The free spectra of semigroups (Part I)

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Aim

To find similar structure theorems for other classes of algebras.

Free spectrum

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- the number of different n-ary terms

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Examples

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Examples

- Vector space over Z_p:
  terms: $\sum \lambda_i x_i$,
  $|\mathbf{F}_{\mathcal{V}}(n)| = p^n$

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Examples

- Vector space over Z_p:
  terms: $\sum \lambda_i x_i$,
  $|\mathbf{F}_{\mathcal{V}}(n)| = p^n$
- Boolean algebra: $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$

Trivial estimate

A algebra, if $|\mathbf{A}| = k > 1$, then

$$n \leq |\mathbf{F}_{\mathcal{V}(\mathcal{A})}(n)| \leq k^{k^n}$$

Trivial estimate

$\mathbf{A}$ algebra, if $|\mathbf{A}| = k > 1$, then

$$n \leq |\mathbf{F}_{\mathcal{V}(\mathcal{A})}(n)| \leq k^{k^n}$$

n: the number of projections
k^{k^n}: the number of functions

Known spectra

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polynomial

exponential

double exponential

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nothing in between (gap theorem)
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exponential	Then: G is nilpotent
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The free spectra of semigroups

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Simple semigroups

Will be discussed by Kamilla Kátai!

p_n sequence

Let $t = t(x_1, \ldots, x_n)$ be an n-ary term. A term operation $t^{\mathbf{A}}$ is said to be essentially n-ary, if it depends on all of its variables, i.e. if for all $1 \leq i \leq n$ there exist $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n \in A$ such that

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \neq t(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

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$p_n(\mathbf{A})$: the number of essentially n-ary terms over $\mathbf{A}$

$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$$

Band varieties

Band: idempotent semigroup ($x^2 = x$)

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Examples

- semilattice ($xy = yx$),
- left-zero semigroup ($xy = x$),
- adding a formal identity element to a band we get another band

The lattice

Birjukov, Fennemore, Gerhard (1970-71)

The description of the lattice of the band varieties:

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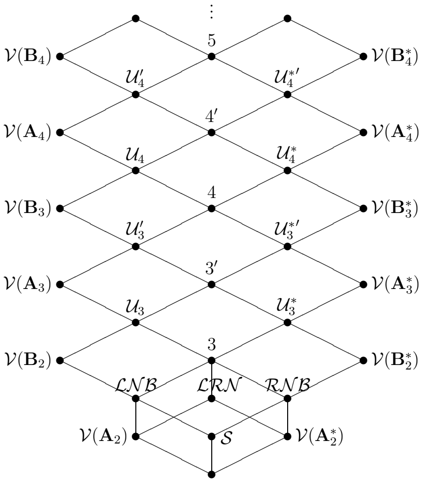
The description of the lattice of the band varieties:

- with relations,
- with identities,
- with generating semigroups

The lattice


⋮
5
$\mathcal{V}(\mathbf{B}_4)$
$\mathcal{V}(\mathbf{B}_4^*)$
$\mathcal{U}_4'$
$\mathcal{U}_4^{*\prime}$
$4'$
$\mathcal{V}(\mathbf{A}_4)$
$\mathcal{V}(\mathbf{A}_4^*)$
$\mathcal{U}_4$
$\mathcal{U}_4^*$
4
$\mathcal{V}(\mathbf{B}_3)$
$\mathcal{V}(\mathbf{B}_3^*)$
$\mathcal{U}_3'$
$\mathcal{U}_3^{*\prime}$
$3'$
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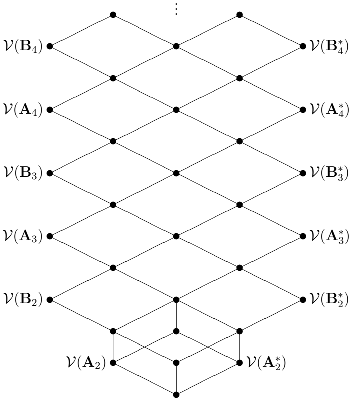
$\mathcal{V}(\mathbf{A}_3^*)$
$\mathcal{U}_3$
$\mathcal{U}_3^*$
3
$\mathcal{V}(\mathbf{B}_2)$
$\mathcal{V}(\mathbf{B}_2^*)$
$\mathcal{LNB}$
$\mathcal{LRN}$
$\mathcal{RNB}$
$\mathcal{V}(\mathbf{A}_2)$
$\mathcal{S}$
$\mathcal{V}(\mathbf{A}_2^*)$

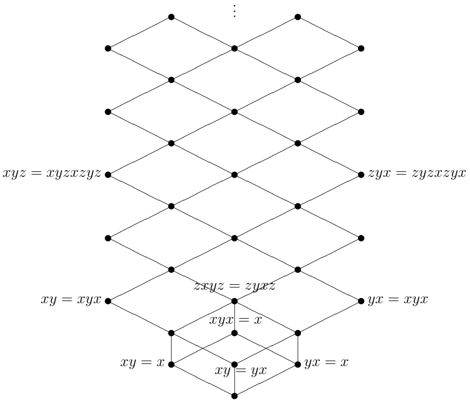
With generating algebras

$\mathbf{A_i}$ and $\mathbf{B_i}$ are the generating algebras.
$\mathbf{B_i} = \mathbf{A_i} \cup \{\mathbf{1}\}$ and $|\mathbf{A_i}| = \frac{\mathbf{n^2+n-2}}{\mathbf{2}}$

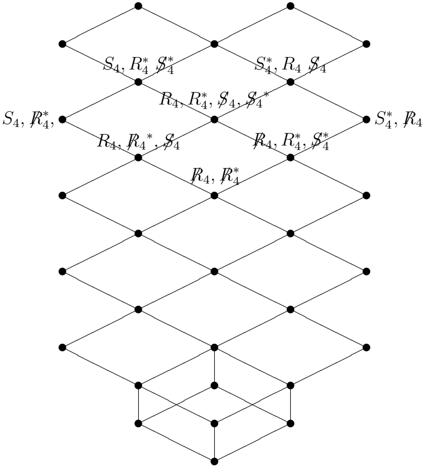


With equations

Each variety can be defined with a single equation additionally to $x(yz) = (xy)z$ and $x^2 = x$.



With relations



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- left-normal bands: $xyz = xzy$; $|\mathbf{F}_{\mathcal{V}}(n)| = n2^{n-1}$,
  first variable and set of variables

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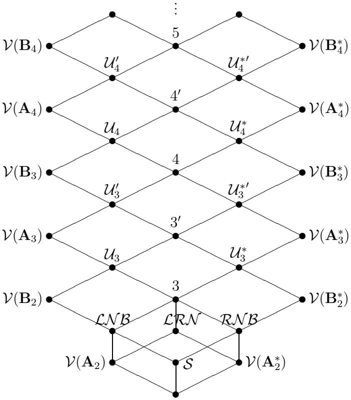
- rectengular bands: $xyz = xz$; $\quad |\mathbf{F}_{\mathcal{V}}(n)| = n^2$,
  first and last variables

The lattice

$$p_n(\mathcal{V}) = \sqrt{p_n(\overline{\mathcal{V}})p_n(\underline{\mathcal{V}})}$$



Gerhard

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- $p_n(k) = n^2 p^2_{k-2}(\infty) \prod_{j=k-1}^{n-1} j^2 p_j(k-1),\ n \geq k$ and $k \geq 4$,

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Our recurrence formula

$p_n(k) = n^2 p_{n-1}(k) p_{n-1}(k-1)$, $n \geq 1$ and $k \geq 4$

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The logarithm of $p_n(k)$

$\log p_n(k) = \log p_{n-1}(k) + \log p_{n-1}(k-1) + 2 \log n, \quad n \geq 1$ and $k \geq 4$

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The initial values

$\log p_n(3) = 2\log n,$
$\log p_1(k) = 0$

Closed form

Explicit form for $p_n(k)$

$$\log p_n(k) = 4 \sum_{m=1}^{n} \sum_{t=0}^{k-4} \binom{n-m-1}{t} \log m + 2 \log n$$

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$$\log p_n(k) = \frac{4}{(k-3)!} n^{k-3} \log n - \frac{4}{(k-3)!} n^{k-3} \sum_{j=1}^{k-3} \frac{1}{j} + O(n^{k-4} \log n)$$

Free spectra of band varieties

Free spectrum (J. Wood, G. Pluhár)

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$\log p_n(\infty)$

$\log p_n(\infty) = 2\log n + 2^2 \log(n-1) + \ldots + 2^{n-1}\log 2 =$

$$= 2^{n+1} \sum_1^n \frac{\log k}{2^k}$$

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Free spectrum (Cs. Szabó, G. Pluhár)

$$|\mathbf{F}_{\mathcal{V}}(n)| \approx p_n(\infty) \sim \frac{1}{n^2}(1.661687)^{2^{n+1}}$$

The spectra

polinomial (n^k)

nothing in between (gap theorem)

exponential (2^{n^k}) Then: G is nilpotent

NEW!!! $2^{n^k \log n}$

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Problems

- how this new kind of spectrum fits into the picture,
- the characterization of the free spectra of semigroup varieties