ON COMPLETING PARTIAL GROUPOIDS TO SEMIGROUPS

Alexei Vernitski
Department of Mathematical Sciences
University of Essex

- A **groupoid** is a set considered with a binary operation
- A **partial groupoid** is a set considered with a partial binary operation

Example

The multiplication table
of a groupoid

	0	1
0	0	0
1	0	1

The multiplication table
of a partial groupoid

	0	1
0	0	
1		1

Example (continued)

- The operation of the partial groupoid is a subset of the operation of the groupoid

The multiplication table of a groupoid

	0	1
0	0	0
1	0	1

deleting →

← completing

The multiplication table of a partial groupoid

	0	1
0	0	
1		1

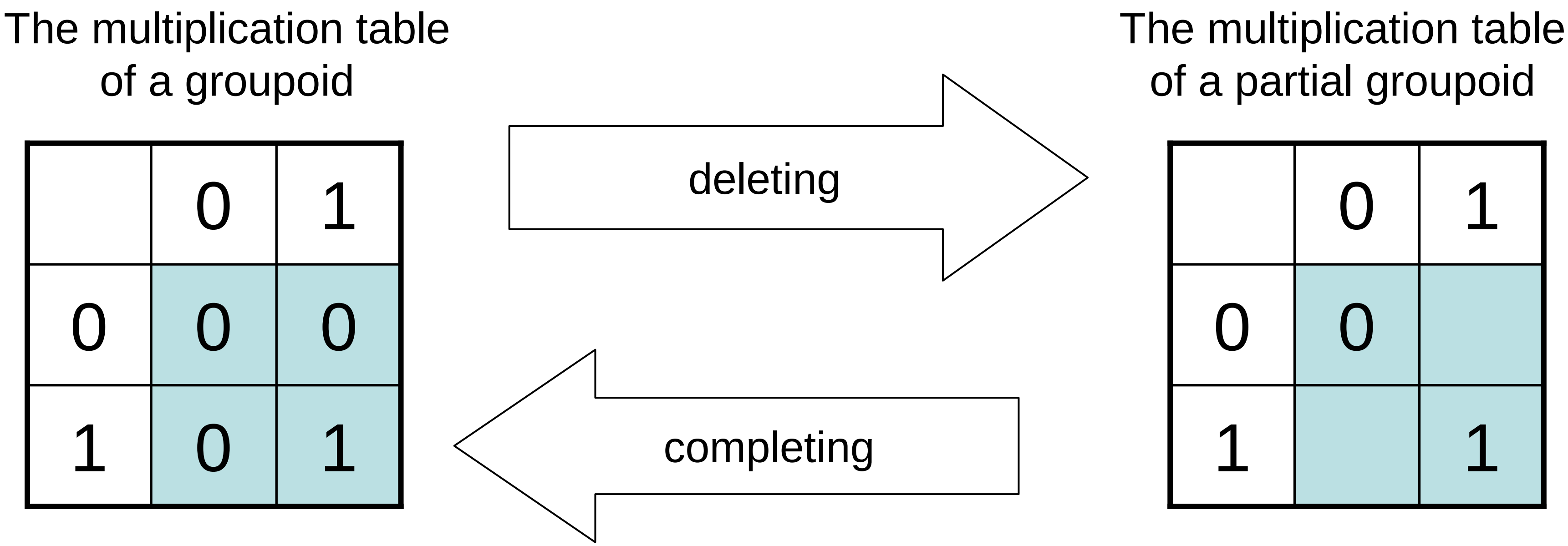
- A semigroup is a groupoid in which the operation is associative, that is,
  $(xy)z = x(yz)$
- A semilattice is a semigroup in which the operation is commutative and idempotent, that is,
  $xy=yx$ and $xx=x$

- Let V be a class of finite semigroups
- **Problem**: for a given partial groupoid, determine whether it can be completed to a semigroup belonging to the class V
- What is the algorithmic complexity of this problem? (Is it polynomial? NP-hard? Unsolvable?)

- An important example: Let V be a class of all finite semilattices
- Problem: for a given partial groupoid, determine whether it can be completed to a semilattice
- What is the algorithmic complexity of this problem? (Is it polynomial? NP-hard? Unsolvable?)

- We have not defined **completing** formally!
- Completing can be understood in at least three different ways

- Instead of explaining three ways of completing, it is easier to explain the opposite direction: three ways of deleting elements in the multiplication table



Funayama, 1953

- You are allowed to delete **all** occurrences of some elements (rows, columns and inside the multiplication table)

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

deleting 2

	0	1
0	0	1
1	1	

Funayama, 1953

- Funayama studied *partial semilattices*, that is, partial groupoids which can be obtained from semilattices in the described way
- Funayama did not consider questions of algorithmic complexity related to partial semilattices
- We shall not consider Funayama’s construction in this talk

Goralčík and Koubek, 2006

- You are allowed to delete some cells inside the multiplication table, but you are not allowed to delete rows and columns

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

deleting some cells →

	0	1	2
0		1	2
1	1	2	
2			

Goralčík and Koubek, 2006

- Let V be a class of finite semigroups containing all finite semilattices. Then the problem of completing a partial groupoid to a semigroup in V is NP-hard.
- The problem of completing a partial groupoid to a semilattice is NP-complete.

Vernitski, new result

- You are allowed to delete some cells inside the multiplication table and all occurrences of some elements (rows, columns and inside the multiplication table)

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

deleting 2 and some cells

	0	1
0		
1	1	

Vernitski, new result

- The problem of completing a partial groupoid to a semilattice is polynomial.

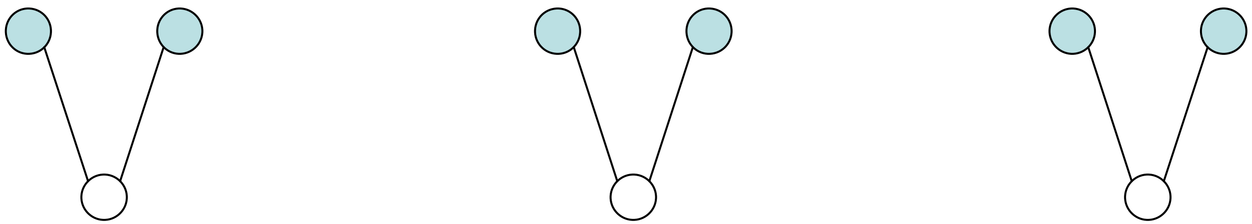
Comparison

- Goralčík and Koubek:
  If you are allowed to fill in empty cells, but not to add extra elements, the problem of completing a partial groupoid to a semilattice is NP-complete.
- Vernitski:
  If you are allowed to fill in empty cells and to add extra elements, the problem of completing a partial groupoid to a semilattice is polynomial.

- The partial operation establishes a partial order on the elements:
  $x \leq y$ if $x = xy = yx$

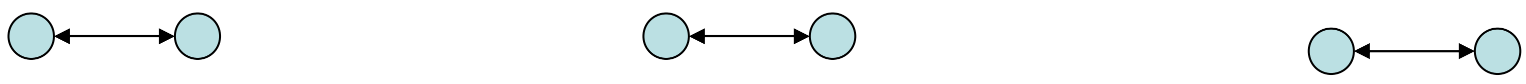
(here the order is shown as an antichain)

- If we are allowed to add new elements, incomparable elements may remain incomparable
- We simply imagine that some new elements will act as meets



(not all meets shown)

- But if we are not allowed to add new elements, then each pair of elements without a meet must decide which of them is less and which is greater
- Because there are many such pairs, the problem becomes NP-complete



Comparison

- Goralčík and Koubek:
  If you are allowed to fill in empty cells, but not to add extra elements,
  the problem … is NP-complete.
- Vernitski:
  If you are allowed to fill in empty cells and to add extra elements,
  the problem … is polynomial.

- Vernitski’s problem is always ‘easier’ than Goralčík-Koubek’s problem

Vernitski: further research 1

- I know a class of semigroups for which both the problem of completing *without new elements* and the problem of completing *with new elements* are NP-complete
- This is the class of semigroups embeddable into semigroups of order-preserving mappings on finite chains

Vernitski: further research 2

- The class of finite semilattices consists of inverse semigroups
- I think that also for other important classes of finite inverse semigroups the problem of completing *with new elements* is polynomial
- For example: all finite groups, all finite Abelian groups
- We do not know anything about the problem of completing *without new elements* for these classes