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# Lyapunov functionals and practical stability for stochastic differential delay equations with general decay rate

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**Abstract.** This paper stands for the almost sure practical stability of nonlinear stochastic differential delay equations (SDDEs) with a general decay rate. We establish some sufficient conditions based upon the construction of appropriate Lyapunov functionals. Furthermore, we provide some numerical examples to validate the effectiveness of the abstract results of this paper.

**Keywords: Keywords:** stochastic delay systems, almost sure practical stability, decay function, Lyapunov functional, Itô formula.

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## 1 Introduction

Several applied problems are modeled by non-delay systems. Non-delay systems are governed by the assumption that the future evolution of the system is determined by the present state. Moreover, it is independent of the past states. In reality, such an assumption is the only a first approximation to the real system. A more realistic model assumes that the evolution of the future states depends not only on the current state but also on their past history. Delay differential equations (DDEs) (also called hereditary systems, systems with aftereffect, functional differential equations, retarded differential equations, differential difference equations) provide an appropriate model for physical processes whose time evolution depends on their history.

The stochastic delay differential equations (SDDEs) have been extensively used in many branches of physics, biology, as well as in dynamical structures in engineering, mechanics, automatic regulation, economy finance, ecology, sociology, medicine, etc. The stability of SDDEs has become a very prevalent theme of recent research in Mathematics and its applications. An

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important direction in the study of equations with delays is the analysis of stability. The corresponding study of the stability properties of solutions has received much attention during the last decades. The reader is referred to [15,16,20–22,24], for more details.

As it is well known, in the case without any hereditary features, Lyapunov's technique is available to obtain sufficient conditions for the stability of the solutions of stochastic differential equations. These sufficient conditions are obtained using the construction of some Lyapunov functions of functionals, being the latter a method which provides better conditions than using Lyapunov functions. Moreover, the construction of Lyapunov functionals is more complicated as Krasovskii [19] pointed out.

In this us, the construction of different Lyapunov functionals for one SDDEs allows to establish several stability conditions for the solution of this equation. There exist numerous works that tackle the construction of Lyapunov functionals for a wide range of equations containing some hereditary properties, see [10,17,23].

Several fundamental variants to Lyapunov's original concepts of practical stability were introduced in [1–6,9,11,12]. When the origin is not necessarily an equilibrium point, we can study the asymptotic stability of solutions of the SDDEs in a small neighborhood of the origin. In the investigation of the asymptotic behavior of solutions to stochastic differential systems, one can find that a solution is asymptotically stable but may not necessarily exponentially stable. Further, in the nonlinear and/or nonautonomous situations, it may happen that the stability cannot always be exponential but can be sub or super-exponential, see [7,8]. For this reason, the main aim of this paper is to discuss the almost sure practical stability with a general decay rate of stochastic delay differential equations.

The general method of Lyapunov functionals construction, which was proposed by V. Kolmanovskii and L. Shaikhet [17,18,23], is used here for stochastic differential equations with delay. This approach has already been successfully used for functional differential equations, for difference equations with discrete time, for difference equations with continuous time. Our interest in this paper is to investigate the practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay by using the general method of Lyapunov functionals construction.

In [11], Caraballo et al. investigated the practical convergence to zero with a general decay rate of stochastic delay evolution equation by using Lyapunov functions. To the best of our knowledge, no work has been published about the practical stability of SDDEs in the literature by using Lyapunov functionals, which is our research topic in our paper. The novelty of our work is to investigate the practical convergence to a small ball centered at the origin with a general decay rate in terms of the existence and construction of Lyapunov functionals. Furthermore, we construct Lyapunov functionals for stochastic differential equations with constant and time-varying delay to obtain sufficient conditions ensuring the practical convergence to a small ball centered at the origin with a general decay rate. The contents of this paper are as follows: in Section 2, we introduce the necessary notations and preliminaries. In Section 3, we establish several sufficient criteria for almost sure practical stability of the stochastic delay systems with a general decay rate utilizing Lyapunov's functional. In Section 4, we aim to analyze the almost sure practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay by constructing suitable Lyapunov functionals. Moreover, we exhibit some examples to illustrate the theoretical findings. Eventually, some conclusions are included in the last section.

## Notations

Throughout this paper, unless otherwise specified, we use the following notations.

Let $\{\Omega, \mathcal{F}, \mathrm{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))$ be an $m$-dimensional Brownian motion defined on the probability space. Let $\mathbb{R}_+ = [0, +\infty)$ and $\tau > 0$. We denote by $C([-\tau, 0], \mathbb{R}^n)$ the family of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Let $p > 0$, and denote by $L^p_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$ the family of all $\mathcal{F}_t$-measurable $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$-valued random variables $\xi$, such that $\mathbb{E}(\|\xi\|^p) < \infty$. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, +\infty)$, for each $t \geq 0$ we define $x_t$ by $x_t(\theta) = x(t+\theta) : -\tau \leq \theta \leq 0$ for $t \geq 0$, which is a $C([-\tau, 0], \mathbb{R}^n)$valued process.

Let us consider the following $n$-dimensional stochastic differential delay equation (SDDE):

$$dx(t) = F(t, x_t)dt + G(t, x_t)dB(t), \quad t \geq 0, \tag{1.1}$$

where $F : [0, +\infty) \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n,\ G : [0, +\infty) \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^{n\times m}$.

We assume that there exits $t \in \mathbb{R}_+$, such that $F(t, 0) \neq 0$ or $G(t, 0) \neq 0$, i.e., the stochastic differential delay equation (1.1) does not have the trivial solution $x \equiv 0$.

In order to solve equation (1.1), we require to know the initial data, then we assume that they are given by

$$x_0 = \xi, \quad \text{i.e.,} \quad x_0(\theta) = \xi(\theta) = x(\theta), \quad \forall\theta \in [-\tau, 0], \tag{1.2}$$

where $\xi$ is a $C([-\tau, 0], \mathbb{R}^n)$ – valued random variable such that $\mathbb{E}(\|\xi\|^2) < \infty$.

For the well-posedness of system (1.1), we impose the following assumptions.

**Assumptions:**

1. A local Lipschitz condition:

   For every real number $T > 0$ and integer $i \geq 1$, there exists a positive constant $K_{T,i}$, such that for all $t \in [0, T]$ and all $\varphi,\ \bar{\varphi} \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\varphi\| \vee \|\bar{\varphi}\| \leq i$,

   $$\|F(t, \varphi) - F(t, \bar{\varphi})\|^2 \vee \|G(t, \varphi) - G(t, \bar{\varphi})\|^2 \leq K_{T,i}\ (\|\varphi - \bar{\varphi}\|^2).$$

2. A linear growth condition:

   For every real number $T > 0$, there exists a positive constant $K_T$, such that for all $t \in [0, T]$ and all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$,

   $$\|F(t, \varphi)\|^2 \vee \|G(t, \varphi)\|^2 \leq K_T\ (1 + \|\varphi\|^2).$$

Then, under assumptions (1) and (2), the stochastic differential delay equation (1.1) with the given initial data (1.2) has a unique global solution $x(\cdot) = x(\cdot, 0, \xi) \in \mathcal{M}^2([-\tau, +\infty), \mathbb{R}^n)$, (see Mao [21], for more details). Moreover, $x(\cdot)$ satisfies the following integral equation:

$$\begin{cases} x(t) = \xi(0) + \displaystyle\int_0^t F(s, x_s)ds + \int_0^t G(s, x_s)dB(s), \quad \text{a.s.,} \quad \text{and} \\ x(t) = \xi(t), \quad t \in [-\tau, 0]. \end{cases}$$

To calculate the stochastic differential of the process $\eta(t) = v(t, x(t))$, where $x(t)$ is a solution of the SDDE (1.1), and the function $v : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ has continuous partial derivatives

$$v_t(t,x) = \frac{\partial v}{\partial t}(t,x); \quad v_x(t,x) = \left(\frac{\partial v}{\partial x_1}(t,x), \ldots, \frac{\partial v}{\partial x_n}(t,x)\right); \quad v_{xx}(t,x) = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}(t,x)\right)_{n\times n}.$$

The following Itô's formula [14] is used:

$$d\eta(t) = \mathcal{L}v(t, x(t))dt + v_x(t, x(t))G(t, x_t)dB(t).$$

The operator $\mathcal{L}v$ is called the generator of (1.1) and is defined in the following way:

$$\mathcal{L}v(t, x(t)) = v_t(t, x(t)) + v_x(t, x(t))F(t, x_t) + \frac{1}{2}\operatorname{trace}\left(G^T(t, x_t)v_{xx}(t, x(t))G(t, x_t)\right).$$

The generator $\mathcal{L}$ can be applied also for some functionals $V(t,\varphi)\colon [0,+\infty)\times C([-\tau,0],\mathbb{R}^n)\to \mathbb{R}_+$. Suppose that a functional $V(t,\varphi)$ can be represented in the form $V(t, \varphi(0), \varphi(\theta))$, $\theta < 0$, and for $\varphi = x_t$, put

$$\begin{aligned} V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+\theta)), \quad \theta < 0, \\ x = \varphi(0) = x(t). \end{aligned} \tag{1.3}$$

Denote by $D$ the set of the functionals for which the function $V_\varphi(t,x)$ defined by (1) has a continuous derivative with respect to $t$ and two continuous derivatives with respect to $x$ (see [23]). For functionals from $D$, the generator $\mathcal{L}$ of (1.1) has the following form:

$$\mathcal{L}V(t, x_t) = V_{\varphi t}(t, x(t)) + V_{\varphi x}(t, x(t))F(t, x_t) + \frac{1}{2}\operatorname{trace}\left(G^T(t, x_t)V_{\varphi xx}(t, x(t))G(t, x_t)\right).$$

From the Itô formula it follows that for a functional $V$ from $D$,

$$dV(t, x_t) = \mathcal{L}V(t, x_t)dt + V_{\varphi x}(t, x(t))G(t, x_t)dB(t).$$

The following lemma is known as the exponential martingale inequality, and will be useful in our analysis.

**Lemma 1.1** (See [21]). *Let $g = (g_1, \ldots, g_m) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, and let $\tau$, $\mu$, $\eta$ be any positive numbers. Then,*

$$\mathbb{P}\left(\sup_{0\le t\le\tau}\left[\int_0^t g(s)dB_s - \frac{\mu}{2}\int_0^t \|g(s)\|^2 ds\right] > \eta\right) \le \exp(-\mu\eta).$$

## 2 Practical stability of stochastic delay equations

First, we define the practical uniform exponential stability of a stochastic delay equation.

**Definition 2.1.**

*i*) The ball $B_r := \{x \in \mathbb{R}^n : \|x\| \le r\}$, $r > 0$ is said to be almost surely globally uniformly exponentially stable, if for any initial data $\xi \in C([-\tau, 0], \mathbb{R}^n)$, such that $0 < \|x(t, 0, \xi)\| - r$, for all $t \ge 0$,

$$\lim_{t\to+\infty} \sup \frac{1}{t}\ln(\|x(t, 0, \xi)\| - r) < 0, \quad \text{a.s.}$$

*ii*) The system (1.1) is said to be almost surely practically uniformly exponentially stable, if there exists $r > 0$, such that $B_r$ is almost surely uniformly exponentially stable.

Now, we state the definition of practical convergence to the ball $B_r$ with a general decay function $\lambda(t)$.

**Definition 2.2.** Let $\lambda(t)$ be a positive function defined for sufficiently large $t > 0$, such that $\lambda(t) \to +\infty$ as $t \to +\infty$. A solution $x(t)$ to system (1.1) is said to decay to the ball $B_r$ almost surely practically with decay function $\lambda(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, i.e.,

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x(t,0,\xi)\| - r)}{\ln \lambda(t)} \leq -\gamma, \quad \text{a.s.}$$

If in addition, 0 is a solution to system (1.1), the zero solution is said to be almost surely practically asymptotically stable with decay function $\lambda(t)$ and order at least $\gamma$, if every solution to system (1.1) decays to the ball $B_r$ almost surely practically with decay function $\lambda(t)$ and order at least $\gamma$, for all $r > 0$ sufficiently small.

**Remark 2.3.** Clearly, replacing in the above definition, the decay function $\lambda(t)$ by $O(e^t)$ leads to the almost sure practical exponential stability.

**Remark 2.4.** Here we should mention that in [11] we establish sufficient conditions for practical decay to zero by using Lyapunov functions but now we will use Lyapunov functionals and decay to ball $B_r$.

Now, we aim to prove the practical stability of stochastic differential delay equations with general decay rate in terms of Lyapunov functionals.

**Theorem 2.5.** *Let $V : \mathbb{R}_+ \times C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ be a functional from $D$. Assume that $\ln \lambda(t)$ is uniformly continuous on $t \geq 0$, and there exists a constant $\sigma \geq 0$, such that*

$$\lim_{t\to+\infty} \frac{\ln \ln t}{\ln \lambda(t)} \leq \sigma.$$

*Let $x(\cdot) = x(\cdot,0,\xi)$ be a solution to system (1.1) and assume that there exist constants $q \in \mathbb{N}^\star$, $m \geq 0$, $\beta_1 \in \mathbb{R}$, $\beta_2 \geq 0$, a non-increasing function $\phi(t) > 0$ and a continuous non-negative function $\psi(t)$, such that, for all $t \geq 0$, the following assumptions hold:*

$(\mathcal{H}_1)$
$$\lambda^m(t)\|x(t)\|^q \leq V(t,x_t).$$

$(\mathcal{H}_2)$
$$\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \phi(s)\|V_x(s,x_s)G(s,x_s)\|^2 ds \leq \int_0^t \psi(s)\lambda^m(s)\|x(s)\|^q ds + \rho(t),$$

*where $\rho(t)$ is a continuous non-negative function.*

$(\mathcal{H}_3)$
$$\lim_{t\to+\infty} \sup \frac{\int_0^t \psi(s)ds}{\ln \lambda(t)} \leq \beta_1,$$
$$\lim_{t\to+\infty} \inf \frac{\ln \phi(t)}{\ln \lambda(t)} \geq -\beta_2,$$
$$\lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = v > 0.$$

$(\mathcal{H}_4)$

$$\|x(t,0,\xi)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}, \quad \textit{for all } t \geq 0.$$

*Then,*

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln \lambda(t)} \leq -\left[m - (\beta_1 + (\beta_2+\sigma)\vee m)\right], \quad a.s.$$

*Proof.* Observe that we have

$$\begin{aligned}\lambda^m(t)\|x(t)\|^q - \rho(t) &= \lambda^m(t)\left(\|x(t)\|^q - \frac{\rho(t)}{\lambda^m(t)}\right)\\ &= \lambda^m(t)\left(\|x(t)\|^q - \left(\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)^q\right).\end{aligned}$$

Using the inequality

$$a^q - b^q = (a-b)(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + a^0 b^{q-1}),$$

we conclude

$$\begin{aligned}\lambda^m(t)\|x(t)\|^q - \rho(t) &= \lambda^m(t)\left(\|x(t)\|^q - \left(\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)^q\right)\\ &= \lambda^m(t)\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)\\ &\quad\times\left(\|x(t)\|^{q-1} + \|x(t)\|^{q-2}\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}} + \cdots + \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{q-1}{q}}\right)\\ &= \lambda^m(t)\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)\sum_{k=1}^{q}\|x(t)\|^{q-k}\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{k-1}{q}}.\end{aligned}$$

From condition $(\mathcal{H}_3)$, we have $\lim_{t\to+\infty}\frac{\rho(t)}{\lambda^m(t)} = v > 0$. That is, for $0 < v_0 < v$, there exits $\bar{T} \geq 0$, such that $\frac{\rho(t)}{\lambda^m(t)} \geq v_0$ for all $t \geq \bar{T}$. Then, as we are assuming that $\|x(t)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}$, for all $t \geq 0$, it holds

$$\begin{aligned}\sum_{k=1}^{q}\|x(t)\|^{q-k}\left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{k-1}{q}} &= \|x(t)\|^{q-1} + \|x(t)\|^{q-2}\left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}} + \cdots + \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{q-1}{q}}\\ &\geq v' = q\,(v_0)^{(q-1)/q}, \quad \forall t \geq \bar{T} \geq 0.\end{aligned}$$

Therefore,

$$\lambda^m(t)\|x(t)\|^q - \rho(t) \geq \lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right]v', \quad \text{for all } t \geq \bar{T} \geq 0.$$

Hence, we see that

$$V(t,x_t) \geq \lambda^m(t)\|x(t)\|^q \geq \lambda^m(t)\|x(t)\|^q - \rho(t)$$
$$\geq \lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] v'.$$

That is,

$$v'\lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] \leq V(t,x_t),$$

and,

$$\ln v' + \ln\left[\lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right]\right] \leq \ln\left[V(t,x_t)\right].$$

Consequently, it follows that

$$\ln v' + m\ln\lambda(t) + \ln\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] \leq \ln\left[V(t,x_t)\right], \quad \forall t \geq \bar{T} \geq 0.$$

Applying the Itô formula, we obtain

$$V(t,x_t) = V(0,x_0) + \int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t V_x(s,x_s)G(s,x_s)dB(s). \tag{2.1}$$

Based upon the uniform continuity of $\ln\lambda(t)$, we can ensure that for each $\varepsilon > 0$ there exists two positive integers $N = N(\varepsilon)$ and $k_1(\varepsilon)$, such that if $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}$, $k \geq k_1(\varepsilon)$, it follows that

$$\left|\ln\lambda\left(\frac{k}{2^N}\right) - \ln\lambda(t)\right| \leq \varepsilon.$$

On the other side, owing to the exponential martingale inequality from Lemma 1.1, we have

$$\mathbb{P}\left\{\omega : \sup_{0\leq t\leq\tau}\left[M(t) - \frac{\mu}{2}\int_0^t \|V_x(s,x_s)G(s,x_s)\|^2 ds\right] > \eta\right\} \leq e^{-\mu\eta},$$

for any positive constants $\mu, \eta$ and $\tau$, where

$$M(t) = \int_0^t V_x(s,x_s)G(s,x_s)dB(s).$$

In particular, for the preceding $\varepsilon > 0$, we set

$$\mu = 2\phi\left(\frac{k-1}{2^N}\right), \quad \eta = \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N}, \quad \tau = \frac{k}{2^N}, \quad k = 2,3,\ldots$$

Then, we apply the well-known Borel–Cantelli lemma to obtain that, for almost all $\omega \in \Omega$, there exists an integer $k_0 = k(\varepsilon,\omega) > 0$, such that

$$M(t) \leq \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N} + \phi\left(\frac{k-1}{2^N}\right)\int_0^t \|V_x(s,x_s))G(s,x_s)\|^2 ds$$
$$\leq \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N} + \int_0^t \phi(s)\|V_x(s,x_s)G(s,x_s)\|^2 ds,$$

for $0 \le t \le \frac{k}{2^N}, \quad k \ge k_0(\varepsilon, \omega)$.

Substituting the last inequality into Eq. (2.1), we obtain

$$V(t, x_t) \le V(0, x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln \frac{k-1}{2^N} + \int_0^t \mathcal{L}V(s, x_s)ds + \int_0^t \phi(s)\|V_s(s, x_s)G(s, x_s)\|^2 ds,$$

for $0 \le t \le \frac{k}{2^N}$, $k \le k_0(\varepsilon, \omega)$.

Using conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, it follows that

$$\begin{aligned} V(t, x_t) &\le V(0, x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln \frac{k-1}{2^N} + \rho(t) + \int_0^t \psi(s)\lambda^m(s)\|x(s)\|^q ds \\ &\le V(0, x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln \frac{k-1}{2^N} + \rho(t) + \int_0^t \psi(s)V(s, x_s)ds, \end{aligned}$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon, \omega)$.

Applying now the Gronwall lemma [13],

$$V(t, x_t) \le \left(V(0, x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln \frac{k-1}{2^N} + \rho(t)\right) \exp\left(\int_0^t \psi(s)ds\right). \tag{2.2}$$

Based upon condition $(\mathcal{H}_3)$ we have that, for any $\varepsilon > 0$, $\lim_{t\to+\infty} \sup \frac{\int_0^t \psi(s)ds}{\ln \lambda(t)} < \beta_1 + \varepsilon$, and $\lim_{t\to+\infty} \inf \frac{\ln \phi(t)}{\ln \lambda(t)} > -\beta_2 - \varepsilon$. Thanks also to the uniform continuity of $\ln \lambda(t)$, there exists a positive integer $k_1(\varepsilon)$, such that whenever $t \ge k_1(\varepsilon)$,

$$\begin{gathered} \int_0^t \psi(s)ds \le (\beta_1 + \varepsilon) \ln \lambda(t), \\ \phi\left(\frac{k-1}{2^N}\right)^{-1} \le \phi(t) \le \lambda(t)^{\beta_2+\varepsilon}, \end{gathered}$$

for $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_1(\varepsilon)$.

Furthermore, we have

$$\ln \frac{k-1}{2^N} \le \ln t \le \ln \frac{k}{2^N}, \quad \text{for} \quad \frac{k-1}{2^N} \le t \le \frac{k}{2^N}.$$

Based on inequality (2.2), and the standing assumptions, we obtain for almost all $\omega \in \Omega$,

$$\ln V(t, x_t) \le \ln\left(V(0, x_0) + \lambda(t)^{\beta_2+\sigma+2\varepsilon} + \rho(t)\right) + (\beta_1 + \varepsilon) \ln \lambda(t),$$

for $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}, \quad k \ge k_1(\varepsilon)$.

Hence, we deduce that

$$\lim_{t\to+\infty} \sup \frac{\ln V(t, x_t)}{\ln \lambda(t)} \le (\beta_2 + \sigma + 2\varepsilon) \vee m + \beta_1 + \varepsilon, \quad \text{a.s.}$$

Recall that, for $t \ge \bar{T} \ge 0$ and $q \in \mathbb{N}^\star$, we have

$$\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right) \le \ln V(t, x_t) - m \ln \lambda(t) - \ln v'.$$

Taking into account that $\varepsilon > 0$ is arbitrary, we derive that,

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\left[m - (\beta_1 + (\beta_2+\sigma)\vee m)\right], \quad \text{a.s.},$$

as required. □

In the next corollary, we will deduce the practical convergence to the ball $B_r$ with a general decay rate of stochastic differential delay equations.

**Corollary 2.6.** *Let $V : \mathbb{R}_+ \times C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ be a functional from D. Assume that $\ln\lambda(t)$ is uniformly continuous on $t \geq 0$, and there exists a constant $\sigma \geq 0$, such that*

$$\lim_{t\to+\infty} \frac{\ln\ln t}{\ln\lambda(t)} \leq \sigma.$$

*Let $x(\cdot) = x(\cdot,0,\xi)$ be a solution to system (1.1), and assume that there exist constants $q \in \mathbb{N}^\star$, $m \geq 0$, $\beta_1 \in \mathbb{R}$, $\beta_2 \geq 0$, a non–increasing function $\phi(t) > 0$ and a continuous non–negative function $\psi(t)$, such that for all $t \geq 0$, assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ are satisfied. Then, if in addition there exists $\widetilde{v} > v > 0$, such that $\|x(t,0,\xi)\| > \widetilde{v}$ for all $t \geq 0$, it follows*

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad ;a.s.,$$

*where $\gamma = m - (\beta_1 + (\beta_2+\sigma)\vee m)$.*

*In particular, if $m > \beta_1 + (\beta_2+\sigma)\vee m$, then the solution to system (1.1) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{q}}$ almost surely practically with decay function $\lambda(t)$ and order at least $\gamma$.*

**Remark 2.7.** Observe that the condition $m > \beta_1 + (\beta_2+\sigma)\vee m$ (or equivalently $\gamma > 0$) in the corollary holds in the following cases:

- If $\beta_2 + \sigma \leq m$, then the condition becomes $m > \beta_1 + m$. Therefore, this requires $\beta_1 < 0$.
- If $\beta_2 + \sigma > m$, then the condition turns to $m > \beta_1 + \beta_2 + \sigma$ which again requires $\beta_1 < 0$. As a conclusion, the condition ensuring that $\gamma$ is positive requires that $\beta_1 < 0$, and this implies that when $\beta_2 + \sigma \leq m$, then $\gamma > 0$, and when $\beta_2 + \sigma > m$, then $\beta_1$ must be smaller than $m - \beta_2 - \sigma$.

*Proof.* From Theorem 2.5, it follows that

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad \text{a.s.}$$

Since, we have $\lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = v < \widetilde{v}$, then there exists $\bar{T} \geq 0$, such that $\frac{\rho(t)}{\lambda^m(t)} \leq \widetilde{v}$, for all $t \geq \bar{T} \geq 0$. Hence, we obtain

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - (\widetilde{v})^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq \limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad textrm{a.s.}$$

Hence, if $m > \beta_1 + (\beta_2+\sigma)\vee m$, then the solution to system (1.1) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{q}}$ almost surely practically with decay function $\lambda(t)$ and order at least $\gamma$. □

**Example 2.8.** Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[-\dfrac{b+1}{2(1+t)}x(t) + \dfrac{1}{1+t}x(t-\tau)\right]dt + (1+t)^{-\frac{1}{2}}dB(t), \quad t \geq 0, \\ x(t) = \xi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.3}$$

where $b \in \mathbb{R}_+$, $B(t)$ is a one-dimensional Brownian motion and $\tau$ is a positive constant.

Define for $\Phi \in C([-\tau, 0], \mathbb{R})$:

$$F(t,\Phi) = -\frac{b+1}{2(1+t)}\Phi(0) + \frac{1}{1+t}\Phi(-\tau), \quad G(t,\Phi) = (1+t)^{-\frac{1}{2}}, \quad t \geq 0.$$

Now, we proceed to investigate the practical stability with a general decay rate of system (2.3) by using a Lyapunov functional.

Consider the following functional,

$$V(t, x_t) := (1+t)|x(t)|^2 + \int_{t-\tau}^{t} |x(u)|^2 du.$$

Then, it is easy to check that for arbitrary $\alpha > 1$, $\phi(t) = \frac{b}{4(1+t)^\alpha}$, we have

$$\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{b}{4(1+s)^\alpha}|V_x(s,x_s)G(s,x_s)|^2 ds \\
&\quad \leq \int_0^t ds - \int_0^t b|x(s)|^2 ds + \int_0^t 2|x(s)||x(s-\tau)|ds \\
&\qquad + \int_0^t |x(s)|^2 ds - \int_0^t |x(s-\tau)|^2 ds + \int_0^t \frac{b}{(1+s)^{\alpha-2}}|x(s)|^2 ds \\
&\quad \leq \int_0^t ds + \int_0^t (1-b)|x(s)|^2 ds + \int_0^t |x(s-\tau)|^2 ds \\
&\qquad + \int_0^t |x(s)|^2 ds - \int_0^t |x(s-\tau)|^2 ds + \int_0^t \frac{b}{(1+s)^{\alpha-2}}|x(s)|^2 ds.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{1}{b(1+s)^\alpha}|V_x(s,x_s)G(s,x_s)|^2 ds \\
&\quad \leq t + \int_0^t \left[\frac{2-b}{1+s} + \frac{b}{(1+s)^{\alpha-1}}\right](1+s)|x(s)|^2 ds.
\end{aligned}$$

Hence, we see that

$$\psi(t) = \frac{b}{(1+t)^{\alpha-1}} + \frac{2-b}{1+t}, \quad \rho(t) = t.$$

Taking $\lambda(t) = (1+t)$, then by some easy computations, we can check that,

$$\sigma = 0, \quad \beta_1 = 2-b, \quad \beta_2 = \alpha, \quad v = 1, \quad m = 1.$$

Finally, Corollary 2.6 allows us to conclude that

$$\lim_{t\to+\infty} \sup \frac{\ln(|x(t)| - 1)}{\ln(1+t)} \leq -\gamma, \quad \text{a.s.}$$

Hence, we deduce that the solution to system (2.3) decays to the ball $B_r$ almost surely practically with decay function $\lambda(t) = 1+t$, $r = 1$, and order at least $\gamma = b-1-\alpha$, whenever $b > 1+\alpha$.

# 3 Method of Lyapunov functionals construction in practical stability of stochastic delay differential equations

Notice that Corollary 2.6 implies that the almost sure practical stability with a general decay rate of SDDE (1.1) can be reduced to the construction of appropriate Lyapunov functionals.

A formal procedure to construct Lyapunov functionals is described below, (see Krasovskii [19], and V. Kolmanovskii and L. Shaikhet [17, 18, 23], for more details).

## 3.1 The formal procedure of constructing Lyapunov functionals

The formal procedure for constructing Lyapunov functionals consists of four steps.

*Step* 1 : Let us represent (1.1) in the following form:

$$dz(t,x_t) = (F_1(t,x(t)) + F_2(t,x_t))\,dt + (G_1(t,x(t)) + G_2(t,x_t))\,dB(t), \tag{3.1}$$

where $z(t,x_t)$ is some functional of $x_t$, the functions $F_1(t,x(t))$ and $G_1(t,x(t))$, depend on $t$ and $x(t)$ only and do not depend on the previous values $x(t+\theta)$, $\theta < 0$, of the solution, and there exists $t \in \mathbb{R}_+$, such that $F_1(t,\cdot) \neq 0$ or $G_1(t,\cdot) \neq 0$.

*Step* 2 : Consider the auxiliary differential equation without memory

$$dy(t) = F_1(t,y(t))dt + G_1(t,y(t))dB(t). \tag{3.2}$$

Assume that the system (3.2) is almost sure practical stable with a general decay rate and there exists a Lyapunov function $v(t,y(t))$, which satisfies the conditions of Corollary 2.6.

*Step* 3 : A Lyapunov functional $V(t,x_t)$ for Eq.(1.1) is constructed in the form $V = V_1 + V_2$, where $V_1(t,x_t) = v(t,z(t,x_t))$. Here the argument $y$ of the function $v(t,y)$ is replaced on the functional $z(t,x_t)$ from the left–hand part of Eq.(3.1).

*Step* 4 : Usually, the functional $V_1(t,x_t)$ almost satisfies the conditions of Corollary 2.6. To fully satisfy these conditions, it is necessary to calculate $\mathcal{L}V_1(t,x_t)$ and estimate it. Then, we choose the additional functional $V_2(t,x_t)$ in a standard way.

**Remark 3.1.** The representation (3.1) is not unique. This fact allows, using different representations of the type of (3.1) or different ways to estimate $\mathcal{L}V_1(t,x_t)$, to construct different Lyapunov functionals and, as a result to obtain different sufficient conditions for the practical stability with a general decay rate.

## 3.2 Construction of Lyapunov functionals for stochastic differential equations with constant delay

Consider the following stochastic differential equation with constant delay:

$$\begin{aligned} dx(t) &= (f(t,x(t)) + F(t,x(t),x(t-h)))\,dt + G(t,x(t),x(t-\tau))dB(t), \\ x(s) &= \xi(s), \quad s \in [-\widetilde{h},0], \end{aligned} \tag{3.3}$$

where,

$$\widetilde{h} = \max[h,\tau], \quad \text{and} \quad f : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n,$$

$$F : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad G : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

$B(t)$ is an $m$-dimensional Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathrm{P}\}$.

Observe that Eq. (3.3) is a particular case of Eq. (1.1).

We will apply the method described above to construct Lyapunov functionals for Eq. (3.3), and, as a consequence, to obtain sufficient conditions ensuring the almost sure practical stability with decay function $\lambda(t)$, where $\lambda(\cdot) \in C^1(\mathbb{R}_+)$.

**Theorem 3.2.** *Assume that $\ln \lambda(t)$ is uniformly continuous on $t \geq 0$, and there exists a constant $\sigma \geq 0$, such that*

$$\lim_{t\to+\infty} \frac{\ln \ln t}{\ln \lambda(t)} \leq \sigma.$$

*Let $\psi_1(t)$ be a continuous non–negative function, and $\rho(t)$ a non-negative continuous differentiable function, such that for all $t \geq 0 \geq 0$ the following assumptions hold:*

$(\mathcal{A}_1)$

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\leq (\psi_1(t) - K)\|x\|^2 + \frac{\rho'(t)}{\lambda^m(t)}, \quad K > 0, \\ \|\widetilde{F}(t,\Phi)\| &\leq \alpha_1 \|\Phi(-h)\|, \\ \|\widetilde{G}(t,\Phi)\| &\leq \alpha_2 \|\Phi(-\tau)\|, \\ \|\Phi(0)\widetilde{G}(t,\Phi)\| &\leq \alpha_3 \|\Phi(-\tau)\|, \end{aligned} \tag{3.4}$$

*where* $\widetilde{F}(t,\Phi) = F(t,\Phi(0),\Phi(-h)), \quad \widetilde{G}(t,\Phi) = G(t,\Phi(0),\Phi(-\tau))$.

$(\mathcal{A}_2)$

$$\lim_{t\to+\infty} \sup \frac{\int_0^t \psi_1(s)ds}{\ln \lambda(t)} \leq \alpha, \quad \alpha \in \mathbb{R}.$$

$$\lim_{t\to+\infty} \sup \frac{t}{\ln \lambda(t)} = C \geq 0, \quad \lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = v > 0.$$

$(\mathcal{A}_3)$

$$\|x(t,0,\xi)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{2}}, \quad \text{for all } t \geq 0.$$

*Then, if in addition there exists $\widetilde{v} \geq v > 0$, such that $\|x(t,0,\xi)\| > \widetilde{v}$ for all $t \geq 0$, it follows*

$$\lim_{t\to+\infty} \sup \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln \lambda(t)} \leq -\gamma, \quad a.s.,$$

*where $\gamma = KC - (m + \alpha + \sigma) - (2\alpha_1 + \widetilde{\alpha})C$, $\widetilde{\alpha} = \alpha_1^2 + \alpha_2^2$.*

*In particular, if $KC > m + \alpha + \sigma + (2\alpha_1 + \widetilde{\alpha})C$, then the solution to system (3.3) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically, with decay function $\lambda(t)$ and order at least $\gamma$.*

*Proof.* Based upon the procedure of Lyapunov functionals construction, we consider the auxiliary equation without memory of the type (3.2) as

$$\dot{y}(t) = f(t, y(t)). \tag{3.5}$$

Our target now is to prove that the solution to system (3.5) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely with decay function $\lambda(t)$. To this end, we consider the function $v(t,y) = \lambda^m(t)\|y\|^2$, $m \geq 0$ as a Lyapunov function for Eq. (3.5). Then, we have to prove that $v(t,y)$ satisfies all conditions of Corollary 2.6.

Using (3.4), it follows that

$$\begin{aligned}
&\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \\
&\quad \le \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_0^t 2\lambda^m(s)\langle y(s), f(s,y(s))\rangle ds \\
&\quad \le \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_0^t \left(\lambda^m(s)\left[\psi_1(s) - K\right]\|y(s)\|^2 + \rho'(s)\right) ds \\
&\quad \le \int_0^t \left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\lambda^m(s)\|y(s)\|^2 ds + \rho(t) - \rho(0).
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \\
&\quad \le \int_0^t \left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\lambda^m(s)\|y(s)\|^2 ds + \rho(t).
\end{aligned}$$

Thus, setting

$$\psi(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi_1(t) - K.$$

Then, using assumption $(\mathcal{A}_2)$, one obtains

$$\lim_{t\to+\infty}\sup \frac{\int_0^t \psi(s)ds}{\ln\lambda(t)} \le m + \alpha - KC.$$

Consequently, Corollary 2.6 allows us to conclude that,

$$\lim_{t\to+\infty}\sup \frac{\ln\left(\|y(t)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \le -\gamma, \quad \text{a.s.,}$$

where $\gamma = KC - (\alpha + \sigma \vee m)$. Hence, if $KC > \alpha + \sigma \vee m$, then the solution to system (3.4) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically with decay function $\lambda(t)$ and order at least $\gamma$.

Based on the procedure, now we construct a Lyapunov functional $V$ for Eq. (3.3) in the form $V = V_1 + V_2$, where $V_1(t, x_t) = \lambda^m(t)\|x(t)\|^2$.

Following Corollary 2.6, we consider the function $\phi(t) = \frac{1}{4\lambda^m(t)}$, $t \ge 0$, then, it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \phi(s)\|V_{1x}(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad = \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|x(s)\|^2 ds + \int_0^t 2\lambda^m(s)\langle f(s,x(s)), x(s)\rangle ds \\
&\qquad + \int_0^t 2\lambda^m(s)\langle \widetilde{F}(s,x(s),x(s-h)), x(s)\rangle ds + \int_0^t \lambda^m(s)\|\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\qquad + \int_0^t \lambda^m(s)\|x(s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds.
\end{aligned}$$

Taking into account assumptions (3.4), we obtain

$$
\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \frac{1}{4\lambda^m(s)}\|V_{1x}(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\|x(s)\|^2 ds + \rho(t) \\
&\qquad + \int_0^t 2\alpha_1\lambda^m(s)\|x(s)\|\|x(s-h)\|ds + \int_0^t \alpha_2^2\lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\qquad + \int_0^t \alpha_3^2\lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left(\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right] + \alpha_1\right)\|x(s)\|^2 ds \\
&\qquad + \int_0^t \alpha_1\lambda^m(s)\|x(s-h)\|^2 ds + \int_0^t \widetilde{\alpha}\lambda^m(s)\|x(s-\tau)\|^2 ds + \rho(t),
\end{aligned}
$$

where, $\widetilde{\alpha} = \alpha_2^2 + \alpha_3^2$.

Set now

$$
V_2(t,x_t) = \alpha_1\int_{t-h}^t \lambda^m(u+h)\|x(u)\|^2 du + \widetilde{\alpha}\int_{t-\tau}^t \lambda^m(u+\tau)\|x(u)\|^2 du.
$$

Then,

$$
\begin{aligned}
\int_0^t \mathcal{L}V_2(s,x_s)ds &= \alpha_1\int_0^t \lambda^m(s+h)\|x(s)\|^2 ds - \alpha_1\int_0^t \lambda^m(s)\|x(s-h)\|^2 ds \\
&\quad + \widetilde{\alpha}\int_0^t \lambda^m(s+\tau)\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\simeq \alpha_1\int_0^t \lambda^m(s)\|x(s)\|^2 ds - \alpha_1\int_0^t \lambda^m(s)\|x(s-h)\|^2 ds \\
&\quad + \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s-\tau)\|^2 ds.
\end{aligned}
$$

That is, for $V = V_1 + V_2$, we obtain

$$
\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{1}{4\lambda^m(s)}\|V_x(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) + 2\alpha_1 + \widetilde{\alpha} - K\right]\|x(s)\|^2 ds + \rho(t).
\end{aligned}
$$

That is, we have

$$
\psi(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi_1(t) + 2\alpha_1 + \widetilde{\alpha} - K, \quad \phi(t) = \frac{1}{4\lambda^m(t)}.
$$

Therefore, we obtain

$$
\begin{aligned}
\limsup_{t\to+\infty} \frac{\int_0^t \psi(s)ds}{\ln\lambda(t)} &\le m + \alpha + (2\alpha_1 + \widetilde{\alpha} - K)C, \\
\liminf_{t\to+\infty} \frac{\ln\phi(t)}{\ln\lambda(t)} &\ge -m.
\end{aligned}
$$

Finally, Corollary 2.6 allows us to conclude that,

$$\lim_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \le -\gamma,$$

where $\gamma = KC - (m+\alpha+\sigma) - (2\alpha_1+\widetilde{\alpha})C$. Thus, if $KC > (m+\alpha+\sigma) + (2\alpha_1+\widetilde{\alpha})C$, the solution to system (3.3) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically with decay function $\lambda(t)$. □

Now, we provide an illustrative example that implements the previous result.

**Example 3.3.** Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[\left(a + e^{-\frac{3}{2}t} - 4K\right)x(t) + \dfrac{1}{2(1+|x(t)|)} + \cos(t)x(t-h)\right]dt \\ \qquad +g(x(t))\dfrac{x(t-h)}{1+|x(t)|}dB(t), \quad t \ge 0, \\ x(t) = \xi(t), \quad t \in [-h,0], \end{cases} \tag{3.6}$$

where $a, K \in \mathbb{R}_+$, $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz continuous function, such that $g(0) \neq 0$, and $|g(x)| \le L$, $x \in \mathbb{R}$, $L > 0$, $B(t)$ is a one-dimensional Brownian motion and $h$ is a positive constant.

We can set this problem in our formulation by taking,

$$\begin{aligned} f(t,x) &= \frac{1}{2}\left(a + e^{-\frac{3}{2}t} - 4K\right)x + \frac{1}{2(1+|x(t)|)}, \\ \widetilde{F}(t,\Phi) &= \cos(t)\Phi(-h), \\ \widetilde{G}(t,\Phi) &= g(\Phi(0))\frac{\Phi(-h)}{1+|\Phi(0)|}, \end{aligned}$$

$x \in \mathbb{R}$, $\Phi \in C([-h,0],\mathbb{R})$.

We will consider the decay function $\lambda(t) = e^t$ and $m = 1$. Indeed, we can apply Theorem 3.2 in a straightforward way since,

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\le \left(a + e^{-\frac{3}{2}t} - 4K\right)|x|^2 + \frac{e^t}{e^t}, \\ |\widetilde{F}(t,\Phi)| &\le |\Phi(-h)|, \\ |\widetilde{G}(t,\Phi)| &\le L|\Phi(-h)|, \\ |\Phi(0)\widetilde{G}(t,\Phi)| &\le L|\Phi(-h)|. \end{aligned}$$

Therefore, we can set

$$\rho(t) = e^t, \quad \psi_1(t) = (a + e^{-\frac{3}{2}t}).$$

Then, we can choose constants in Theorem 3.2 as follows:

$$\sigma = 0, \quad C = 1, \quad \alpha = a, \quad \alpha_1 = 1, \quad \alpha_2 = \alpha_3 = L, \quad v = 1.$$

Eventually, we deduce that

$$\lim_{t\to+\infty}\sup \frac{\ln(|x(t)|-1)}{t} \le -\gamma, \quad \text{a.s.},$$

where $\gamma = 4K - (3 + a + 2L^2)$. Hence, if $4K > 3 + a + 2L^2$, we deduce that the solution to system (3.6) is almost surely practically exponentially stable with decay function $\lambda(t) = e^t$ and order at least $\gamma$.

### 3.3 Construction of Lyapunov functionals for stochastic differential equations with time-varying delay

Consider the following stochastic differential equation with time-varying delay:

$$\begin{aligned} &dx(t) = [f(t, x(t)) + F(t, x(t), x(t - h(t)))]\, dt + G(t, x(t), x(t - \tau(t)))dB(t),\\ &h(t) \in [0, h_0], \quad \tau(t) \in [0, \tau_0], \quad h = \max[h_0, \tau_0],\\ &x(s) = \xi(s), \quad s \in [-h, 0], \end{aligned} \tag{3.7}$$

where,

$$f : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n, \quad F : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad G : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

$B(t)$ is an $m$–dimensional Brownian motion defined on the probability $\{\Omega, \mathcal{F}, \mathrm{P}\}$.
Observe that Eq.(3.7) is a particular case of Eq. (1.1).

Now, we aim to apply the procedure of constructing Lyapunov functionals for Eq. (3.7), in order to obtain sufficient conditions ensuring the almost sure practical uniform exponential stability, with decay function $\lambda(t) = e^t$. The construction of Lyapunov functionals for general decay functions will be analyzed elsewhere.

**Theorem 3.4.** *Let $\psi_1(t)$ be a continuous non–negative function, $\psi_2(t) > 0$ a non-increasing function and $\rho(t)$ a continuous non-negative differentiable function, such that for all $t \geq 0 \geq 0$ the following assumptions hold:*

$(\mathcal{A}_1')$

$$\begin{aligned} 2\langle x, f(t, x)\rangle &\leq (\psi_1(t) - K)\|x\|^2 + \frac{\rho'(t)}{e^{mt}}, \quad K > 0,\\ \|\widetilde{F}(t, \Phi)\| &\leq \psi_2(t)\|\Phi(-h(t))\|,\\ \|\widetilde{G}(t, \Phi)\| &\leq \alpha_2\|\Phi(-\tau(t))\|,\\ \|\Phi(0)\widetilde{G}(t, \Phi)\| &\leq \alpha_3\|\Phi(-\tau(t))\|, \end{aligned} \tag{3.8}$$

*where $\widetilde{F}(t, \Phi) = F(t, \Phi(0), \Phi(-h(t)))$, $\widetilde{G}(t, \Phi) = G(t, \Phi(0), \Phi(-\tau(t)))$, and*

$$\begin{aligned} &h(t) \in [0, h_0], \quad \dot{h}(t) \leq h_1 \leq 1,\\ &\tau(t) \in [0, \tau_0], \quad \dot{\tau}(t) \leq \tau_1 \leq 1. \end{aligned} \tag{3.9}$$

$(\mathcal{A}_2')$

$$\begin{aligned} &\lim_{t\to+\infty} \sup \frac{\int_0^t \psi_1(s)ds}{t} \leq \alpha, \quad \alpha > 0,\\ &\lim_{t\to+\infty} \sup \frac{\int_0^t \psi_2(s)ds}{t} \leq a, \quad a > 0,\\ &\lim_{t\to+\infty} \frac{\rho(t)}{e^{mt}} = v > 0. \end{aligned}$$

$(\mathcal{A}_3')$

$$\|x(t, 0, \xi)\| > \left(\frac{\rho(t)}{e^{mt}}\right)^{\frac{1}{2}}, \quad \textit{for all } t \geq 0.$$

*Then, if in addition there exists $\widetilde{v} \geq v > 0$, such that $\|x(t,0,\xi)\| > \widetilde{v}$ for all $t \geq 0$, it follows*

$$\lim_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln \lambda(t)} \leq -\gamma, \quad a.s.,$$

*where* $\gamma = K - (m+\alpha) - \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a - \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, $\widetilde{\alpha} = \alpha_1^2 + \alpha_2^2$.

*In particular, if* $K > m + \alpha + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, *the solution to system (3.7) decays to the ball* $B_r$, *with* $r = (\widetilde{v})^{\frac{1}{2}}$ *almost surely uniformly practically exponentially stable, i.e., with decay function* $\lambda(t) = e^t$, *and order at least* $\gamma$.

*Proof.* Proceeding as in the proof of Theorem 3.2, we consider the auxiliary equation without memory of the type (3.2) as

$$\dot{y}(t) = f(t, y(t)). \tag{3.10}$$

We have to prove that the solution to system (3.10) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely with decay function $\lambda(t)$. To this end, we consider the function $v(t,y) = e^{mt}\|y\|^2$, $m \geq 0$ as a Lyapunov function for Eq. (3.10).

Then, we have to prove that $v(t,y)$ satisfies all conditions of Corollary 2.6.

On account of (3.8), it follows that

$$\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \leq \int_0^t [m + \psi_1(s) - K]\, e^{ms}\|y(s)\|^2 ds + \rho(t).$$

Thus, setting

$$\psi(t) = m + \psi_1(t) - K.$$

and using Corollary 2.6, it follows that

$$\lim_{t\to+\infty} \sup \frac{\ln\left(\|y(t)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{t} \leq -\gamma, \quad \text{a.s.},$$

where $\gamma = K - (\alpha + \sigma \vee m)$. Hence, if $K > \alpha + \sigma \vee m$, then the solution to system (3.10) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically uniformly exponentially stable with order at least $\gamma$. Based on the procedure, now we construct a Lyapunov functional $V$ for Eq. (3.7) in the form $V = V_1 + V_2$, where $V_1(t,x_t) = e^{mt}\|x(t)\|^2$.

Following Corollary 2.6, we consider the function $\phi(t) = \frac{1}{4e^{mt}}$, $t \geq 0$, then it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \phi(s)\|V_{1x}(s,x_s)\widetilde{G}(s,x_s)\|^2 ds \\
&\quad = \int_0^t e^{ms}\|x(s)\|^2 ds + \int_0^t 2e^{ms}\langle f(s,x(s)), x(s)\rangle ds \\
&\qquad + \int_0^t 2e^{ms}\langle \widetilde{F}(s,x_s), x(s)\rangle ds + \int_0^t e^{ms}\|\widetilde{G}(s,x_s)\|^2 ds + \int_0^t e^{ms}\|x(s)\widetilde{G}(s,x_s)\|^2 ds.
\end{aligned}$$

Taking into account assumption $(\mathcal{A}_1')$, it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \frac{1}{4e^{ms}}\|V_{1x}(s,x_s)\widetilde{G}(s,x_s)\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left[m+\psi_1(s)-K\right]\|x(s)\|^2 ds + \rho(t) \\
&\qquad + \int_0^t 2\psi_2(s)e^{ms}\|x(s)\|\|x(s-h(s))\|ds + \int_0^t \alpha_2^2 e^{ms}\|x(s-\tau(s))\|^2 ds \\
&\qquad + \int_0^t \alpha_3^2 e^{ms}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left(m+\psi_1(s)-K+\psi_2(s)\right)\|x(s)\|^2 ds \\
&\qquad + \int_0^t \psi_2(s)e^{ms}\|x(s-h(s))\|^2 ds + \int_0^t \widetilde{\alpha}e^{ms}\|x(s-\tau(s))\|^2 ds + \rho(t),
\end{aligned}$$

where $\widetilde{\alpha} = \alpha_2^2 + \alpha_3^2$.

Set now

$$V_2(t,x_t) = \frac{1}{1-h_1}\int_{t-h(t)}^t e^{m(u+h_0)}\psi_2(u)\|x(u)\|^2 du + \frac{\widetilde{\alpha}}{1-\tau_1}\int_{t-\tau(t)}^t e^{m(u+\tau_0)}\|x(u)\|^2 du.$$

Then,

$$\begin{aligned}
&\int_0^t \mathcal{L}V_2(s,x_s)ds \\
&\quad = \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds \\
&\qquad - \frac{1}{1-h_1}\int_0^t (1-\dot{h}(s))e^{m(s-h(s)+h_0)})\psi_2(s-h(s))\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t (1-\dot{\tau}(s))e^{m(s-\tau(s)+\tau_0)}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds \\
&\qquad - \frac{1}{1-h_1}\int_0^t (1-h_1)e^{ms}e^{m(h_0-h(s))}\psi_2(s-h(s))\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t (1-\tau_1)e^{ms}e^{m(\tau_0-\tau(s))}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds - \int_0^t e^{ms}\psi_2(s)\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t e^{ms}\|x(s-\tau(s))\|^2 ds.
\end{aligned}$$

That is, for $V = V_1 + V_2$, we obtain

$$\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{1}{4e^{ms}}\|V_x(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left(m+\psi_1(s)-K+\left(1+\frac{e^{mh_0}}{1-h_1}\right)\psi_2(s)+\widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}\right)\|x(s)\|^2 ds + \rho(t).
\end{aligned}$$

That is, we have

$$\psi(t) = m + \psi_1(t) - K + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)\psi_2(t) + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1},$$
$$\phi(t) = \frac{1}{4e^{mt}}.$$

Therefore, we obtain

$$\lim_{t\to+\infty}\sup\frac{\int_0^t \psi(s)ds}{t} \le m + \alpha - K + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1},$$
$$\lim_{t\to+\infty}\inf\frac{\ln\phi(t)}{t} = -m.$$

Using Corollary 2.6, we infer that

$$\lim_{t\to+\infty}\frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \le -\gamma, \quad \text{a.s.},$$

where $\gamma = K - (m+\alpha) - \big(1 + \frac{e^{mh_0}}{1-h_1}\big)a - \widetilde{\alpha}\dfrac{e^{m\tau_0}}{1-\tau_1}$.

Then, if $K > m + \alpha + \big(1 + \frac{e^{mh_0}}{1-h_1}\big)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, the solution to system (3.7) decays to the ball $B_r$, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely uniformly practically exponentially stable with decay function $\lambda(t) = e^t$, and order at least $\gamma$. □

We analyze now an example to show how the previous theorem can be implemented.

**Example 3.5.** Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[\frac{1}{2}\left(b + |\cos(t)| - K\right)x(t) + \frac{1}{2}\frac{e^{-t}}{1+|x(t)|} + \frac{1}{t+1}x(t-h(t))\right]dt \\ \qquad\qquad + \frac{x(t-\tau(t))}{1+|x(t)|}dB(t), \quad t \ge 0, \\ x(t) = \xi(t), \quad t \in [-h, 0], \end{cases} \tag{3.11}$$

with the conditions,

$$h(t) \in [0, h_0], \quad \dot{h}(t) \le h_1 \le 1,$$
$$\tau(t) \in [0, \tau_0], \quad \dot{\tau}(t) \le \tau_1 \le 1,$$

where $b,\ K \in \mathbb{R}_+,\ x \in \mathbb{R},\ B(t)$ is a one-dimensional Brownian motion, and $h = \max[h_0, \tau_0]$.

We can set this problem in our formulation by taking,

$$f(t,x) = \frac{1}{2}\left(a + |\cos(t)| - K\right)x + \frac{1}{2}\frac{e^{-t}}{1+|x|},$$
$$\widetilde{F}(t,\Phi) = \frac{1}{t+1}\Phi(-h(t)),$$
$$\widetilde{G}(t,\Phi) = \frac{\Phi(-\tau(t))}{1+|\Phi(0)|},$$

$x \in \mathbb{R}, \quad \Phi \in C([-h,0],\mathbb{R})$.

For $m = 2$, we can check that

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\leq (b-K)|x|^2 + \frac{e^t}{e^{2t}}, \\ |\widetilde{F}(t,\Phi)| &\leq \frac{1}{t+1}|\Phi(-h(t))|, \\ |\widetilde{G}(t,\Phi)| &\leq |\Phi(-\tau(t))|, \\ |\Phi(0)\widetilde{G}(t,\Phi)| &\leq |\Phi(-\tau(t))|. \end{aligned}$$

Hence, we see that

$$\psi_1(t) = (b + |\cos(t)|), \quad \psi_2(t) = \frac{1}{t+1}, \quad \rho(t) = e^t.$$

Then, we can choose constants in Theorem 3.4 as follows:

$$\alpha = b, \quad a = 0, \quad \alpha_2 = \alpha_3 = 1, \quad v = 1.$$

Finally, Theorem 3.4 allows us to conclude that,

$$\lim_{t\to+\infty} \sup \frac{\ln(|x(t)| - 1)}{t} \leq -\gamma, \quad \text{a.s.},$$

where $\gamma = K - (2-b) - \frac{e^{2\tau_0}}{1-\tau_1}$. Hence, if $K > 2 + b + 2\frac{e^{2\tau_0}}{1-\tau_1}$, we deduce that the solution to system (3.11) is almost surely practically exponentially stable, i.e., with decay function $\lambda(t) = e^t$, and order at least $\gamma$.

## 4 Conclusion

We investigated herein the practical convergence to a small ball centered at the origin with a general decay rate of stochastic differential delay equations. We then establish sufficient conditions ensuring practical stability with a general decay rate of SDDEs by using Lyapunov functionals. Furthermore, we construct suitable Lyapunov functionals for stochastic differential equations with constant and time–varying delay to obtain sufficient conditions ensuring the practical stability with a general decay rate. Finally, based on the established stability criteria, some examples are given to check the correctness of the derived results.

## Data availability statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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