ON NONNEGATIVE SOLUTIONS OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS WITH ADVANCED ARGUMENTS

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§ 1. Statement of the Problem and Formulation of the Existence and Uniqueness Theorems

Consider the differential system

$$u_i'(t) = f_i\Big(t, u_1(\tau_{i1}(t)), u_2(\tau_{i2}(t))\Big) \quad (i = 1, 2) \tag{1.1}$$

with the boundary conditions

$$\varphi\Big(u_1(0), u_2(0)\Big) = 0, \quad u_1(t) = u_1(a), \quad u_2(t) = 0 \quad \text{for} \quad t \geq a, \tag{1.2}$$

where $f_i : [0, a] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ satisfy the local Carathéodory conditions, while $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\tau_{ik} : [0, a] \to [0, +\infty[$ $(i, k = 1, 2)$ are continuous functions. We are interested in the case, where

$$f_i(t, 0, 0) = 0, \quad f_i(t, x, y) \leq 0 \quad \text{for} \quad 0 \leq t \leq a, \quad x \geq 0, \quad y \geq 0 \quad (i = 1, 2) \tag{1.3}$$

and the function φ satisfies one of the following two conditions:

$$\varphi(0, 0) < 0, \quad \varphi(x, y) > 0 \quad \text{for} \quad x > r, \quad y \geq 0 \tag{1.4}$$

and

$$\varphi(0, 0) < 0, \quad \varphi(x, y) > 0 \quad \text{for} \quad x \geq 0, \quad y \geq 0, \quad x + y > r, \tag{1.5}$$

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where r is a positive constant.

Put

$$\tau_{ik}^0(t) = \begin{cases} \tau_{ik}(t) & \text{for} \quad \tau_{ik}(t) \leq a \\ a & \text{for} \quad \tau_{ik}(t) > a \end{cases} \quad (i, k = 1, 2). \tag{1.6}$$

Under a solution of the problem (1.1), (1.2) is understood an absolutely continuous vector function $(u_1, u_2) : [0, a] \to \mathbb{R}^2$ satisfying almost everywhere on $[0, a]$ the differential system

$$u_i'(t) = f_i\Big(t, u_1(\tau_{i1}^0(t)), u_2(\tau_{i2}^0(t))\Big) \quad (i = 1, 2) \tag{1.1$'$}$$

and also the boundary conditions

$$\varphi\Big(u_1(0), u_2(0)\Big) = 0, \quad u_2(a) = 0. \tag{1.2$'$}$$

A solution (u_1, u_2) of the problem (1.1), (1.2) is called nonnegative if

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq a.$$

If (1.3) holds, then it is obvious that each component of a nonnegative solution of the problem (1.1), (1.2) is a nonincreasing function.

For the case, where $\tau_{ik}(t) \equiv t$ $(i, k = 1, 2)$, the boundary value problems of the type (1.1), (1.2) have been investigated by quite a number of authors (see, e.g., [4–8, 14–21] and the references therein). In this paper the case, where

$$\tau_{ik}(t) \geq t \quad \text{for} \quad 0 \leq t \leq a \quad (i, k = 1, 2), \tag{1.7}$$

is considered and the optimal, in a certain sense, sufficient conditions are established for the existence and uniqueness of a solution of the problem (1.1), (1.2). Some of these results (see, e.g., Corollaries 1.1 and 1.4) are specific for advanced differential systems and have no analogues for the system

$$u_i'(t) = f_i\Big(t, u_1(t), u_2(t)\Big) \quad (i = 1, 2).$$

Theorems 1.1–1.4 proven below and also their corollaries make the previous well-known results [1–3, 9–12] on the solvability and unique solvability of the boundary value problems for the differential systems with deviated arguments more complete.

Along with (1.1) we will consider its perturbation

$$u_i'(t) = f_i\Big(t, u_1(\tau_{i1}(t) + \varepsilon), u_2(\tau_{i2}(t) + \varepsilon)\Big) \quad (i = 1, 2), \tag{1.1$_\varepsilon$}$$

where $\varepsilon > 0$. As it will be proved below[1], for every $\varepsilon > 0$ the problem (1.1$_\varepsilon$), (1.2) has at least one nonnegative solution provided the conditions (1.3), (1.4), and (1.7) are fulfilled.

[1] See Lemma 2.4.

Theorem 1.1 below reduces the question of the solvability of the problem (1.1), (1.2) to obtaining uniform a priori estimates of second components of solutions of the problem (1.1_ε), (1.2) with respect to the parameter ε. Such estimates can be derived in rather general situations and therefore from Theorem 1.1 the effective and optimal in a sense conditions are obtained for the solvability of the problem (1.1), (1.2) (see Corollaries 1.1–1.5 and Theorem 1.2).

Theorem 1.1. *Let the conditions* (1.3), (1.4), *and* (1.7) *be fulfilled and let there exist positive numbers* ε_0 *and* ρ_0 *such that for any* $\varepsilon \in]0, \varepsilon_0]$ *the second component of an arbitrary nonnegative solution* (u_1, u_2) *of the problem* (1.1_ε), (1.2) *admits the estimate*

$$u_2(0) \le \rho_0. \tag{1.8}$$

Then the problem (1.1), (1.2) *has at least one nonnegative solution.*

Corollary 1.1. *Let the conditions* (1.3), (1.4) *be fulfilled and let*

$$\tau_{1i}(t) \ge t \quad (i = 1, 2), \quad \tau_{21}(t) \ge t, \quad \tau_{22}(t) > t \quad \text{for} \quad 0 \le t \le a. \tag{1.9}$$

Then the problem (1.1), (1.2) *has at least one nonnegative solution.*

Remark 1.1. The condition (1.9) in Corollary 1.1 is essential and it cannot be replaced by the condition (1.7). To convince ourselves that this is so, consider the boundary value problem

$$u_1'(t) = 0, \quad u_2'(t) = -\Big(|u_1(t)| + |u_2(t)|\Big)^{\lambda}, \tag{1.10}$$

$$u_1(0) = 1, \quad u_2(a) = 0, \tag{1.11}$$

where

$$\lambda \ge \frac{1}{a} + 1. \tag{1.12}$$

It is seen that for that problem all the conditions of Corollary 1.1, except (1.9), are fulfilled. Instead of (1.9) there takes place the condition (1.7). Nevertheless, the problem (1.10), (1.11) has no solution. Indeed, should this problem have a solution (u_1, u_2), the function u_2 would be positive on $[0, a[$ and

$$a = -\int_0^a \Big(1 + u_2(s)\Big)^{-\lambda} du_2(s) = \frac{1}{\lambda - 1} - \frac{1}{\lambda - 1}\Big(1 + u_2(0)\Big)^{1-\lambda} < \frac{1}{\lambda - 1}\,.$$

But the latter inequality contradicts (1.12).

Corollary 1.2. *Let the conditions* (1.3), (1.4), *and* (1.7) *be fulfilled and let there exist* $y_0 > 0$ *such that*

$$f_2(t,x,y) \geq -h(t)\omega(y) \quad \text{for} \quad 0 \leq t \leq a, \quad 0 \leq x \leq r, \quad y \geq y_0, \tag{1.13}$$

where $h : [0,a] \to [0,+\infty[$ *is a summable function and* $\omega : [y_0,+\infty[\,\to\,]0,+\infty[$ *is a nondecreasing continuous function satisfying the condition*

$$\int_{y_0}^{+\infty} \frac{dy}{\omega(y)} = +\infty. \tag{1.14}$$

Then the problem (1.1), (1.2) *has at least one nonnegative solution.*

Remark 1.2. As is shown above, if $a = 1/\varepsilon$ and $\lambda = 1+\varepsilon$, then the problem (1.10), (1.11) has no solution. This fact shows that the condition (1.14) in Corollary 1.2 cannot be replaced by the condition

$$\int_{y_0}^{+\infty} \frac{y^\varepsilon\, dy}{\omega(y)} = +\infty$$

no matter how small $\varepsilon > 0$ is.

Corollary 1.3. *Let the conditions* (1.3), (1.4), *and* (1.7) *be fulfilled and let there exist numbers* $a_i \in\,]0,a]$ $(i = 1,2)$ *and* $y_0 > 0$ *such that*

$$\tau_{12}(t) \leq a_2 \quad \text{for} \quad 0 \leq t \leq a_1, \tag{1.15}$$

$$f_1(t,x,y) \leq -\delta(t,y) \quad \text{for} \quad 0 \leq t \leq a_1, \quad 0 \leq x \leq r, \quad y \geq y_0, \tag{1.16}$$

and

$$f_2(t,x,y) \geq -\Big[h(t) + |f_1(t,x,y)|\Big]\omega(y) \quad \text{for} \quad 0 \leq t \leq a_2, \ \ 0 \leq x \leq r, \ \ y \geq y_0, \tag{1.17}$$

where $\delta : [0,a_1] \times [y_0,+\infty[\,\to [0,+\infty[$ *is a summable in the first and nondecreasing in the second argument function, while* $h : [0,a_2] \to [0,+\infty[$ *and* $\omega : [y_0,+\infty[\,\to\,]0,+\infty[$ *are summable and nondecreasing continuous functions, respectively. Let, moreover,* $\tau_{1i}(t) \equiv \tau_{2i}(t)$ $(i = 1,2)$,

$$\lim_{y \to +\infty} \int_0^{a_1} \delta(t,y)\, dt > r, \tag{1.18}$$

and ω *satisfy the condition* (1.14). *Then the problem* (1.1), (1.2) *has at least one nonnegative solution.*

Remark 1.3. The condition (1.15) in Corollary 1.3 cannot be replaced by the condition

$$\tau_{12}(t) \leq a_2 + \varepsilon \quad \text{for} \quad 0 \leq t \leq a_1 \tag{1.19}$$

no matter how small $\varepsilon > 0$ is. As an example verifying this fact, consider the differential system

$$u_1'(t) = -u_2(a), \quad u_2'(t) = -g(t)\Big(|u_1(t)| + |u_2(t)|\Big)^{1+\frac{1}{\varepsilon}} \tag{1.20}$$

with the boundary conditions (1.11), where $\varepsilon \in]0, a[$ and

$$g(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq a - \varepsilon \\ 1 & \text{for} \quad a - \varepsilon < t \leq a \end{cases}.$$

It is seen that for this problem the conditions (1.16)–(1.18) hold and instead of (1.15) the condition (1.19) is fulfilled, where $\tau_{12}(t) \equiv a$, $a_1 = a$, $a_2 = a - \varepsilon$, $\delta(t, y) \equiv y$, $h(t) \equiv 0$, $\omega(y) \equiv 1$. But the problem (1.20), (1.11) has no solution. Indeed, if we assume that (1.20), (1.11) has a solution (u_1, u_2), then we will obtain the contradiction, i.e.,

$$\varepsilon = -\int\limits_{a-\varepsilon}^{a} \Big(1 + u_2(s)\Big)^{-\frac{1}{\varepsilon}-1} du_2(s) = \varepsilon - \varepsilon\Big(1 + u_2(a - \varepsilon)\Big)^{-\frac{1}{\varepsilon}} < \varepsilon.$$

In contrast to Corollary 1.3, Corollaries 1.4 and 1.5 below catch the effect of an advanced argument τ_{22}.

Corollary 1.4. *Let the conditions* (1.3), (1.4), *and* (1.7) *be fulfilled and let for some* $a_i \in]0, a]$ $(i = 1, 2)$ *and* $y_0 > 0$ *the inequalities* (1.15), (1.16), *and*

$$\tau_{22}(t) \geq a_2 \quad \textit{for} \quad 0 \leq t \leq a \tag{1.21}$$

hold, where $\delta : [0, a_1] \times [y_0, +\infty[\to [0, +\infty[$ *is a summable in the first and nondecreasing in the second argument function satisfying the condition* (1.18). *Then the problem* (1.1), (1.2) *has at least one nonnegative solution.*

Remark 1.4. It is obvious from the example (1.20), (1.11) that it is impossible in Corollary 1.4 to replace the condition (1.21) by the condition

$$\tau_{22}(t) \geq a_2 - \varepsilon \quad \text{for} \quad 0 \leq t \leq a$$

no matter how small $\varepsilon > 0$ is.

Corollary 1.5. *Let the conditions* (1.3), (1.4), *and* (1.7) *hold and let, moreover, for some* $a_0 \in]0, 1] \cap]0, a^{1/\alpha}]$ *the inequalities*

$$\tau_{12}(t) \leq t^{\alpha} \quad \textit{for} \quad 0 \leq t \leq a_0, \tag{1.22}$$

$$f_1(t, x, y) \leq -l t^{\beta} y^{\lambda_1} \quad \textit{for} \quad 0 \leq t \leq a_0, \quad 0 \leq x \leq r, \quad y \geq 0, \tag{1.23}$$

and

$$f_2(t,x,y) \geq -h(t)(1+y)^{1+\lambda_2} \quad \textit{for} \quad 0 \leq t \leq a_0, \quad 0 \leq x \leq r, \quad y \geq 0 \tag{1.24}$$

be fulfilled, where $0 < \alpha \leq 1$, $\beta > -1$, $l > 0$, $\lambda_1 > 0$, $\lambda_2 \geq 0$ *and* $h : [0, a_0] \to [0, +\infty[$ *is a measurable function satisfying the condition*

$$\int_0^{a_0} [\tau_{22}(t)]^{-\frac{(1+\beta)\lambda_2}{\alpha\lambda_1}} h(t)\,dt < +\infty. \tag{1.25}$$

Then the problem (1.1), (1.2) *has at least one nonnegative solution.*

Remark 1.5. The condition (1.25) in Corollary 1.5 cannot be replaced by the condition

$$\int_0^{a_0} [\tau_{22}(t)]^{-\frac{(1+\beta)\lambda_2}{\alpha\lambda_1}+\varepsilon} h(t)\,dt < +\infty \tag{1.26}$$

no matter how small $\varepsilon > 0$ is. As an example, consider the differential system

$$u_1'(t) = -u_2(t), \quad u_2'(t) = -\gamma t^{\lambda-1-\delta}\Big(|u_1(t)| + |u_2(t)|\Big)^{1+\lambda} \tag{1.27}$$

with the boundary conditions (1.11), where $0 < \delta < \varepsilon < \lambda$,

$$\gamma > \frac{\lambda-\delta}{\lambda}\left(\frac{\lambda}{\delta}\right)^{\lambda}\eta^{\delta-\lambda} \tag{1.28}$$

and $\eta = \min\{a, 1\}$. For the system (1.27) the conditions (1.22)–(1.24) hold and instead of (1.25) there takes place the condition (1.26), where $\tau_{12}(t) \equiv \tau_{22}(t) \equiv t$, $a_0 = \eta$, $\alpha = 1$, $\beta = 0$, $l = 1$, $\lambda_1 = 1$, $\lambda_2 = \lambda$, $r = 1$, and $h(t) = 2^{1+\lambda}\gamma t^{\lambda-1-\delta}$. Show that the problem (1.27), (1.11) has no solution. Assume the contrary that this problem has a solution (u_1, u_2). Then $u_1(t) > u_1(\eta) > 0$, $u_2(t) > 0$ for $0 \leq t < \eta$,

$$\int_0^{\eta} u_2(t)\,dt = 1 - u_1(\eta),$$

and

$$-\Big(u_1(\eta) + u_2(t)\Big)^{-1-\lambda} u_2'(t) > \gamma t^{\lambda-1-\delta} \quad \text{for} \quad 0 < t < \eta.$$

The integration of the latter inequality from 0 to t yields

$$\Big(u_1(\eta) + u_2(t)\Big)^{-\lambda} - \Big(u_1(\eta) + u_2(0)\Big)^{-\lambda} > \frac{\lambda\gamma}{\lambda-\delta}\, t^{\lambda-\delta} \quad \text{for} \quad 0 < t < \eta$$

and hence

$$u_1(\eta) + u_2(t) < \left(\frac{\lambda-\delta}{\lambda\gamma}\right)^{\frac{1}{\lambda}} t^{-1+\frac{\delta}{\lambda}} \quad \text{for} \quad 0 < t < \eta.$$

Integrating this inequality from 0 to η, we obtain

$$\eta u_1(\eta) + 1 - u_1(\eta) < \frac{\lambda}{\delta}\left(\frac{\lambda-\delta}{\lambda\gamma}\right)^{\frac{1}{\lambda}} \eta^{\frac{\delta}{\lambda}}.$$

However, since $0 < u_1(\eta) < 1$, we have $\eta u_1(\eta) + 1 - u_1(\eta) > \eta$. Thus

$$\frac{\lambda}{\delta}\left(\frac{\lambda-\delta}{\lambda\gamma}\right)^{\frac{1}{\lambda}} \eta^{\frac{\delta}{\lambda}} > \eta,$$

which contradicts (1.28).

Theorem 1.2. *If the conditions* (1.3), (1.5), *and* (1.7) *are fulfilled, then the problem* (1.1), (1.2) *has at least one nonnegative solution.*

The uniqueness of a solution of the problem (1.1), (1.2) is closely connected with the uniqueness of a solution of the system (1.1) with the Cauchy conditions

$$u_1(t) = c, \quad u_2(t) = 0 \quad \text{for} \quad t \geq a. \tag{1.29}$$

The following theorem is valid.

Theorem 1.3. *Let the condition* (1.7) *be fulfilled and for any* $c \in \mathbb{R}$ *the Cauchy problem* (1.1), (1.29) *have no more than one solution. Let, moreover, the functions* f_i $(i = 1, 2)$ *not increase in the last two arguments, while the function* φ *increase in the first argument and not decrease in the second argument. Then the problem* (1.1), (1.2) *has no more than one solution.*

Theorem 1.4. *Let* (1.7) *be fulfilled and for any* $c \in \mathbb{R}$ *the Cauchy problem* (1.1), (1.29) *have no more than one solution. Let, moreover, the function* f_1 *not increase in the last two arguments,* f_2 *decrease in the second argument and not increase in the third argument, while the function* φ *be such that*

$$\varphi(x, y) < \varphi(\overline{x}, \overline{y}) \quad \textit{for} \quad x < \overline{x}, \quad y < \overline{y}. \tag{1.30}$$

Then the problem (1.1), (1.2) *has no more than one solution.*

Remark 1.6. For the uniqueness of a solution of the problem (1.1), (1.29) it is sufficient that either the functions f_i $(i = 1, 2)$ satisfy in the last two arguments the local Lipschitz condition or the functions τ_{ik} $(i, k = 1, 2)$ satisfy the inequalities

$$\tau_{ik}(t) > t \quad \text{for} \quad 0 \leq t \leq a \quad (i, k = 1, 2).$$

As an example, consider the boundary value problem

$$u_i'(t) = -\sum_{k=1}^{2} p_{ik}(t)\Big|u_k(\tau_{ik}(t))\Big|^{\lambda_{ik}} \operatorname{sgn}(u_k(\tau_{ik}(t))) \quad (i = 1, 2),$$

$$u_1(0) + \alpha u_2(0) = \beta, \quad u_1(t) = u_1(a), \quad u_2(t) = 0 \quad \text{for} \quad t \geq a,$$

where $\lambda_{ik} > 0$ $(i,k = 1,2)$, $\alpha \geq 0$, $\beta > 0$, $p_{ik} : [0,a] \to [0,+\infty[$ $(i,k = 1,2)$ are summable functions and $\tau_{ik} : [0,a] \to [0,+\infty[$ $(i,k = 1,2)$ are continuous functions satisfying the inequalities (1.7). By virtue of Corollaries 1.1, 1.3 and Theorems 1.2, 1.3 this problem has the unique nonnegative solution if, besides the above given, one of the following three conditions is fulfilled:

(i) $\tau_{ik}(t) > t$ for $0 \leq t \leq a$ $(i,k = 1,2)$;

(ii) $\alpha > 0$, $\lambda_{ik} \geq 1$ $(i,k = 1,2)$;

(iii) $\alpha = 0$, $\lambda_{ik} \geq 1$ $(i,k = 1,2)$, $\lambda_{22} \leq 1 + \lambda_{12}$ and there exist $a_i \in]0,a]$ $(i = 1,2)$ such that $\tau_{12}(t) \leq a_2$ for $0 \leq t \leq a_1$, $p_{12}(t) > 0$ for $0 < t < a_2$, and

$$\operatorname{vrai\,max}\left\{\frac{p_{22}(t)}{p_{12}(t)} : \ 0 < t < a_2\right\} < +\infty.$$

§ 2. Some Auxiliary Statements

2.1. Lemmas on properties of solutions of an auxiliary Cauchy problem.

Lemma 2.1. *If the conditions* (1.3) *and*

$$\tau_{ik}(t) > t \quad \textit{for} \quad 0 \leq t \leq a \quad (i,k = 1,2) \tag{2.1}$$

are fulfilled, then for any nonnegative c the problem (1.1), (1.29) *has the unique solution* $(u_1(\cdot,c), u_2(\cdot,c))$. *Moreover, the functions* $(t,c) \to u_i(t,c)$ $(i = 1,2)$ *are continuous on the set* $[0,a] \times [0,+\infty[$ *and satisfy on the same set the inequalities*

$$u_1(t,c) \geq c, \quad u_2(t,c) \geq 0. \tag{2.2}$$

Proof. In view of (2.1) there exists a natural number m such that

$$\tau_{ik}(t) - t > \frac{a}{m} \quad \text{for} \quad 0 \leq t \leq a \quad (i,k = 1,2).$$

Put

$$t_j = \frac{ja}{m} \quad (j = 0,\dots,m).$$

Then

$$\tau_{ik}(t) > t_j \quad \text{for} \quad t_{j-1} \leq t \leq t_j \quad (i,k = 1,2; \ \ j = 1,\dots,m). \tag{2.3}$$

Suppose that for some $c \geq 0$ the problem (1.1), (1.29) has a solution $(u_1(\cdot,c), u_2(\cdot,c))$. Then on account of

$$u_i(t,c) = u_{ij}(t,c) \quad \text{for} \quad t_j \leq t \leq t_{j+1} \quad (i = 1,2; \ \ j = m-1,\dots,0),$$

where

$$u_{1m}(t,c)=c,\quad u_{2m}(t,c)=0\quad\text{for}\quad t\geq a,\tag{2.4}$$

$$u_{ij}(t,c)=u_{i\,j+1}(t_{j+1},c)-$$
$$-\int_t^{t_{j+1}} f_i\Big(s,u_{1\,j+1}(\tau_{i1}(s),c),u_{2\,j+1}(\tau_{i2}(s),c)\Big)\,ds\quad\text{for}\quad t_j\leq t\leq t_{j+1},\tag{2.5}$$
$$u_{ij}(t,c)=u_{i\,j+1}(t,c)\quad\text{for}\quad t>t_{j+1}\quad(i=1,2;\ \ j=m-1,\dots,0),$$

we conclude that if the problem (1.1), (1.29) has a solution, then this solution is unique since the functions u_{ij} $(i=1,2;\ j=m,\dots,0)$ are defined uniquely.

Let us now suppose that $u_{ij}:[t_j,+\infty[\times[0,+\infty[\to\mathbb{R}$ $(i=1,2;\ j=m-1,\dots,0)$ are the functions given by (2.4) and (2.5). Then proceeding from the conditions (1.3) and (2.3) we prove by the induction that these functions are continuous,

$$u_{1j}(t,c)\geq c,\quad u_{2j}(t,c)\geq 0\quad\text{for}\quad t\geq t_j,\quad c\geq 0\quad(j=m-1,\dots,0),$$

and for any $j\in\{0,\dots,m-1\}$ and $c\geq 0$ the restriction of the vector function $(u_{1j}(\cdot,c),u_{2j}(\cdot,c))$ on $[t_j,a]$ is a solution of the problem (1.1), (1.29) on $[t_j,a]$. Consequently, the vector function $(u_1(\cdot,c),u_2(\cdot,c))$ with the components

$$u_i(t,c)=u_{i0}(t,c)\quad\text{for}\quad 0\leq t\leq a\quad(i=1,2)$$

is the unique solution of the problem (1.1), (1.29). Moreover, $u_i:[0,a]\times[0,+\infty[\to\mathbb{R}$ $(i=1,2)$ are continuous and satisfy (2.2). □

Lemma 2.2. *Let* (1.7) *be fulfilled and the functions* f_i $(i=1,2)$ *be nonincreasing in the last two arguments. Let, moreover, there exist* $c_1\geq 0$ *and* $c_2>c_1$ *such that for* $c\in\{c_1,c_2\}$ *the problem* (1.1)*,* (1.29) *has the unique solution* $(u_1(\cdot,c),u_2(\cdot,c))$*. Then*

$$u_1(t,c_2)\geq c_2-c_1+u_1(t,c_1),\quad u_2(t,c_2)\geq u_2(t,c_1)\quad\textit{for}\quad 0\leq t\leq a.\tag{2.6}$$

Proof. For each $j\in\{1,2\}$ the vector function $(u_1(\cdot,c_j),u_2(\cdot,c_j))$ is a solution of the differential system (1.1′) under the conditions

$$u_1(a,c_j)=c_j,\quad u_2(a,c_j)=0.$$

Using the transformation

$$s=a-t,\quad v_i(s)=u_i(t)\quad(i=1,2)$$

we can rewrite the system (1.1′) in the form

$$v_i'(s)=\widetilde{f}_i\Big(s,v_1(\zeta_{i1}(s)),v_2(\zeta_{i2}(s))\Big)\quad(i=1,2),\tag{2.7}$$

where

$$\widetilde{f}_i(s,x,y)=-f_i(a-s,x,y),\quad \zeta_{ik}(s)=a-\tau_{ik}^0(a-s)\quad(i,k=1,2).$$

On the basis of (1.6) and (1.7) we have

$$0 \le \zeta_{ik}(s) \le s \quad \text{for} \quad 0 \le s \le a \quad (i,k=1,2). \tag{2.8}$$

According to one of the conditions of the lemma, for each $j \in \{1,2\}$ the system (2.7) under the initial conditions

$$v_1(0) = c_j, \quad v_2(0) = 0$$

has the unique solution $(v_1(\cdot, c_j), v_2(\cdot, c_j))$, and

$$v_i(s, c_j) = u_i(a-s, c_j) \quad (i=1,2).$$

On the other hand,

$$c_1 < c_2,$$

the functions $\widetilde{f}_i$ $(i=1,2)$ do not decrease in the last two arguments, and the functions ζ_{ik} $(i,k=1,2)$ satisfy (2.8). By virtue of Corollary 1.9 from [13] the above conditions guarantee the validity of the inequalities

$$v_i(s, c_1) \le v_i(s, c_2) \quad \text{for} \quad 0 \le s \le a \quad (i=1,2).$$

It is obvious that

$$v_i'(s, c_1) \le v_i'(s, c_2) \quad \text{for} \quad 0 \le s \le a \quad (i=1,2).$$

Consequently,

$$u_i(t, c_1) \le u_i(t, c_2), \quad u_i'(t, c_1) \ge u_i'(t, c_2) \quad \text{for} \quad 0 \le t \le a \quad (i=1,2).$$

Hence the inequalities (2.6) follow immediately. $\square$

For every natural m consider the Cauchy problem

$$u_i'(t) = f_i\Big(t, u_1(\tau_{i1m}(t)), u_2(\tau_{i2m}(t))\Big) \quad (i=1,2), \tag{2.9}$$

$$u_1(t) = c_m, \quad u_2(t) = 0 \quad \text{for} \quad t \ge a, \tag{2.10}$$

where $\tau_{ikm} : [0,a] \to [0,+\infty[$ $(i,k=1,2)$ are continuous functions.

The following lemma holds.

Lemma 2.3. *Let*

$$\lim_{m\to+\infty} \tau_{ikm}(t) = \tau_{ik}(t) \quad \textit{uniformly on} \quad [0,a] \quad (i,k=1,2), \tag{2.11}$$

$$\lim_{m\to+\infty} c_m = c \tag{2.12}$$

and let there exist positive number ρ such that for every natural m the problem (2.9), (2.10) *has a solution (u_{1m}, u_{2m}) satisfying the inequality*

$$|u_{1m}(t)| + |u_{2m}(t)| \le \rho \quad \textit{for} \quad 0 \le t \le a. \tag{2.13}$$

Then from the sequences $(u_{im})_{m=1}^{+\infty}$ $(i = 1,2)$ *we can choose uniformly converging subsequences* $(u_{im_j})_{j=1}^{+\infty}$ $(i = 1,2)$ *such that the vector function* (u_1, u_2)*, where*

$$u_i(t) = \lim_{j\to+\infty} u_{im_j}(t) \quad \textit{for} \quad 0 \le t \le a \quad (i = 1,2), \tag{2.14}$$

is a solution of the problem (1.1), (1.29).

Proof. According to (2.13), for every m we have

$$|u'_{1m}(t)| + |u'_{2m}(t)| \le f^*(t) \quad \text{for} \quad 0 \le t \le a,$$

where

$$f^*(t) = \max\Big\{ \sum_{i=1}^{2} |f_i(t,x,y)| : \quad |x| + |y| \le \rho \Big\}$$

and, as is evident, f^* is summable on $[0,a]$. Therefore the sequences $(u_{im})_{m=1}^{+\infty}$ $(i = 1,2)$ are uniformly bounded and equicontinuous on $[0,a]$. By virtue of the Arzela–Ascoli lemma, these sequences contain subsequences $(u_{im_j})_{j=1}^{+\infty}$ $(i = 1,2)$ converging uniformly on $[0,a]$. Let u_1 and u_2 be functions given by (2.14). If in the equalities

$$u_{1m_j}(t) = c_{m_j} - \int_t^a f_1\Big(s, u_{1m_j}(\tau_{11m_j}(s)), u_{2m_j}(\tau_{12m_j}(s))\Big)\, ds,$$

$$u_{2m_j}(t) = -\int_t^a f_2\Big(s, u_{1m_j}(\tau_{21m_j}(s)), u_{2m_j}(\tau_{22m_j}(s))\Big)\, ds \quad \text{for} \quad 0 \le t \le a$$

we pass to the limit as $j \to +\infty$, then by virtue of (2.10)–(2.12) and the Lebesgue theorem concerning the passage to the limit under the integral sign we find that

$$u_1(t) = c - \int_t^a f_1\Big(s, u_1(\tau_{11}(s)), u_2(\tau_{12}(s))\Big)\, ds,$$

$$u_2(t) = -\int_t^a f_2\Big(s, u_1(\tau_{21}(s)), u_2(\tau_{22}(s))\Big)\, ds \quad \text{for} \quad 0 \le t \le a$$

and the vector function (u_1, u_2) satisfies (1.29). Consequently, (u_1, u_2) is a solution of the problem (1.1), (1.29). □

2.2. Lemma on the solvability of the problem (1.1), (1.2).

Lemma 2.4. *If the conditions* (1.3), (1.4)*, and* (2.1) *are fulfilled, then the problem* (1.1), (1.2) *has at least one nonnegative solution.*

Proof. By Lemma 2.1, for every nonnegative c the problem (1.1), (1.29) has the unique solution $(u_1(\cdot,c),u_2(\cdot,c))$, while the functions $(t,c)\to u_i(t,c)$ $(i=1,2)$ are continuous on the set $[0,a]\times[0,+\infty[$ and satisfy on the same set the inequalities (2.2). Hence, according to (1.3), it becomes evident that

$$u_i(t,0)=0 \quad \text{for} \quad 0\le t\le a \quad (i=1,2) \tag{2.15}$$

and

$$u_1(0,c)\ge c, \quad u_2(0,c)\ge 0 \quad \text{for} \quad c>0. \tag{2.16}$$

Put

$$\varphi_0(c)=\varphi\Big(u_1(0,c),u_2(0,c)\Big).$$

It is clear that $\varphi_0:[0,+\infty[\,\to\mathbb{R}$ is a continuous function. On the other hand, in view of (1.4), (2.15), and (2.16) we have

$$\varphi_0(0)<0, \quad \varphi_0(r)\ge 0.$$

Thus there exists $c_0\in\,]0,r]$ such that

$$\varphi_0(c_0)=0.$$

Consequently,

$$\varphi\Big(u_1(0,c_0),u_2(0,c_0)\Big)=0$$

and thus the vector function $(u_1(\cdot,c_0),u_2(\cdot,c_0))$ is a nonnegative solution of the problem (1.1), (1.2). $\square$

2.3. Lemmas on a priori estimates. First of all consider the system of differential inequalities

$$\begin{aligned} u_1'(t)&\le-\delta(t,u_2(a_0)),\\ u_2'(t)&\ge-\Big[h(t)+|u_1'(t)|\Big]\omega(u_2(t)) \end{aligned} \tag{2.17}$$

with the initial condition

$$u_1(0)\le r, \tag{2.18}$$

where $\delta:[0,a_0]\times[0,+\infty[\,\to[0,+\infty[$ is a continuous in the first and nondecreasing in the second argument function, $h:[0,a_0]\to[0,+\infty[$ is a summable function and $\omega:[0,+\infty[\,\to\,]0,+\infty[$ is a nondecreasing continuous function.

A vector function (u_1,u_2) with the nonnegative components $u_i:[0,a_0]\to[0,+\infty[$ $(i=1,2)$ is said to be a nonnegative solution of the problem (2.17), (2.18) if the functions u_1 and u_2 are absolutely continuous, the function u_1 satisfies the inequality (2.18), and the system of differential inequalities (2.17) holds almost everywhere on $[0,a_0]$.

Lemma 2.5. *Let*

$$\lim_{y\to+\infty}\int_0^{a_0}\delta(s,y)\,ds>r \tag{2.19}$$

and

$$\int_0^{+\infty}\frac{dy}{\omega(y)}=+\infty. \tag{2.20}$$

Then there exists a positive number ρ_0 such that the second component of an arbitrary nonnegative solution (u_1,u_2) of the problem (2.17), (2.18) *admits the estimate*

$$u_2(t)\le\rho_0 \quad \textit{for} \quad 0\le t\le a_0. \tag{2.21}$$

Proof. By virtue of (2.19) and (2.20) there exist positive numbers ρ_0 and ρ_1 such that

$$\int_0^{a_0}\delta(s,y)\,ds>r \quad \text{for} \quad y>\rho_1 \tag{2.22}$$

and

$$\int_{\rho_1}^{\rho_0}\frac{dy}{\omega(y)}=r+\int_0^{a_0}h(s)\,ds. \tag{2.23}$$

Let (u_1,u_2) be an arbitrary nonnegative solution of the problem (2.17), (2.18). Then

$$\int_0^{a_0}|u_1'(s)|\,ds=-\int_0^{a_0}u_1'(s)\,ds=u_1(0)-u_1(a_0)\le r,$$

which, owing to (2.17), results in

$$r\ge-\int_0^{a_0}u_1'(s)\,ds\ge\int_0^{a_0}\delta(s,u_2(a_0))\,ds, \tag{2.24}$$

$$\int_{u_2(a_0)}^{u_2(t)}\frac{dy}{\omega(y)}=-\int_t^{a_0}\frac{u_2'(s)\,ds}{\omega(u_2(s))}\le\int_0^{a_0}h(s)\,ds+\int_0^{a_0}|u_1'(s)|\,ds\le$$

$$\le r+\int_0^{a_0}h(s)\,ds \quad \text{for} \quad 0\le t\le a_0. \tag{2.25}$$

Taking into account (2.22), from (2.24) we get

$$u_2(a_0)\le\rho_1.$$

On the basis of the last inequality and (2.23), from (2.25) we find the estimate (2.21). $\square$

Finally, on the segment $[0, a_0]$ consider the system of differential inequalities

$$\begin{aligned} u_1'(t) &\le -lt^\beta u_2^{\lambda_1}(t^\alpha), \\ -h(t)\Big(1 + u_2(\tau(t))\Big)^{1+\lambda_2} &\le u_2'(t) \le 0, \end{aligned} \tag{2.26}$$

where

$$a_0 \in]0, 1], \tag{2.27}$$

$0 < \alpha \le 1$, $\beta > -1$, $l > 0$, $\lambda_1 > 0$, $\lambda_2 \ge 0$, while $h : [0, a_0] \to [0, +\infty[$ and $\tau : [0, a_0] \to [0, a_0]$ are measurable functions.

Lemma 2.6. *Let $\tau(t) \ge t$ for $0 \le t \le a_0$ and*

$$\int_0^{a_0} [\tau(t)]^{-\frac{(1+\beta)\lambda_2}{\alpha\lambda_1}} h(t)\, dt < +\infty. \tag{2.28}$$

Then there exists a positive number ρ_0 such that the second component of an arbitrary nonnegative solution (u_1, u_2) of the problem (2.26), (2.18) *admits the estimate*

$$u_2(0) < \rho_0.$$

Proof. Put

$$h_0(t) = \left(1 + \left[\frac{r(1+\beta)}{l}\right]^{\frac{1}{\lambda_1}} [\tau(t)]^{-\frac{1+\beta}{\alpha\lambda_1}}\right)^{\lambda_2} h(t)$$

and

$$\rho_0 = \left(1 + \left[\frac{r(1+\beta)}{l}\right]^{\frac{1}{\lambda_1}} a_0^{-\frac{1+\beta}{\alpha\lambda_1}}\right) \exp\Big(\int_0^{a_0} h_0(s)\, ds\Big).$$

Then by (2.28), $\rho_0 < +\infty$.

Let (u_1, u_2) be an arbitrary nonnegative solution of the problem (2.26), (2.18). Then

$$\begin{aligned} r \ge u_1(t) - \int_0^t u_1'(s)\, ds &\ge l \int_0^t s^\beta u_2^{\lambda_1}(s^\alpha)\, ds \ge \\ &\ge u_2^{\lambda_1}(t^\alpha) l \int_0^t s^\beta\, ds = \frac{l}{1+\beta} t^{1+\beta} u_2^{\lambda_1}(t^\alpha) \quad \text{for} \quad 0 < t \le a_0. \end{aligned}$$

Thus

$$u_2(t^\alpha) \le \left(\frac{r(1+\beta)}{l}\right)^{\frac{1}{\lambda_1}} t^{-\frac{1+\beta}{\lambda_1}} \quad \text{for} \quad 0 < t \le a_0,$$

whence by virtue of (2.27) we get

$$u_2(t) \le \left(\frac{r(1+\beta)}{l}\right)^{\frac{1}{\lambda_1}} t^{-\frac{1+\beta}{\alpha\lambda_1}} \quad \text{for} \quad 0 < t \le a_0.$$

According to the latter estimate and the inequalities (2.26), we have

$$u_2(a_0) \le \left(\frac{r(1+\beta)}{l}\right)^{\frac{1}{\lambda_1}} a_0^{-\frac{1+\beta}{\alpha\lambda_1}},$$
$$u_2(\tau(t)) \le u_2(t),$$

and

$$\begin{aligned}\bigl(1+u_2(t)\bigr)' &\ge -h(t)\bigl[1+u_2(\tau(t))\bigr]^{\lambda_2}\bigl[1+u_2(\tau(t))\bigr] \ge \\ &\ge -h_0(t)\bigl(1+u_2(t)\bigr) \quad \text{for} \quad 0 < t \le a_0.\end{aligned}$$

Thus

$$1+u_2(t) \le \bigl(1+u_2(a_0)\bigr)\exp\Bigl(\int_t^{a_0} h_0(s)\,ds\Bigr) \le \rho_0 \quad \text{for} \quad 0 \le t \le a_0. \qquad \square$$

§ 3. Proofs of the Existence and Uniqueness Theorems

Proof of Theorem 1.1. Let

$$\tau_{ikm}(t) = \tau_{ik}(t) + \frac{\varepsilon_0}{m} \quad (i,k = 1,2; \;\; m = 1,2,\dots).$$

Then by virtue of Lemma 2.4, for every natural m the system (2.9) has a nonnegative solution (u_{1m}, u_{2m}) satisfying the boundary conditions

$$\varphi\bigl(u_{1m}(0), u_{2m}(0)\bigr) = 0, \quad u_{1m}(t) = u_{1m}(a), \quad u_{2m}(t) = 0 \quad \text{for} \quad t \ge a. \tag{3.1}$$

By the condition of the theorem,

$$u_{2m}(0) \le \rho_0 \quad (m = 1,2,\dots). \tag{3.2}$$

On the other hand, taking into account (1.4) and (3.1), we find

$$u_{1m}(0) \le r \quad (m = 1,2,\dots). \tag{3.3}$$

In view of (1.3) and the fact that u_{1m} and u_{2m} are nonnegative, we can conclude that these functions are nonincreasing. Thus it becomes clear from (3.2) and (3.3) that for every natural m the estimate (2.13), where $\rho = r + \rho_0$, is valid. It is also obvious that (u_{1m}, u_{2m}) is a solution of the problem (2.9), (2.10), where $c_m = u_{1m}(a)$. Without loss of generality it can be assumed that the sequence $(c_m)_{m=1}^{+\infty}$ is convergent. Denote by c the limit of that sequence.

According to Lemma 2.3, we can choose from $(u_{im})_{m=1}^{+\infty}$ $(i = 1,2)$ uniformly converging subsequences $(u_{im_j})_{j=1}^{+\infty}$ $(i = 1,2)$, and the vector function (u_1, u_2) whose components are given by (2.14) is a solution of the problem (1.1), (1.29). On the other hand, if in the equality

$$\varphi\Big(u_{1m_j}(0), u_{2m_j}(0)\Big) = 0$$

we pass to the limit as $j \to +\infty$, then, taking into account the fact that φ is continuous, we obtain

$$\varphi\Big(u_1(0), u_2(0)\Big) = 0.$$

Therefore (u_1, u_2) is a nonnegative solution of the problem (1.1), (1.2). $\square$

Proof of Corollary 1.1. Choose a natural number m so that

$$\tau_{22}(t) > t + \frac{a}{m} \quad \text{for} \quad 0 \le t \le a.$$

Put

$$t_j = \frac{ja}{m}, \quad \tau(t) = t_{j+1} \quad \text{for} \quad t_j < t \le t_{j+1} \quad (j = 0, \dots, m-1). \tag{3.4}$$

Then

$$\tau_{22}(t) > \tau(t) \quad \text{for} \quad 0 \le t \le a. \tag{3.5}$$

Introduce the function

$$h(t, y) = \max\Big\{|f_2(t, x, z)| : \ 0 \le x \le r, \ 0 \le z \le y\Big\} \tag{3.6}$$

and the numbers

$$\rho_m = 0, \quad \rho_j = \rho_{j+1} + \int_{t_j}^{t_{j+1}} h(s, \rho_{j+1})\, ds \quad (j = m-1, \dots, 0). \tag{3.7}$$

Let $\varepsilon \in]0, 1]$ be an arbitrarily fixed number and (u_1, u_2) be a nonnegative solution of the problem (1.1_ε), (1.2). By Theorem 1.1, to prove Corollary 1.1 it is sufficient to show that u_2 admits the estimate (1.8).

By virtue of (1.3), the functions u_1 and u_2 are nonincreasing. From this fact, on account of (1.4) and (3.5) it follows that

$$u_1(t) \le u_1(0) \le r \quad \text{for} \quad 0 \le t \le a \tag{3.8}$$

and

$$u_2(\tau_{22}(t) + \varepsilon) \le u_2(\tau(t)) \quad \text{for} \quad 0 \le t \le a. \tag{3.9}$$

In view of (3.6), (3.8), and (3.9), from (1.1_ε) we get

$$u_2'(t) \ge -h(t, u_2(\tau(t))) \quad \text{for} \quad 0 \le t \le a,$$

whence with regard for (3.4) and since $u_2(t_m) = 0$, we find that

$$u_2(t) \le \rho_j \quad \text{for} \quad t_j \le t \le a \quad (j = m-1, \dots, 0).$$

Therefore the estimate (1.8) is valid. $\square$

Proof of Corollary 1.2. Without loss of generality it can be assumed below that

$$h(t) > \frac{1}{\omega(y_0)} \max\Big\{|f_2(t,x,y)| : \;\; 0 \le x \le r, \;\; 0 \le y \le y_0\Big\}. \tag{3.10}$$

If now we suppose

$$\omega(y) = \omega(y_0) \quad \text{for} \quad 0 \le y \le y_0,$$

then owing to (1.13), we will have

$$f_2(t,x,y) \ge -h(t)\omega(y) \quad \text{for} \quad 0 \le t \le a, \quad 0 \le x \le r, \quad y \ge 0. \tag{3.11}$$

On the other hand, by virtue of (1.14) there exists $\rho_0 > 0$ such that

$$\int\limits_0^{\rho_0} \frac{dy}{\omega(y)} = \int\limits_0^a h(s)\,ds. \tag{3.12}$$

Let (u_1, u_2) be a nonnegative solution of the problem (1.1_ε), (1.2) for some $\varepsilon \in]0,1]$. By Theorem 1.1, to prove Corollary 1.2 it suffices to show that u_2 admits the estimate (1.8).

By (1.3), the functions u_1 and u_2 are nonincreasing. Hence, taking into account (1.4) and (1.7), we get (3.8) and

$$u_2(\tau_{22}(t) + \varepsilon) \le u_2(t) \quad \text{for} \quad 0 \le t \le a. \tag{3.13}$$

According to (3.8), (3.11), and (3.13), for almost all $t \in [0,a]$ we have

$$-u_2'(t) \le h(t)\omega(u_2(t))$$

as long as ω is a nondecreasing function. Moreover, $u_2(a) = 0$. Thus

$$\int\limits_0^{u_2(0)} \frac{dy}{\omega(y)} = -\int\limits_0^a \frac{u_2'(s)}{\omega(u_2(s))}\,ds \le \int\limits_0^a h(s)\,ds.$$

Hence by virtue of (3.12) we obtain the estimate (1.8). $\square$

Proof of Corollary 1.3. Without loss of generality we assume that the function h satisfies the inequality (3.10) on $[0, a_2]$. If we now suppose $a_0 = a_2$,

$$\omega(y) = \omega(y_0) \quad \text{for} \quad 0 \le y \le y_0, \quad \delta(t,y) = 0 \quad \text{for} \quad 0 \le t \le a_1, \quad 0 \le y \le y_0$$

and

$$\delta(t,y) = 0 \quad \text{for} \quad a_1 < t \le a_0, \quad y \ge 0,$$

then in view of (1.15)–(1.17) we get

$$\begin{gathered}
\tau_{12}(t) \le a_0 \quad \text{for} \quad 0 \le t \le a_1, \\
f_1(t,x,y) \le -\delta(t,y) \quad \text{for} \quad 0 \le t \le a_0, \quad 0 \le x \le r, \quad y \ge 0, \\
f_2(t,x,y) \ge -\Big[h(t) + |f_1(t,x,y)|\Big]\omega(y) \quad \text{for} \quad 0 \le t \le a_0, \quad 0 \le x \le r, \quad y \ge 0.
\end{gathered} \tag{3.14}$$

On the other hand, by (1.18) and (1.14) the functions δ and ω satisfy the conditions (2.19) and (2.20), respectively.

Let ρ_0 be the positive constant appearing in Lemma 2.5. Put

$$\psi(y) = \begin{cases} 1 & \text{for} \quad |y| \le \rho_0 \\ 2 - \dfrac{|y|}{\rho_0} & \text{for} \quad \rho_0 < |y| < 2\rho_0 \\ 0 & \text{for} \quad |y| \ge 2\rho_0 \end{cases}, \tag{3.15}$$

$$\widetilde{f}_2(t,x,y) = \psi(y) f_2(t,x,y) \tag{3.16}$$

and consider the differential system

$$u_1'(t) = f_1\Big(t, u_1(\tau_{11}(t)), u_2(\tau_{12}(t))\Big), \quad u_2'(t) = \widetilde{f}_2\Big(t, u_1(\tau_{21}(t)), u_2(\tau_{22}(t))\Big). \tag{3.17}$$

In view of (3.15) and (3.16)

$$\widetilde{f}_2(t,x,y) \ge -h^*(t) \quad \text{for} \quad 0 \le t \le a, \quad 0 \le x \le r, \quad y \ge 0, \tag{3.18}$$

where

$$h^*(t) = \max\Big\{|f_2(t,x,y)| : \;\; 0 \le x \le r, \;\; 0 \le y \le 2\rho_0\Big\} \tag{3.19}$$

and h^* is summable on $[0,a]$. By virtue of Corollary 1.2 the condition (3.18) guarantees the existence of a nonnegative solution (u_1,u_2) of the problem (3.17), (1.2).

By the conditions (1.3) and (1.4), the functions u_1 and u_2 are nonincreasing and u_1 satisfies the inequalities (3.8). If along with this fact we take into account the conditions (1.7), (3.14)–(3.16), then we will see that the restriction of (u_1,u_2) on $[0,a_0]$ is a solution of the problem (2.17), (2.18). Hence if we take into consideration how ρ_0 is, then we will get the estimate (2.21). Consequently,

$$u_2(\tau_{22}(t)) \le \rho_0 \quad \text{for} \quad 0 \le t \le a. \tag{3.20}$$

According to this estimate, from (3.15)–(3.17) follows that (u_1,u_2) is a solution of the system (1.1). $\square$

Proof of Corollary 1.4. By virtue of (1.18) we can find $\rho_1 \ge y_0$ so that

$$\int_0^{a_1} \delta(s,\rho_1)\, ds > r. \tag{3.21}$$

Put

$$h(t) = \max\Big\{|f_2(t,x,y)|: \quad 0 \le x \le r, \quad 0 \le y \le \rho_1\Big\} \tag{3.22}$$

and

$$\rho_0 = \rho_1 + \int_0^{a_2} h(s)\,ds. \tag{3.23}$$

Let ψ, $\widetilde{f}_2$ and h^* be the functions defined by (3.15), (3.16) and (3.19), respectively. Then $\widetilde{f}_2$ satisfies the condition (3.18). By this condition and Corollary 1.2 the problem (3.17), (1.2) has a nonnegative solution (u_1, u_2). By virtue of (1.3) and (1.4) the functions u_1 and u_2 do not increase and u_1 admits the estimate (3.8).

Let us now show that

$$u_2(a_2) < \rho_1. \tag{3.24}$$

Assume the contrary that $u_2(a_2) \ge \rho_1$. Then in view of (1.15) and (1.16) we get

$$u_2(\tau_{12}(t)) \ge u_2(a_2) \ge \rho_1 \quad \text{for} \quad 0 \le t \le a_1$$

and

$$-u_1'(t) \ge \delta(t, \rho_1) \quad \text{for} \quad 0 \le t \le a_1.$$

Integrating the latter inequality from 0 to a_1 and taking into account (3.8), we obtain

$$r \ge \int_0^{a_1} \delta(s, \rho_1)\,ds.$$

But this contradicts (3.21). The contradiction obtained proves that the estimate (3.24) is valid. Hence by virtue of (1.21) we get

$$u_2(\tau_{22}(t)) < \rho_1 \quad \text{for} \quad 0 \le t \le a.$$

According to this estimate and the conditions (3.8), (3.22), we have

$$|u_2'(t)| \le h(t) \quad \text{for} \quad 0 \le t \le a.$$

If along with the above inequality we take into account (3.23) and (3.24), then we will get

$$u_2(0) \le u_2(a_2) + \int_0^{a_2} h(s)\,ds < \rho_0.$$

Consequently, the estimate (3.20) is valid. In view of this estimate, from (3.15)–(3.17) follows that (u_1, u_2) is a solution of the system (1.1). □

Proof of Corollary 1.5. Introduce the function

$$\tau(t) = \min\Big\{a_0, \tau_{22}(t)\Big\}.$$

Then by virtue of (1.25), the condition (2.28) is fulfilled.

Let ρ_0 be the positive constant appearing in Lemma 2.6 and let ψ and $\widetilde{f}_2$ be the functions defined by (3.15) and (3.16). By the conditions (1.3), (1.4) and Corollary 1.2, the problem (3.17), (1.2) has a nonnegative solution (u_1, u_2), the functions u_1 and u_2 do not increase and u_1 admits the estimate (3.8). If along with this fact we take into account the conditions (1.7), (1.22)–(1.24), then it will become evident that the restriction of (u_1, u_2) on $[0, a_0]$ is a solution of the problem (2.26), (2.18). Hence if we take into consideration how ρ_0 is, then we obtain the estimate (3.20). According to this estimate, from (3.15)–(3.17) follows that (u_1, u_2) is a solution of the system (1.1). $\square$

Proof of Theorem 1.2. First of all note that (1.4) follows from (1.5).

Let us now suppose that (u_1, u_2) is an arbitrary nonnegative solution of the problem (1.1_ε), (1.2) for some $\varepsilon \in]0, 1]$. Then according to (1.5) we have

$$u_1(0) + u_2(0) \leq r.$$

Therefore the estimate (1.8), where $\rho_0 = r$, is valid.

From the above reasoning it is clear that all the conditions of Theorem 1.1 are fulfilled, which guarantees the existence of at least one nonnegative solution of the problem (1.1), (1.2). $\square$

Proof of Theorem 1.3. Assume the contrary that the problem (1.1), (1.2) has two different solutions (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$. Suppose

$$u_1(a) = c_0, \quad \overline{u}_1(a) = \overline{c}_0. \tag{3.25}$$

Then (u_1, u_2) is a solution of the problem (1.1), (1.29) with $c = c_0$, while $(\overline{u}_1, \overline{u}_2)$ is a solution of the same problem with $c = \overline{c}_0$. Thus $\overline{c}_0 \neq c_0$, since according to one of the conditions of the theorem the problem (1.1), (1.29) has no more than one solution for an arbitrarily fixed c.

Without loss of generality it can be assumed that

$$\overline{c}_0 > c_0.$$

Then by virtue of Lemma 2.2 we have

$$\overline{u}_1(t) \geq \overline{c}_0 - c_0 + u_1(t) > u_1(t), \quad \overline{u}_2(t) \geq u_2(t) \quad \text{for} \quad 0 \leq t \leq a. \tag{3.26}$$

Consequently,

$$\overline{u}_1(0) > u_1(0), \quad \overline{u}_2(0) \geq u_2(0)$$

and

$$\varphi\Big(\overline{u}_1(0), \overline{u}_2(0)\Big) > \varphi\Big(u_1(0), u_2(0)\Big), \tag{3.27}$$

since φ is an increasing in the first argument and nondecreasing in the second argument function. But the latter inequality contradicts the equalities

$$\varphi\Big(\overline{u}_1(0), \overline{u}_2(0)\Big) = 0, \quad \varphi\Big(u_1(0), u_2(0)\Big) = 0. \tag{3.28}$$

The contradiction obtained proves the validity of the theorem. $\square$

Proof of Theorem 1.4. Assume the contrary that the problem (1.1), (1.2) has two different solutions (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$. Then, as is shown when proving Theorem 1.3, $\overline{c}_0 \neq c_0$, where c_0 and $\overline{c}_0$ are the numbers defined by (3.25). For the sake of definiteness we assume that $\overline{c}_0 > c_0$. Then by virtue of Lemma 2.2 the inequalities (3.26) are fulfilled. Thus

$$\begin{aligned} \overline{u}_2{}'(t) = f_2\Big(t, \overline{u}_1(\tau_{21}(t)), \overline{u}_2(\tau_{22}(t))\Big) < \\ < f_2\Big(t, u_1(\tau_{21}(t)), u_2(\tau_{22}(t))\Big) = u_2'(t) \quad \text{for} \quad 0 \le t \le a, \end{aligned} \tag{3.29}$$

since f_2 is a function decreasing in the second and nonincreasing in the third argument.

From (3.26) and (3.29) we have

$$\overline{u}_1(0) > u_1(0), \quad \overline{u}_2(0) > u_2(0).$$

Hence by virtue of (1.30) we obtain the inequality (3.27), which contradicts (3.28). The contradiction obtained proves the validity of the theorem. $\square$

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