THE IMPLICIT FUNCTION THEOREM AND CONVEXITY

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Let $\mathbf{f} : \mathbf{R}^{\mathrm{n+m}} \to \mathbf{R}^{\mathrm{m}}$ be a continuously differentiable function. $\mathbf{R}^{\mathrm{n+m}}$ is the direct sum $\mathbf{R}^{\mathrm{n}} + \mathbf{R}^{\mathrm{m}}$, a point of this is $(\ \mathbf{x},\ \mathbf{y}) = (\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m})$. Starting from the given function $\mathbf{f}$, the goal is to construct a function $\mathbf{g}\mathbf{R}^{\mathrm{n}} \to \mathbf{R}^{\mathrm{m}}$ whose graph $(\mathbf{x}, g(\mathbf{x}))$ is precisely the set of all $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

Fix a point $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ with $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, where $\mathbf{0}$ is the element of $\mathbf{R}^m$. If the matrix $[(\partial f_\mathrm{i}/\partial y_j)\,(\mathbf{a},\ \mathrm{b})\ ]$ is invertible, then there exists an open set U containing $\mathbf{a}$, an open set V containing $\mathbf{b}$, and a unique function g $U \to V$ such that

$$\{\mathbf{x}, \mathbf{g}(\mathbf{x}) \mid \mathbf{x} \in \mathbf{U}\} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbf{U}, \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$$

which is continuously differentiable.

In this work, it is stated that there always exists a scalar, C^2 function $\mathrm{F} : \mathbf{R}^{\mathrm{n+m}} \to \mathbf{R}^1$ whose partial derivative with respect to the m parameters of $\mathbf{y}$ is $\mathbf{f} : \mathbf{R}^{n+m} \to \mathbf{R}^m$ and whose second partial derivative with respect to the m parameters of $\mathbf{y}$ is the Hessian according to the conditions of the theorem. Depending on the Hessian, the critical point where $\mathbf{f} = \mathbf{0}$ can be a local sub-minimum, a sub-maximum and may have several other local sub-geometries according to the Morse theorem. The implicit function collects the critical points of a local projection map.

Applications:

1. Any division $(\mathbf{x}, \mathbf{y}) = (\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m})$ where the type of the critical point is a minimum for every value of $\mathbf{x}$ may lead to a hierarchical minimisation consisting of two kinds of smaller dimensional minimisations.

2. The graph of the implicit function assigns a transformed merit function (a section of the original merit function) which is minimised in the second minimisation with respect to $\mathbf{x}$ (during of which the $\mathbf{y}$-part is eliminated in every iteration step with respect to $\mathbf{x}$ by sub-minimisation with respect to $\mathbf{y}$).

3. The solution of hierarchical minimisation is identical to the solution of the original minimisation if the type of the critical point is a global minimum for every value of $\mathbf{x}$

The globally positive definite cases:

5. In case of strictly convex $\mathrm{F} : \mathbf{R}^{\mathrm{n+m}} \to \mathbf{R}^1$, for any division $(\mathbf{x}, \mathbf{y}) = (\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m})$, the Hessian is globally positive definite for every value of $\mathbf{x}$.

6. In case of linearly dependent parameters, the second partial derivative with respect to the m parameters of $\mathbf{y}$ is globally positive definite for every value of $\mathbf{x}$.