Partial orders on algebras and related structures

Reinhard Pöschel,
Institute of Algebra, TU Dresden, Germany

Sándor Radeleczki,
Institute of Mathematics, University of Miskolc, Hungary

(A, F, r) - is a **partialy ordered algebra** if $r \subsetneqq A \times A$ is a compatible partial order on (A, F)

Related structures: $\mathrm{Con}(A, F)$, $\mathrm{Quord}(A, F)$, $\mathrm{Tol}(A, F)$, $\mathrm{Refl}(A, F)$

OUR SOURCE:

Gábor Czédli and László Szabó proved:

If (A, F, r) is a lattice ordered majority algebra, then $\mathrm{Quord}(A, F) \cong \mathrm{Con}^2(A, F)$.

Involution Lattices

A quadruplet $(L, \wedge, \vee, \iota)$ is called an *involution lattice*, if ι is an atomorphism of the the lattice $(L, \wedge, \vee)$ such that $\iota^2(x) = x$, $\forall x \in L$.

If $(L, \wedge, \vee, \iota)$ is an involution lattice, then $(I, \wedge, \vee, \iota)$, where

$$I = \{x \in L \mid \iota(x) = x\}$$

is a subalgebra of it - called *its invariant part*. Of course, $\iota_{|I} = id_{|I}$. Moreover, if $(L, \wedge, \vee)$ is a complete lattice, then $(I, \wedge, \vee)$ is a complete sublattice of it.

EXAMPLES:

- $(\mathrm{Quord}(A, F), \cap, \vee, \iota)$ is an involution lattice with $q \overset{\iota}{\longmapsto} q^{-1}$ and its invariant part is $(\mathrm{Con}(A, F), \cap, \vee, id)$,
- $(\mathrm{Refl}(A, F), \cap, \sqcup, \iota)$ is an involution lattice with $q \overset{\iota}{\longmapsto} q^{-1}$ and its invariant part is $(\mathrm{Tol}(A, F), \cap, \sqcup, id)$.
- For any lattice L its direct square L^2 becomes an involution lattice, by defining: $\tau(x, y) = (y, x)$, $\forall (x, y) \in L^2$.

CENTRAL ELEMENTS IN A LATTICE

An element $c \in L$ of a bounded lattice L is called a *central elment* of L, if

1) c is complemented (i.e. $\exists c' \in L$ such that $c \wedge c' = 0$ and $c \vee c' = 1$).
2) c is neutral, i.e. for any $x, y \in L$ the sublattice $\langle c, x, y \rangle$ is distributive.

The central elements of L form a Boolean sublattice of it denoted by $\mathrm{Cen}(L)$.
If $c \in \mathrm{Cen}(L)$, then $c' \in \mathrm{Cen}(L)$.

$$c, c' \in \mathrm{Cen}(L) \overset{F.M \;\&\; S.M}{\Longleftrightarrow} \begin{cases} x = (x \wedge c) \vee (x \wedge c') \\ x = (x \vee c) \wedge (x \vee c') \end{cases}, \forall x \in L.$$

(Notins: central pair, semicentral pair, dually semicentral pair)

THEOREM (R.P. & S.R.):
(i) $(\mathrm{Quord}(A,F),\cap,\vee,\iota) \cong (\mathrm{Con}^2(A,F),\cap,\vee,\tau) \iff (A,F)$ has a compatible partial order $r \subsetneqq A \times A$ such that $r \in \mathrm{Cen}(\mathrm{Quord}(A,F))$.
(ii) $(\mathrm{Refl}(A,F),\cap,\sqcup,\iota) \cong (\mathrm{Tol}^2(A,F),\cap,\sqcup,\tau) \iff \exists r \in \mathrm{Cen}(\mathrm{Refl}(A,F))$, such that r is antisymmetric and $r \sqcup r^{-1} = \nabla$.

MAJORITY ALGEBRAS

(A,F) is a majority algebra if $\exists m \in T^3(A,F)$ such that

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x, \ \forall x,y \in A$$

$\mathrm{Refl}(A,F)$, $\mathrm{Tol}(A,F)$ 0-modular and pseudocomplemented lattices
$\mathrm{Quord}(A,F)$, $\mathrm{Con}(A,F)$ distributive lattices.

$\mathrm{Tol}(A,F)$ has same additional properties, too:

- *Strong 0-distributivity*: For any $\alpha,\beta,\gamma \in \mathrm{Tol}(A,F)$,
$$\alpha \cap \beta = \triangle \implies \gamma \cap (\alpha \sqcup \beta) = (\gamma \cap \alpha) \sqcup (\gamma \cap \beta)$$
- Any $\alpha,\beta,\gamma \in \mathrm{Tol}(A,F)$, with $\alpha \cap \beta = \alpha \cap \gamma = \beta \cap \gamma = \triangle$, is a *Von Neumann triple*, i.e.: $\langle \alpha,\beta,\gamma \rangle$ is a distributive lattice.

New results

THEOREM. *Let* (A, F) *be a majority algebra. Then*
(i) $(\mathrm{Quord}(A, F), \cap, \vee, \iota) \cong (\mathrm{Con}^2(A, F), \cap, \vee, \tau) \iff (A, F)$ *has a connected compatible partial order* $r \subsetneqq A \times A$ *(the transitive closure of* $r \cup r^{-1}$ *is* ∇*).*
(ii) $(\mathrm{Refl}(A, F), \cap, \sqcup, \iota) \cong (\mathrm{Tol}^2(A, F), \cap, \sqcup, \tau) \iff (A, F)$ *has a compatible lattice order.*

Corollary 1. (i) If (A, F, r) is a partially ordered majority algebra and r is a connected order, then any homomorphism of (A, F) preserves r.
(ii) If (A, F, r) is a lattice-ordered majority algebra, then $\mathrm{Refl}(A, F)$ is strongly 0-distributive, and any triple $\alpha, \beta, \gamma \in \mathrm{Refl}(A, F)$, with $\alpha \cap \beta = \alpha \cap \gamma = \beta \cap \gamma = \triangle$ is a Von Neumann triple.

An important **proposition**

Proposition 1. *Let* (A, F) *be a majority algebra. Then the following are equivalent:*
(i) ρ *and* $\rho^{-1} \in$Refl(A, F) *are complements each of other.*
(ii) $\rho, \rho^{-1} \in$Cen(Refl(A, F)) *and* $\rho' = \rho^{-1}$.
(iii) ρ *is a compatible lattice order.*

Lattice-orderd majority algebras

Proposition 2. *Let* (A, F, r) *be a lattice-ordered majority algebra. Then there exists a bijection between the pairs* ρ, ρ^{-1} *of compatible lattice orders of* (A, F) *with* $\rho, \rho^{-1} \notin \{r, r^{-1}\}$ *and the factor congruence-pairs of* (A, F).

$$(\rho, \rho^{-1}) \text{ compatible lattice orders} \longmapsto (\theta, \theta') \text{ factor congruece-pair}$$
$$(r, r^{-1}) \longmapsto (\triangle, \nabla)$$

Corollary 2. *Let $\mathcal{A} = (A, F, r)$ be a lattice-ordered majority algebra. Then*

(i) $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i \Longrightarrow \mathrm{Refl}(\mathcal{A}) \cong \prod_{i=1}^{n} \mathrm{Refl}(\mathcal{A}_i)$,

(ii) $\mathrm{Refl}(\mathcal{A}) = \prod_{i=1}^{n} L_i \Longrightarrow \mathcal{A} \cong \prod_{i=1}^{n} \mathcal{A}_i$, with $\mathrm{Refl}(\mathcal{A}_i) \cong L_i$, $1 \leq i \leq n$.

(iii) (A, F) is directly irreducible $\Longleftrightarrow$ (A, F) has no compatible lattice-order different from r and r^{-1}.

Corollary 3. *Any linearly ordered majority algebra is directly irreducible.*

Proposition 3. *Let (A, F, r) be a lattice-ordered majority algebra. Then the following assertions are equivalent:*

(i) $\mathrm{Tol}(A, F) = \mathrm{Con}(A, F)$,

(ii) $\mathrm{Refl}(A, F) = \mathrm{Quord}(A, F)$.

References

[1] I. Chajda and S. Radeleczki, '0-conditions and tolerance schemes', *Acta Math. Univ. Comenianae*, **72**, (2003), No.2. 177-184.

[2] I. Chajda and S. Radeleczki, Congruence schemes and their applications, *Comment. Math. Univ. Carolinae* **46**, I (2005) 1-14.

[3] G. Czédli, Quasiorders of lattices versus pairs of congruences, *Acta Sci. Math. Szeged* (1993) 1-4.

[4] G. Czédli and A. Lenkehegyi, 'On classes of ordered algebras and quasi-order distributivity', *Acta Sci. Math.* **41** (1983), 17-26.

[5] G. Czédli and A. Lenkehegyi, 'On classes of ordered algebras and quasi-order distributivity', *Acta Sci. Math.* **46** (1983), 41-54.

[6] G. Czédli, E. K. Horváth and S. Radeleczki S, 'On tolerance lattices of algebras in congruence modular varieties', *Acta Math. Hungar.*, **100 (1-2)** (2003), 9-17.

[7] A. G. Pinus and I. Chajda, 'Quasiorders on universal algebras', *Algebra i Logika,* Vol. **32** No. **3** (1993), 308-325 (In Russian).

[8] R. Pöschel, Galois connections for operations and relations. In: K. Denecke, M. Erné and S. L. Wishmath (Eds.) Galois connections and applications, vol. 565 of *Mathematics and its Applications,* Kluwer Acad. Publishers, Dordrecht 2004, pp 231-258.

[9] Á. Szendrei, Clones in universal algebra, vol 90 of *Séminaire de Mathématiques Supérieures.* Les Presses de l'Univ. de Montréal, Montréal, 1986.

Thank You for your kind attention !