}} - \frac{F(w)}{w^{k+1}}\right](\tau).$$
By Lemma 4.1, $Y(\tau) = 0$ implies $v(\tau) \geq w(\tau)$. Since $F(t)/t^{k+1}$ increases on $(0,\infty)$, it follows that
$$\Phi(\tau) \leq 0.$$
- If $\tau = b$, then $Y(b) = 0$ implies that $w(b) = 0$. If, in addition, $v'(b) \neq 0$ then
$$\Phi(\tau) = \lim_{r\to b}\Phi(r) = P(b,w) - \left(\frac{w'(b)}{v'(b)}\right)^{k+1}P(b,v) = 0.$$
If $v'(b) = 0$ then we must have $w'(b) = 0$. Otherwise, since the ratio v/w increases on $(0,b)$, we obtain
$$0 < \frac{v(0)}{w(0)} < \lim_{r\to b}\frac{v}{w}(r) = \frac{v'(b)}{w'(b)} = 0.$$

a contradiction. Thus $P(b, w) = P(b, v) = 0$, and hence

$$|\Phi(\tau)| = \lim_{r\to b} |\Phi(r)| \le \lim_{r\to b} \left[P(r, w) + \left(\frac{w(0)}{v(0)}\right)^{k+1} P(r, v) \right] = 0.$$

Hence $\Phi(\tau) \le 0$ as desired. □

We have shown that if $v(0) < w(0)$ and $b < \infty$ then $c < b$, this holds for any solutions v and w of problem (2.1) in $(0, b)$ and $(0, c)$ respectively. This implies that the set $\mathcal{A} := \{r \in (0, c);\ v(r) = w(r)\}$ is nonempty. Arguing by contradiction, suppose that the set $\mathcal{A}$ contains two points $r_1 < r_2$. Thus

$$\frac{v}{w}(r_1) = 1 = \frac{v}{w}(r_2).$$

This yields the existence of $\tau \in (r_1, r_2)$ such that

$$\left(\frac{v}{w}\right)'(\tau) = 0,$$

or equivalently,

$$\left(v'w - vw'\right)(\tau) = 0.$$

Using Lemma 4.1, we can assume that $v'w - vw' < 0$ on the interval $(0, \tau)$. The contradiction follows now again from Lemmas 4.3 and 4.4 above. Hence v and w intersect exactly once in $(0, c)$. This completes the proof of Theorem 1.2. □

5 Proof of Theorem 1.4

Let $k \in \{1, \cdots, n\}$ be the index of the Hessian operator. Let $f(v)$ be of the form:

(a) $f_1(v) = v^p - v^k,\ \ k < p$ and $p(n - 2k) < k(n + 2)$.

(b) $f_3(v) = v^k \ln v + \beta v^k,\ \ \beta \in \mathbb{R}$.

We shall first prove that problem (2.1) has at most one solution in $(0, \infty)$. The proof proceeds along the same lines as the proof of Theorem 1.2. We shall be brief here and just outline the proof. Let the notation be as in the preceding paragraph. Arguing by contradiction, suppose that problem (2.1) has two solutions v and w in $(0, \infty)$. We can assume that $v(0) < w(0)$. For $r \ge 0$, let

$$Y(r) = w'v - v'w$$

and

$$\Phi(r) = P(r, w) - \left(\frac{w}{v}\right)^{k+1} P(r, v).$$

By Lemma 4.1, we have $Y(r) < 0$ as long as $v(r) < w(r)$.

- Assume that there exists $\tau \in (0, \infty)$ such that

$$Y < 0 \quad \text{on } (0, \tau) \quad \text{and} \quad Y(\tau) = 0.$$

In this case the required contradiction follows from Lemmas 4.3 and 4.4 above.

- Assume that $Y < 0$ on the whole of $(0, \infty)$. Since the ratio w/v decreases on $(0, \infty)$, it follows that
$$|\Phi(r)| \leq |P(r, w)| + \left(\frac{w(0)}{v(0)}\right)^{k+1} |P(r, v)|.$$
This implies together with (2.15) that
$$\lim_{r \to \infty} |\Phi(r)| = 0.$$
On the other hand, following the same steps as in the proof of Lemma 4.3, we conclude that
$$\lim_{r \to \infty} \Phi(r) > 0$$
provided we have
$$\lim_{r \to \infty} w(r) Z(r) = 0, \tag{5.1}$$
where Z is given by (4.2). So it remains to show (5.1). We see that w/v is decreasing on $(0, \infty)$ since $Y < 0$, and so
$$\frac{w}{v} \leq \frac{w(0)}{v(0)}.$$
Thus
$$|Z(r)| \leq \frac{C_{n-1}^{k-1}}{k} r^{n-k} \left[(-w')^k + \left(\frac{w(0)}{v(0)}\right)^k (-v')^k \right]$$
from which we obtain (5.1) using (2.11) and (2.12) when $f(v) = f_1(v)$, and the estimates (2.13) and (2.14) when $f(v) = f_3(v)$. This completes the proof of the uniqueness result.

The upper bound of solutions stated in the first statement of the theorem is given by (2.11). So it remains to show that the function
$$u(r) := \exp\left[-\left(\frac{1}{2\, C_{n-1}^{k-1}}\right)^{\frac{1}{k}} \frac{r^2}{2} + \frac{n}{2k} - \beta \right]$$
is a solution of equation (2.1) in $(0, \infty)$ when $f(u) = u^k(\ln u + \beta)$. Let
$$v(r) = e^{-a\frac{r^2}{2} - b}.$$
Then $v'(r) = -arv(r)$, yielding
$$\begin{aligned}
\frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k\right)' &= \frac{C_{n-1}^{k-1}}{k} a^k \left(r^n v^k\right)' \\
&= \frac{C_{n-1}^{k-1}}{k} a^k n r^{n-1} v^k + C_{n-1}^{k-1} a^k r^n v' v^{k-1} \\
&= 2a^k C_{n-1}^{k-1} v^k \left(\frac{n}{2k} - a\frac{r^2}{2}\right) r^{n-1}.
\end{aligned}$$
Now by taking $2a^k C_{n-1}^{k-1} = 1$ and $b = \beta - \frac{n}{2k}$, we obtain
$$\frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k\right)' = r^{n-1} v^k (\ln v + \beta).$$
This completes the proof of the theorem. □

References

[1] Adimurthi, S. L. Yadava, An elementary proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet problem, *Arch. Rational Mech. Anal.* **127**(1994), No. 3, 219–229. https://doi.org/10.1007/BF00381159; MR1288602; Zbl 0806.35031

[2] H. Berestycki, P. L. Lions, Nonlinear scalar field equations. I, Existence of a ground state, *Arch. Rational Mech. Anal.* **82**(1983), 313–375. https://doi.org/10.1007/BF00250555; MR0695535; Zbl 0533.35029

[3] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations III: Functions of the eigenvalues of the Hessian, *Acta Math.* **155**(1985), No. 3–4, 261–301. https://doi.org/10.1007/BF02392544; MR0806416; Zbl 0654.35031

[4] C. C. Chen, C. S. Lin, Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, $n \geq 3$, *Comm. Partial Differential Equations* **16**(1991), No. 8–9, 1549–1572. https://doi.org/10.1080/03605309108820811; MR1132797; Zbl 0753.35034

[5] P. Clément, D. de Figueiredo, E. Mitidieri, Quasilinear elliptic equations with critical exponents, *Topol. Methods Nonlinear Anal.* **7**(1996), No. 1, 133–170. https://doi.org/10.12775/TMNA.1996.006; MR1422009; Zbl 0939.35072

[6] C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.* **46**(1972), 81–95. https://doi.org/10.1007/BF00250684; MR0333489; Zbl 0249.35029

[7] C. Cortazar, M. Elgueta, P. Felmer, Uniqueness of positive solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, $n \leq 3$, *Arch. Rational Mech. Anal.* **142**(1998), No. 2, 127–141. https://doi.org/10.1007/s002050050086; MR1629650; Zbl 0912.35059

[8] L. Erbe, M. Tang, Structure of positive radial solutions of semilinear elliptic solutions, *J. Differential Equations* **133**(1997), No. 2, 179–202. https://doi.org/10.1006/jdeq.1996.3194; MR1427849; Zbl 0871.34023

[9] L. Erbe, M. Tang, Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball, *J. Differential Equations* **138**(1997), No. 2, 351–379. https://doi.org/10.1006/jdeq.1997.3279; MR1462272; Zbl 0884.34025

[10] R. L. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, *Comm. Pure Appl. Math.* **69**(2016), No. 9, 1671–1726. https://doi.org/10.1002/cpa.21591; MR3530361; Zbl 1365.35206

[11] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68**(1979), No. 3, 209–243. https://doi.org/10.1007/BF01221125; MR0544879; Zbl 0425.35020

[12] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, in: *Mathematical analysis and applications, Part A*, Adv. in Math. Suppl. Stud., Vol. 7, Academic Press, New York–London, 1981, pp. 369–402. https://doi.org/10.1080/03605309308820960; MR0634248; Zbl 0469.35052

[13] J. Jacobsen, Global bifurcation problems associated with k-Hessian operators, *Topol. Methods Nonlinear Anal.* **14**(1999), No. 1, 81–130. https://doi.org/10.12775/TMNA.1999.023; MR1758881; Zbl 0959.35066

[14] J. Jang, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, $n \geq 2$, *Nonlinear Anal.* **73**(2010), No. 7, 2189–2198. https://doi.org/10.1016/j.na.2010.05.045; MR2674195; Zbl 1209.34030

[15] N. V. Krylov, Degenerate nonlinear elliptic equations. II (in Russian), *Mat. Sb. (N.S.)* **121(163)**(1983), No. 2, 211–232. https://doi.org/10.1070/SM1984v048n02ABEH002676; MR0703325; Zbl 0538.35037

[16] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **105**(1989), No. 3, 243–266. https://doi.org/10.1007/BF00251502; MR0969899; Zbl 0676.35032

[17] M. K. Kwong, Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.* **333**(1992), No. 1, 339–363. https://doi.org/10.2307/2154113; MR1088021; Zbl 0785.35038

[18] K. McLeod, J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **99**(1987), No. 2, 115–145. https://doi.org/10.1007/BF00275874; MR0886933; Zbl 0667.35023

[19] K. McLeod, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, II, *Trans. Amer. Math. Soc.* **339**(1993), No. 2, 495–505. https://doi.org/10.2307/2154282; MR1201323; Zbl 0804.35034

[20] M. A. Navarro, J. Sánchez, A characterization of semistable radial solutions of k-Hessian equations, *J. Math. Anal. Appl.* **497**(2021), No. 2, 14 pp. https://doi.org/10.1016/j.jmaa.2020.124902; MR4196574; Zbl 1459.35164

[21] W. M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50**(1983), No. 2, 289–304. https://doi.org/10.1016/0022-0396(83)90079-7; MR0719451; Zbl 0476.35033

[22] W. M. Ni, R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u,r) = 0$, *Comm. Pure Appl. Math.* **38**(1985), No. 1, 67–108. https://doi.org/10.1002/cpa.3160380105; MR0768105; Zbl 0581.35021

[23] L. A. Peletier, J. Serrin, Uniqueness of Positive solutions of semilinear equations in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **81**(1983), No. 2, 181–197. https://doi.org/10.1007/BF00250651; MR0682268; Zbl 0516.35031

[24] J. Serrin, M. Tang, Uniqueness of ground states for quasilinear elliptic equations, *Indiana Univ. Math. J.* **49**(2000), No. 3, 897–923. https://doi.org/10.1512/iumj.2000.49.1893; MR1803216; Zbl 0979.35049

[25] W. C. Troy, Uniqueness of positive ground state solutions of the logarithmic Schrödinger equation, *Arch. Rational Mech. Anal.* **222**(2016), No. 3, 1581–1600. https://doi.org/10.1007/s00205-016-1028-5; MR3544334; Zbl 1348.35247

[26] N. S. Trudinger, X. J. Wang, Hessian measures I, *Topol. Methods Nonlinear Anal.* **10**(1997), No. 2, 225–239. https://doi.org/10.12775/TMNA.1997.030; MR1634570; Zbl 0915.35039

[27] N. S. Trudinger, X. J. Wang, Hessian measures II, *Ann. of Math. (2)* **150**(1999), No. 2, 579–604, 1997. https://doi.org/10.2307/121089; MR1726702; Zbl 0947.35055

[28] K. Tso, Remarks on critical exponents for Hessian operators, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **7**(1990), No. 2, 113–122. https://doi.org/10.1016/S0294-1449(16)30302-X; MR1051232; Zbl 0715.35031

[29] X. J. Wang, A class of fully nonlinear elliptic equations and related functionals, *Indiana Univ. Math. J.* **43**(1994), No. 1, 25–54. https://doi.org/10.1512/iumj.1994.43.43002; MR1275451; Zbl 0805.35036

[30] W. Wei, Uniqueness theorems for negative radial solutions of k-Hessian equations in a ball, *J. Differential Equations* **261**(2016), No. 6, 3756–3771. https://doi.org/10.1016/j.jde.2016.06.004; MR3527645; Zbl 1350.35003

[31] L. Zhang, Uniqueness of positive solutions of $\Delta u + u + u^p = 0$ in a finite ball, *Comm. Partial Differential Equations* **17**(1992), No. 7-8, 1141–1164. https://doi.org/10.1080/03605309208820880; MR1179281; Zbl 0782.35025

[32] X. Zhang, Existence and uniqueness of nontrivial radial solutions for k-Hessian equations, *J. Math. Anal. Appl.* **492**(2020), No. 1, 124439, 19 pp. https://doi.org/10.1016/j.jmaa.2020.124439; MR4128635; Zbl 1460.35149