On lattices embeddable into convexity lattices of some posets

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Definition

Let $\langle P, \trianglelefteq \rangle$ be a poset. A set $A \subseteq P$ is order-convex, if $x \trianglelefteq z \trianglelefteq y$ and $x, y \in A$ imply $z \in A$. The set $\mathrm{Co}(P)$ of all convex subsets of P forms a lattice under inclusion.

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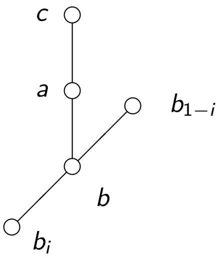
Theorem (Semenova, Wehrung, 2004)

Let L be a lattice. Then L embeds into some lattice of the form $\mathrm{Co}(P)$ iff L satisfies identities (S), (U), (B).

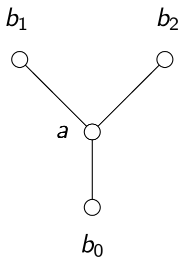
$$x \wedge (y' \vee z) = (x \wedge y') \vee \bigvee_{i<2} \left[x \wedge (y_i \vee z) \wedge \left((y' \wedge (x \vee y_i)) \vee z \right) \right],$$

where $y' = y \wedge (y_0 \vee y_1)$.



Illustrating (S)

$$x \wedge (x_0 \vee x_1) \wedge (x_1 \vee x_2) \wedge (x_0 \vee x_2) =$$
$$\left[x \wedge x_0 \wedge (x_1 \vee x_2)\right] \vee \left[x \wedge x_1 \wedge (x_0 \vee x_2)\right] \vee$$
$$\left[x \wedge x_2 \wedge (x_0 \vee x_1)\right].$$

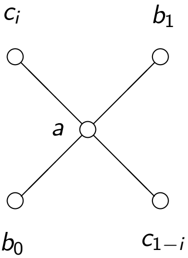


Illustrating (U)

$$x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) =$$

$$\bigvee_{i<2} [x \wedge y_i \wedge (z_0 \vee z_1)] \vee [x \wedge (y_0 \vee y_1) \wedge z_i] \vee$$

$$\bigvee_{i<2} [x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) \wedge (y_0 \vee z_i) \wedge (y_1 \vee z_{1-i})].$$



Illustrating (B)

One of the way of embedding L into $Co(P)$ is to construct tree-like poset P via sequences of join-irreducible elements of L.

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Definition

A poset $\langle P, \trianglelefteq \rangle$ with predecessor relation $\prec$ is tree-like, if it has no infinite bounded chain and between any points a and b of P there exists at most one finite sequence $\langle x_i | i = 0, \ldots, n \rangle$ with distinct entries such that $x_0 = a$, $x_n = b$, and either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$, for all $i = 0, \ldots, n$.

For a class $\mathcal{P}$ of posets, $\mathrm{Co}(\mathcal{P})$ denotes the class of respective convexity lattices and $\mathbf{S}\,\mathrm{Co}(\mathcal{P})$ denotes the class of lattices which embed into those from $\mathrm{Co}(\mathcal{P})$.

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Theorem (Semenova, Wehrung, 2004)

Let $\mathcal{C}$ be the class of posets which are disjoint unions of chains. The class $\mathrm{S}\,\mathrm{Co}(\mathcal{C})$ is a locally finite finitely based variety.

Definition

A poset $\langle P, \trianglelefteq\rangle$ is a forest, if the lower set $\downarrow a = \{x \in P \mid x \trianglelefteq a\}$ is a chain, for any $a \in P$. A connected forest is a tree.

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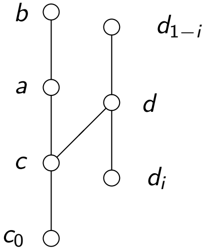
$\mathcal{F}$ denotes the class of forests, while $\mathcal{T}$ denotes the class of trees.

Theorem (Semenova, Zamojska, 2006)

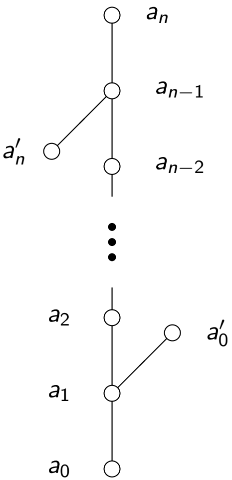
The following are equivalent for a lattice L:

1. $L \in \mathrm{S}\,\mathrm{Co}(\mathcal{F})$;
2. $L \in \mathrm{S}\,\mathrm{Co}(\mathcal{T})$;
3. L *satisfies* (S), (U), (B), (T), (T_2), (T_3), (T_4), (Z).

In particular, $\mathrm{S}\,\mathrm{Co}(\mathcal{F})$ *is a finitely based variety.*



Illustrating (T)



Illustrating (T_n) and (Z)

Corollary

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The class $\mathbf{S}\,\mathrm{Co}(\mathcal{F}_n)$ *is a finitely based variety for any* $n < \omega$.

Problem

Is the variety $\mathbf{S}\,\mathrm{Co}(\mathcal{F})$ *locally finite?*

A generalization of the class of forests is the class of series-parallel posets, i.e. posets that do not contain subposet isomorphic to the letter N (N-free posets). We denote this class by $\neg\mathcal{N}$. Obviously $\mathcal{T} \subset \mathcal{F} \subset \neg\mathcal{N}$.

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Is the class $\mathrm{S}\,\mathrm{Co}(\neg\mathcal{N})$ *a variety?*

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