Modes not embeddable into semimodules

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Example

- commutative cancellative semigroups into commutative groups

Embeddability of algebras

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- integral domains into fields

Embeddability of algebras

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- commutative cancellative semigroups into commutative groups
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Theorem (A.I.Mal'cev)

The class of semigroups embeddable into groups is a quasivariety that cannot be defined by finitely may quasi-identities.

Definition

A groupoid $(G, \cdot)$ is entropic if it satisfies the following entropic law:

$$(x \cdot y) \cdot (z \cdot w) \quad \approx \quad (x \cdot z) \cdot (y \cdot w).$$

Embeddability of entropic groupoids

Definition

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Theorem (M.Sholander)

Each cancellative entropic groupoid embeds into entropic quasigroup.

Modules and semimodules

Definition

A (commutative) semiring $(S, +, \circ)$:

- $(S, +)$ - a commutative semigroup
- $(S, \circ)$ - a (commutative) semigroup
- $(x + y) \circ z \approx (x \circ z) + (y \circ z)$,
- $z \circ (x + y) \approx (z \circ x) + (z \circ y)$.

A semimodule over a semiring $(S, +, \circ)$ - a commutative semigroup $(M, +)$ together with a semiring homomorphism:

$$\begin{aligned} h : (S, +, \circ) &\to (End(M, +), +, \circ), \\ s \mapsto h_s : M \to M; &\quad m \mapsto h_s(m) := sm. \end{aligned}$$

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Example

Affine spaces - the full idempotent reducts of modules over commutative rings.

Reducts and subreducts

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Example

Idempotent subreducts of semimodules over commutative semirings are modes.

Embeddability modes into modules

Definition

A mode (A, Ω) is cancellative if it satisfies the quasi-identity:

$$f(a_1, \ldots, x_i, \ldots, a_n) = f(a_1, \ldots, y_i, \ldots, a_n) \longrightarrow x_i = y_i.$$

for each n-ary operation $f \in \Omega$ and each $i = 1, \cdots, n$.

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Theorem (A.Romanowska and J.D.H.Smith)

Each cancellative mode embeds as a subreduct into an affine space.

Fact

Each reduct of an affine space is abelian.

Definition

An abelian algebra (A, Ω):

$$t(a, x_1, \dots, x_n) = t(a, y_1, \dots, y_n) \rightarrow$$

$$t(b, x_1, \dots, x_n) = t(b, y_1, \dots, y_n).$$

for each Ω-term t.

Not all modes are subreducts of modules

Example

$(\mathbb{Z}_4, \cdot)$ - the reduct of the group $(\mathbb{Z}_4, +_4, -, 0)$, with $x \cdot y := 2y - x$:

$\cdot$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

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The homomorphic image $h(\mathbb{Z}_4, \cdot)$ is not a reduct of any affine space - it is not abelian:

$$0 \cdot 0 = 0 \cdot 1 \quad \textit{but} \quad 1 \cdot 0 \neq 1 \cdot 1.$$

Groupoid modes

Theorem (J.Ježek and T.Kepka)

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Corollary

Each groupoid mode embeds into a semimodule over a commutative semiring.

Semilattice modes

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A mode is a semilattice mode if it has a binary term operation such that is a semilattice operation.

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Theorem (K.Kearnes)

Each semilattice mode is a subreduct of a semimodule over a commutative semiring $(S, +, \cdot)$ with unity 1, satisfying the identities: $0 \cdot x = 0$ and $1 + x = 1$.

Embeddability modes into semimodules

Question

Is it true that each mode is a subreduct of some semimodule over a commutative semiring?

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Embeddability modes into semimodules

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Theorem (M.Stronkowski)

A mode (A, Ω) embeds into a semimodule over a commutative semiring with unity iff it is so-called Szendrei mode - mode satisfying Szendrei identities:

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) \approx$$
$$f(f(x_{\pi(11)}, \dots, x_{\pi(1n)}), \dots, f(x_{\pi(n1)}, \dots, x_{\pi(nn)})),$$

for each n-ary operation $f \in \Omega$ and every transposition $\pi : ij \mapsto ji$ of indices.

Non Szendrei modes

Example

Szendrei identities in the case of one ternary operation $f(x, y, z)$:

$$f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})) \approx$$
$$f(f(x_{11}, x_{21}, x_{13}), f(x_{12}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33}))$$

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Non Szendrei modes

Example

(M.Stronkowski) A free mode with at least one basic operation of arity at least three over a set of cardinality at least two, is not a Szendrei mode.

Ternary modes

Example

(D. Stanovský) The 3-elements algebra $(D = \{0, 1, 2\}, f)$ with one ternary operation $f : D^3 \to D;\ \ (x, y, z) \mapsto f(x, y, z)$

$$f(x,y,z) := \begin{cases} 2 - x, & \text{if } y = z = 1 \\ x & \text{otherwise.} \end{cases}$$

is a mode, but not Szenderi:
$((210)(000)(100)) = (201) = 2 \neq 0 = (200) = ((211)(000)(000))$.

$(D, f) \rightsquigarrow h(\mathbb{Z}_4, \cdot)$

The homomorphic image $h(\mathbb{Z}_4, \cdot)$ belongs to the variety $\mathcal{D}_2$ of differential binary modes defined by two additional identities:

$$\begin{aligned} (x \cdot y) \cdot z &\approx (x \cdot z) \cdot y && (\textit{left normal law}), \\ x \cdot y &\approx x \cdot (y \cdot z) && (\textit{left reduction law}). \end{aligned}$$

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The algebra (D, f) belongs to the variety $\mathcal{D}_3$ of differential ternary modes defined by two additional identities:

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\begin{aligned}
f(f(x, y_1, y_2), z_1, z_2) &\approx f(f(x, z_1, z_2), y_1, y_2),\\
f(x, y_1, y_2) &\approx f(x, f(y_1, z_1, z_2), f(y_2, z_1, z_2)).
\end{aligned}
$$

The variety $\mathcal{D}_2$

Fact

Each algebra in the variety $\mathcal{D}_2$ or $\mathcal{D}_3$ has a left-zero quotient with the corresponding left-zero congruence classes.

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Each proper non-trivial subvariety of $\mathcal{D}_2$ is defined by a unique identity of the form

$$xy^{i+j} \approx xy^i.$$

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The lattice $\mathfrak{L}(\mathcal{D}_2)$ of all subvarieties of $\mathcal{D}_2$ is isomorphic with the direct product of two lattices of natural numbers: one with the divisibility relation as an ordering relation and the other one with the usual linear ordering.

The variety $\mathcal{D}_3$

Fact

The lattice $\mathfrak{L}(\mathcal{D}_3)$ of all subvarieties of $\mathcal{D}_3$ contains sublattices isomorphic to the lattice of proper non-trivial subvarieties of the variety $\mathcal{D}_2$.

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(Each binary term operation of a ternary differential mode is a differential groupoid operation.)

The subvariety $\mathcal{SD}_3$: $f(x,x,y) \approx f(x,y,x) \approx x$

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Proposition

Let B and C be two non-empty disjoint sets. Put $A = B \cup C$. Let $f_{ij} : B \to B$, $i, j \in C$, be a collection of mappings such that $f_{ij} f_{kl} = f_{kl} f_{ij}$ for every $i, j, k, l \in C$. Define a ternary operation by

$$f(x,y,z) := \begin{cases} f_{yz}(x), & \text{if } y, z \in C \text{ and } x \in B \\ x & \text{in all other cases.} \end{cases}$$

The algebra $\mathbf{A} = (A, f)$ belongs to the variety $\mathcal{SD}_3$.
If $f_{ij} \neq id$ for at least one ij, then $\mathbf{A}$ is not a Szendrei mode.

Example

Let $k \geq 0$ and $n > 1$ be natural numbers,
$B = \{-k, \ldots, -1, 0, 1, \ldots, n-1\}$ and $C = \{a\}$. Let

$$f_{aa}(x) = \begin{cases} x +_n 1, & \textit{if } x \in \{0, \ldots, n-1\} \\ x + 1, & \textit{if } x \in \{-1, \ldots, -k\} \end{cases}$$

The algebra $\mathbf{A} \in \mathcal{SD}_3$ is a non-Szendrei mode which satisfies

$$f(x, y, z) \approx f(x, z, y)$$

$$f(f(\ldots f(f(x, \underbrace{y, z), y, z) \ldots), y, z}_{(k+n)-times}) \approx$$

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In particular, for $k = 0$ and $n = 2$, $\mathbf{A} = (D, f)$.

Example

For $B = \mathbb{N}$, $C = \{a\}$ and $f_{aa}(x) = x + 1$ the algebra $\mathbf{A} \in \mathcal{SD}_3$ is a non-Szendrei differential mode.

For any $k, n \in \mathbb{N}$, $n \neq 0$,

$$f(f(\dots f(f(x, \underbrace{y, z), y, z) \dots), y, z)}_{(k+n)-times} \neq$$

$$f(f(\dots f(f(x, \underbrace{y, z), y, z) \dots), y, z)}_{k-times}$$

The End

Thank you for your attention