Full dualities: quasi-primal algebras via relational structures

Todd Niven

Eduard Čech Center
Institute of Mathematics of the Academy of Sciences
Czech Republic

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Natural duality

Given a finite algebra $\underline{\mathbf{M}} = \langle M; F \rangle$, an alter ego of $\underline{\mathbf{M}}$ is a finite structure $\underset{\sim}{\mathbf{M}} = \langle M; H, R, \mathcal{T} \rangle$, where $\mathcal{T}$ is the discrete topology and

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(i) H is a collection of algebraic (partial) operations, i.e. homomorphisms of the form $h : \mathbf{A} \to \underline{\mathbf{M}}$ where $\mathbf{A} \leqslant \underline{\mathbf{M}}^n$, for some $n \in \omega$; and

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The topological quasi-variety generated by $\underset{\sim}{\mathbf{M}}$ is the class $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, of all isomorphic copies of topologically closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$.

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Since the structure on $\underset{\sim}{\mathbf{M}}$ is "compatible" with the algebra $\underline{\mathbf{M}}$, it follows that there are dually adjoint hom-functors

$$\mathrm{D} : \mathbb{ISP}(\underline{\mathbf{M}}) \to \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$$

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$$\mathrm{D}(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}}) \leqslant \underset{\sim}{\mathbf{M}}^A$$

and

$$\mathrm{E}(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \leqslant \underline{\mathbf{M}}^X.$$

Question

Given a finite algebra $\underline{\mathbf{M}}$, does there exist an alter ego $\underset{\sim}{\mathbf{M}}$ such that the algebra $\mathrm{ED}(\mathbf{A})$ is isomorphic to $\mathbf{A}$, for all $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$?

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More specifically, is there an alter ego $\underset{\sim}{\mathbf{M}}$ such that the natural embedding $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$, given by

$$e_{\mathbf{A}}(a)(\alpha) = \alpha(a),$$

is an isomorphism, for all $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$?

Duality

If $\underset{\sim}{\mathbf{M}}$ is an alter ego of $\underline{\mathbf{M}}$ such that $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$, we say "$\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a duality".

Full Duality

Question

Assume that $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a duality. Does there exist an alter ego $\underset{\sim}{\mathbf{M}}$ such that $\mathrm{DE}(\widetilde{\mathbf{X}})$ is isomorphic to $\mathbf{X}$, for all $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$?

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More specifically, is there an alter ego $\underset{\sim}{\mathbf{M}}$ such that the natural embedding $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$, given by

$$\varepsilon_{\mathbf{X}}(x)(\varphi) = \varphi(x),$$

is an isomorphism, for all $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$.

Full Duality

If $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a duality, and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, for all $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, we say "$\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a full duality".

Examples of full dualities

Stone duality for Boolean algebras. In this case

$$\underline{\mathbf{M}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1 \rangle$$

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Priestley duality for distributive lattices,

$$\underline{\mathbf{M}} = \langle \{0,1\}; \vee, \wedge \rangle$$

and

$$\underset{\sim}{\mathbf{M}} = \langle \{0,1\}; \leqslant, 0, 1, \mathcal{T} \rangle.$$

Strong Duality

If $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a full duality and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, we say "$\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a strong duality".

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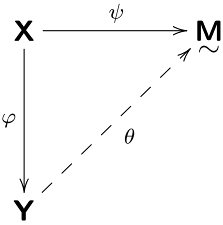
That is,



where $\varphi : \mathbf{X} \to \mathbf{Y}$ is an embedding.

Question (Davey and Werner [3])

Does there exist a finite algebra $\underline{\mathbf{M}}$ and a choice of alter ego $\underset{\sim}{\mathbf{M}}$ such that $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a full but not strong duality?

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Solution (Clark, Davey and Willard 2006 [2])

There is a 4-element quasi-primal algebra $\underline{\mathbf{S}}$ and an alter ego $\underset{\sim}{\mathbf{S}}$ such that $\underline{\mathbf{S}}$ and $\underset{\sim}{\mathbf{S}}$ yield a full but not strong duality.

Clark, Davey and Willard's example

The algebra

$\underline{\mathbf{S}} = \langle \{0, a, b, 1\}; \wedge, \vee, t, 0, 1 \rangle$, where $\langle \{0, a, b, 1\}; \wedge, \vee, 0, 1 \rangle$ is the 4-element bounded chain, with $0 < a < b < 1$, and t is the ternary discriminator function.

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The alter ego

$\underset{\sim}{\mathbf{S}} = \langle \{0, a, b, 1\}; r, \mathcal{T} \rangle$, where $r = \{(0,0), (a,b), (1,1)\}$.

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It so happens that r is the graph of the "partial automorphism" $f : 0 \mapsto 0, a \mapsto b, 1 \mapsto 1$.

Theorem (Niven 2006)

Let $\underline{\mathbf{Q}}$ be a quasi-primal algebra. The following are equivalent.

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(B) for all subalgebras $\mathbf{A}, \mathbf{B} \leqslant \underline{\mathbf{Q}}$, if $\mathbf{C} \leqslant \mathbf{A} \cap \mathbf{B}$ such that

$$C = \bigcap_{1 \leqslant i \leqslant l} \{x \in Q \mid f_i(x) = g_i(x)\},$$

for some homomorphisms $f_1, f_2, \ldots, f_l : \mathbf{A} \to \underline{\mathbf{Q}}$ and $g_1, g_2, \ldots, g_l : \mathbf{B} \to \underline{\mathbf{Q}}$, then every homomorphism $h : \mathbf{C} \to \underline{\mathbf{Q}}$ either extends to a homomorphism on $\mathbf{A}$ or to a homomorphism on $\mathbf{B}$.

Corollary

Let $\underline{\mathbf{Q}}$ be a quasi-primal algebra. The following are equivalent.

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