Minimal clones with few majority operations

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Concrete clones

Definition

A family $\mathcal{C}$ of finitary operations on a set A is a concrete clone, if it is closed under composition of functions and contains all projections.

Example

If $\mathbb{A} = (A; F)$ is an algebra, then the set of its term functions is a clone on A.
This clone is generated by F: $\operatorname{Clo} \mathbb{A} = [F]$.

Minimal clones

All clones on a given set A form a lattice with respect to inclusion.

- least element: $\mathcal{I}_A$, the trivial clone (only projections)
- atoms: minimal clones
- greatest element: $\mathcal{O}_A$, the clone of all operations on A
- coatoms: maximal clones

Definition

A clone is a minimal clone, if its only proper subclone is the trivial one.

Minimality Criterion

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$$\mathcal{I} \neq \mathcal{C} \text{ is minimal } \iff \forall g \in \mathcal{C} \setminus \mathcal{I} : \; \mathcal{C} = [g] \, .$$

Therefore all minimal clones are singly generated.
We usually determine a minimal clone by a generating function of minimal arity.

Definition

An n-ary function f is a minimal function if

- $[f]$ is a minimal clone;
- every nontrivial operation in $[f]$ has arity at least n.

The five types

Theorem (Rosenberg)

Let f be a minimal function on A. Then f satisfies one of the following conditions:

- *f is unary, and $f^2(x) = f(x)$ or $f^p(x) = x$ for some prime p;*
- *f is a binary idempotent operation, i.e. $f(x, x) = x$;*
- *f is a ternary majority operation, i.e. $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*
- *$f(x, y, z) = x + y + z$ for a Boolean group $(A; +)$;*
- *f is a semiprojection, i.e. there exists an i $(1 \leq i \leq n)$ such that $f(x_1, x_2, \ldots, x_n) = x_i$ whenever the arguments are not pairwise distinct.*

The three-element case

Theorem (Csákány)

If f is a minimal majority function on a three-element set A, then $(A; f) \cong (\{1,2,3\}; m)$ for some function m from the the twelve majority functions listed below.
These functions belong to three minimal clones containing $1, 3$ and 8 majority operations respectively, as shown in the table.

	m_1	m_2			m_3							
$(1,2,3)$	1	1	2	3	2	3	2	2	3	2	3	3
$(2,3,1)$	1	2	3	1	2	2	2	3	3	3	3	2
$(3,1,2)$	1	3	1	2	2	2	3	2	3	3	2	3
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$(3,2,1)$	1	3	2	1	3	3	3	2	2	2	2	3

Abstract clones

Definition

An abstract clone $\mathcal{C}$ is given by a family $\mathcal{C}^{(n)}$ $(n \geq 1)$ of sets with distinguished elements $e_i^{(n)} \in \mathcal{C}^{(n)}$ $(1 \leq i \leq n)$ and mappings

$$F_k^n : \mathcal{C}^{(n)} \times \left(\mathcal{C}^{(k)}\right)^n \to \mathcal{C}^{(k)},\ (f, g_1, \ldots, g_n) \mapsto f\,(g_1, \ldots, g_n)$$

such that the following axioms are satisfied.

- $e_i^{(n)}\,(f_1, \ldots, f_n) = f_i$
- $f\left(e_1^{(n)}, \ldots, e_n^{(n)}\right) = f$
- $f\,(g_1, \ldots, g_n)\,(h_1, \ldots, h_k) = f\,(g_1\,(h_1, \ldots, h_k)\,, \ldots, g_n\,(h_1, \ldots, h_k))$

Majority operations

Fact

Majority operations are nice. (Congruence distributivity, Baker-Pixley theorem, . . .)

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Fact

Let $\mathcal{C}$ be a clone generated by a majority operation. If every nontrivial ternary operation in $\mathcal{C}$ generates $\mathcal{C}$, then $\mathcal{C}$ is a minimal clone.

Thus it suffices to verify

$$\forall g \in \mathcal{C}^{(3)} \setminus \mathcal{I} : \ \mathcal{C} = [g] \, .$$

The ternary part of the clone

To study minimal majority clones as abstract clones one can consider the algebra $\left(\mathcal{C}^{(3)};\ F_3^3,\ e_1^{(3)},\ e_2^{(3)},\ e_3^{(3)}\right)$.
This is an algebra with one quaternary and three nullary operations satisfying the following identities.

- $F\left(e_i, f_1, f_2, f_3\right) = f_i \quad (i = 1, 2, 3)$
- $F\left(f, e_1, e_2, e_3\right) = f$
- $F\left(F\left(f, g_1, g_2, g_3\right), h_1, h_2, h_3\right) =$
  $F\left(f, F\left(g_1, h_1, h_2, h_3\right), F\left(g_2, h_1, h_2, h_3\right), F\left(g_3, h_1, h_2, h_3\right)\right)$

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  $F(f, F(g_1, h_1, h_2, h_3), F(g_2, h_1, h_2, h_3), F(g_3, h_1, h_2, h_3))$

Fact

A majority clone $\mathcal{C}$ is minimal iff $\mathcal{C}^{(3)}$ has no proper nontrivial subalgebras.

Median functions

Example

The median function $f(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ on any lattice is a minimal majority function.

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Proof.

Clearly f is a totally symmetric majority operation. One can check that the following identity also holds:

$$f(x,y,z) = f(f(x,y,z),y,z).$$

This ensures that the ternary part of $[f]$ is $\{e_1, e_2, e_3, f\}$. □

Dual discriminator functions

Example

The dual discriminator function d_A on any set A is a minimal majority function.

$$d_A(a,b,c) = \begin{cases} a & \text{if } a = b \\ c & \text{if } a \neq b \end{cases}.$$

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Proof.

One can check that the ternary part of $[d_A]$ is $\{\, e_1,\ e_2,\ e_3,\ d_A\,(x, y, z),\ d_A\,(y, z, x),\ d_A\,(z, x, y)\,\}$. □

Minimal clones with few majority operations

Theorem

If $\mathcal{C}$ is a minimal clone with one majority operation, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$.

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There is no minimal clone with exactly two majority operations.

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If $\mathcal{C}$ is a minimal clone with three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_2]^{(3)}$.

Theorem

There is no minimal clone with exactly four majority operations.