MANY FACES OF DUALITIES

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SZEGED, JULY 19, 2007

$\mathcal{F}$ A FINITE SET OF STRUCTURES (GRAPHS)

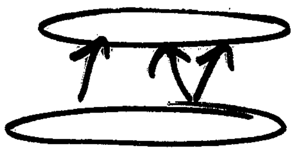
$$\mathrm{FORB}(\mathcal{F}) = \{A \,;\, F \in \mathcal{F} \Rightarrow F \not\to A\}$$

$$= \mathcal{F} \not\to$$

EXAMPLE :

FORB($\triangle$) = ALL TRIANGLE FREE GRAPHS

FORB($\to\to$) = ALL BIPARTITE ORIENTATIONS



$f : A \longrightarrow B \quad \stackrel{\text{DEF}}{\equiv}$ EXISTENCE OF A HOMOMORPHISM

$\mathcal{D}$ A FINITE SET OF STRUCTURES

$$\mathrm{CSP}(\mathcal{D}) = \{ A \, ; \; A \longrightarrow D \text{ FOR SOME } D \in \mathcal{D} \}$$
$$= \longrightarrow \mathcal{D}$$

EXAMPLE:

$\mathrm{CSP}(\Delta)$ = ALL 3-COLORABLE GRAPHS

$\mathrm{CSP}(H)$ = ALL H-COLORABLE GRAPHS

(FINITE) DUALITY

$$\mathcal{F} \not\longrightarrow \; = \; \longrightarrow \mathcal{D}$$

$$\text{FORB}(\mathcal{F}) = \text{CSP}(\mathcal{D})$$

GAME OF ALTERNATIVES

$\forall A$ EITHER $F \longrightarrow A$ FOR SOME $F \in \mathcal{F}$

OR $A \longrightarrow D$ FOR SOME $D \in \mathcal{D}$

NESETRIL + PULTR 78

CATEGORY THEORY CONTEXT

OF "GOOD CHARACTERIZATIONS" (EDMONDS)

NP ∩ coNP

LP DUALITY

(N. GRAPH THEORY 79)

ALL DREAMS REALIZED

LP (N. + HOCHSTÄTTER '02)

NP (N. + KUN '06)

⋮

$\mathcal{F}$ .... FORBIDDEN SET

$\mathcal{D}$ ... DUAL SET

UP TO HOMOMORPHISM EQUIVALENCE

$\mathcal{D}$ UNIQUELLY DETERMINED BY $\mathcal{F}$

$\mathcal{F}$ —— " ———— BY $\mathcal{D}$

(MINIMAL SETS)

KEY CASE :

$\boxed{\text{SINGLETON DUALITY}}$

$|\mathcal{F}| = |\mathcal{D}| = 1$

$$F \not\to A \iff A \to D$$

$\boxed{\text{THM}}$ (N. TARDIF 2000)

1. FOR EVERY RELATIONAL TREE $T = F$ THERE EXISTS DUAL $D_T = D$

2. IF F FAILS TO BE HOM-EQUIVALENT TO A TREE THEN DUAL DOES NOT EXISTS.

THIS LECTURE :

CONTEXT OF EXISTENCE

(4 PROOFS)

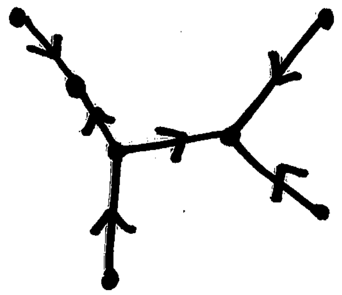
$\boxed{\text{I}}$ SIMPLEST (BEAR) CONSTRUCTION

(N. TARDIF '04)

THM FOR EVERY ORIENTED TREE T THERE EXISTS DUAL

(FOR REL. STRUCTURES SIMILAR VIA INCIDENCE GRAPH)

PROOF $T = (V, E)$



DEFINITION OF DUAL

$$V(D) = \{ f : V \to V ;\ f(v) \sim v \}$$

NEIGHBOURLY MAPS

$(f, g) \in E(D)$ IFF

$$(u, v) \in E \implies (f(u), g(v)) \neq (v, u)$$

NO EDGE IS FLIPPED

(i) $T \not\to D$

(ii) $T \not\to G \implies G \xrightarrow{f} D$

(i) BY CONTRADICTION

(ii) $f(x)(y) = y'$

$$T_y \not\to G_x \implies \exists\ y' \sim y \quad T_{y'} \not\to G_x$$

AMAZING DUALS

- EXPONENTIAL SIZE (N. TARDIF)

- LINEAR DIAMETER (N. ŠVEJDAROVÁ)

- ENP (LAROSE, LOTEN, TARDIF)

⋮

$\boxed{\text{II.}}$ DELETION CONSTRUCTION (LIFTS)

KOMAREK, FEDER+VARDI, ...

ASSUME $T \not\to G$

$x \in V$, B_x BRANCH OF T AT x

PUT $y \in V(G)$ $\quad y \in U_{B_x}$

$$\Updownarrow$$

$$\exists f : B_x \to G_y$$

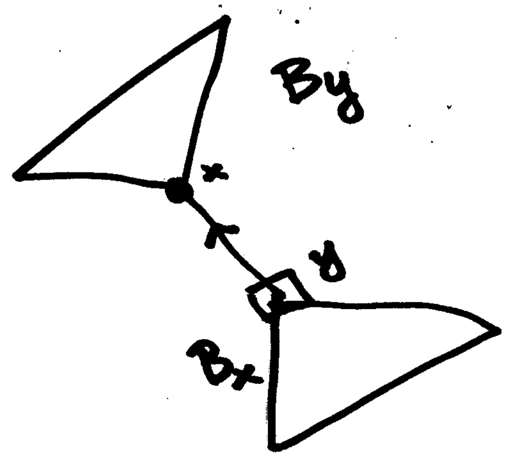
$\leq 2(n-1)$ UNARY OPERATIONS

$V(D)$ .... ALL (CONSISTENT) COMBINATIONS OF UNARY OPERATIONS

$+$

ALL POSSIBLE EDGES

DELETE CONFLICTING TUPLES



(B_y, B_x)

LIFTS $\quad \mathcal{C} \rightsquigarrow \mathcal{C}'$

SHADOWS $\quad \phi(\mathcal{C}') = \mathcal{C}$

(KUN. N.)

A STRUCTURE IN $\mathrm{REL}(\Delta)$

$\parallel$

$(X, (R_i \,;\, i \in I))$

$\boxed{\text{LIFT OF } A}$ IN $\mathrm{REL}(\Delta')$

$I \subseteq I'$

$A' = (X, (R_i \,;\, i \in I'))$

EXAMPLE:

$A = (X, R)$ RELATION

$A' = (X, (R, U_1, U_2, U_3))$

RELATION WITH 3 UNARY OP.

(COLORS)

$\boxed{A \text{ SHADOW OF } A'}$

$\Phi(A') = A$ FORGETFUL FUNCTOR

$\mathcal{F}$ $F_1, \ldots, F_t$ GRAPHS

$\mathcal{F}'$ $F_1', \ldots, F_{t'}'$ LIFTS IN $REL(\Delta')$

$FORB(\mathcal{F}')$ IN $REL(\Delta')$

$\Phi(FORB(\mathcal{F}'))$ IN $REL(\Delta)$

$\boxed{\text{THM}}$ (KUN, N. '06)

EVERY NP PROBLEM IS POLYNOMIALLY EQUIVALENT TO THE MEMBERSHIP PROBLEM FOR A CLASS

$$\Phi(FORB(\mathcal{F}')).$$

$\boxed{\text{III}}$ FIRST CONSTRUCTION

(VIA CORRESPONDENCE
GAP $\times$ DUAL)

(N. TARDIF '00)

HOMOMORPHISM POSET (FOR CORES)

$A \leq B$ IFF $A \longrightarrow B$

(A, B) $\boxed{\text{GAP}}$ IFF 1) $A < B$

2) $A < C < B$ FOR NO C

B CONNECTED $\equiv$ $\boxed{\text{CONNECTED GAP}}$

IN THIS CASE A UNIQUE
THE $\boxed{\text{PREDECESSOR}}$ OF B

$\boxed{\text{PROP 1}}$

(A, B) CONNECTED GAP $\Rightarrow (B, A^B)$ DUALITY PAIR

$\boxed{\text{PROP 2}}$

(F, D) DUALITY $\Rightarrow (D \times F, F)$ CONNECTED GAP

HOLDS VERY GENERALLY (N., PULTR, TARDIF '06)
"HEYTING POSETS"

NO GAPS $\Leftrightarrow$ NO DUALITIES

(WELZL) (N. PULTR)

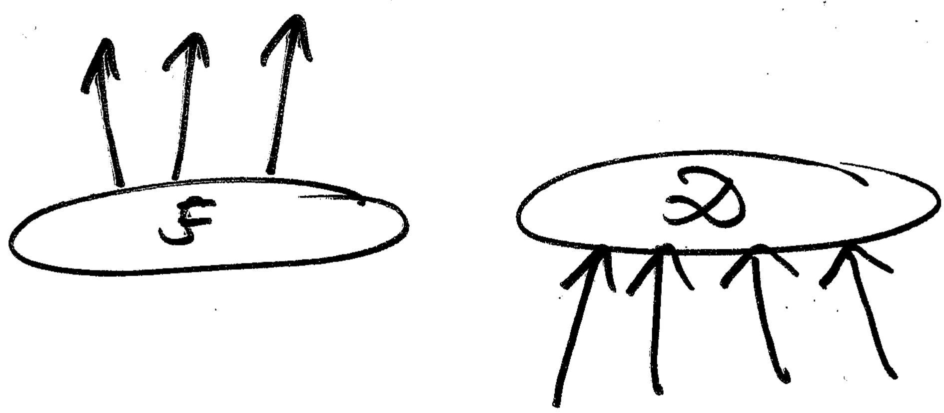
MAC $\equiv$ MAXIMAL FINITE ANTICHAIN IN THE HOMOMORPHISM ORDER.

⊙ : $(\mathcal{F}, \mathcal{D})$ FINITE DUALITY

$$\Downarrow$$

$\mathcal{F} \cup \mathcal{D}$ IS MAC

PROOF :



(SPLIT MAC)

$\boxed{\text{THM}}$ (FONIOK, N., TARDIF '06)

$\Delta = (k)$

THERE IS ONE-TO-ONE CORRESPONDENCE BETWEEN MAC's AND FINITE DUALITIES

IV. UNIVERSALITY

$\Downarrow$

DUALITY (HUBIČKA, N. '07)

D DUAL OF $\mathcal{F}$

$\Updownarrow$

D MAXIMUM (IN THE HOMOMORPHISM ORDER) OF FORB($\mathcal{F}$)

DUALITY PROBLEM :

WHEN FORB($\mathcal{F}$) HAS FINITELY MANY MAXIMAL ELEMENTS ?

CLASSICAL RELATED :

? WHEN FORB($\mathcal{F}$) HAS COUNTABLE EMBEDDING UNIVERSAL STRUCTURE ?

CHERLIN, SHELAH, SHI

YES FOR ANY SET $\mathcal{F}$ OF CONNECTED GRAPHS.

THM (HUBIČKA, N. 07)

FOR ANY SET $\mathcal{F}$ OF RELATIONAL STR. THERE EXISTS A LIFT-SET $\mathcal{F}'$ (IN REL(Δ')) SUCH THAT FORB($\mathcal{F}'$) HAS ULTRAHOMOGENEOUS UNIVERSAL.

THM' (HN)

IF $\mathcal{F}$ IS A FINITE SET OF TREE-STR. THEN THERE EXISTS LIFT SET $\mathcal{F}'$ SUCH THAT FORB($\mathcal{F}'$) HAS ULTRAHOMOGENEOUS UNIVERSAL WITH A FINITE RETRACT $\equiv$ $\boxed{\text{DUAL}}$

V. NEW DICHOTOMY CONJECTURE (N., SIGGERS '07)

CONSIDER H, $CSP(H)$

$\boxed{\text{FIBRE GADGET}}$ $M \in CSP(H)$

(1) $W_1 \subset W(M)$, $W_2 \subset V(M)$

$W_1 \cap W_2 = \phi$

$|W_1| = |W_2|$ (COPIES OF INDEX SET W^*)

(2) $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ DISJOINT SETS OF H-PATTERNS $(W^* \to V(H))$

(3) FOR EVERY $f: M \longrightarrow H$

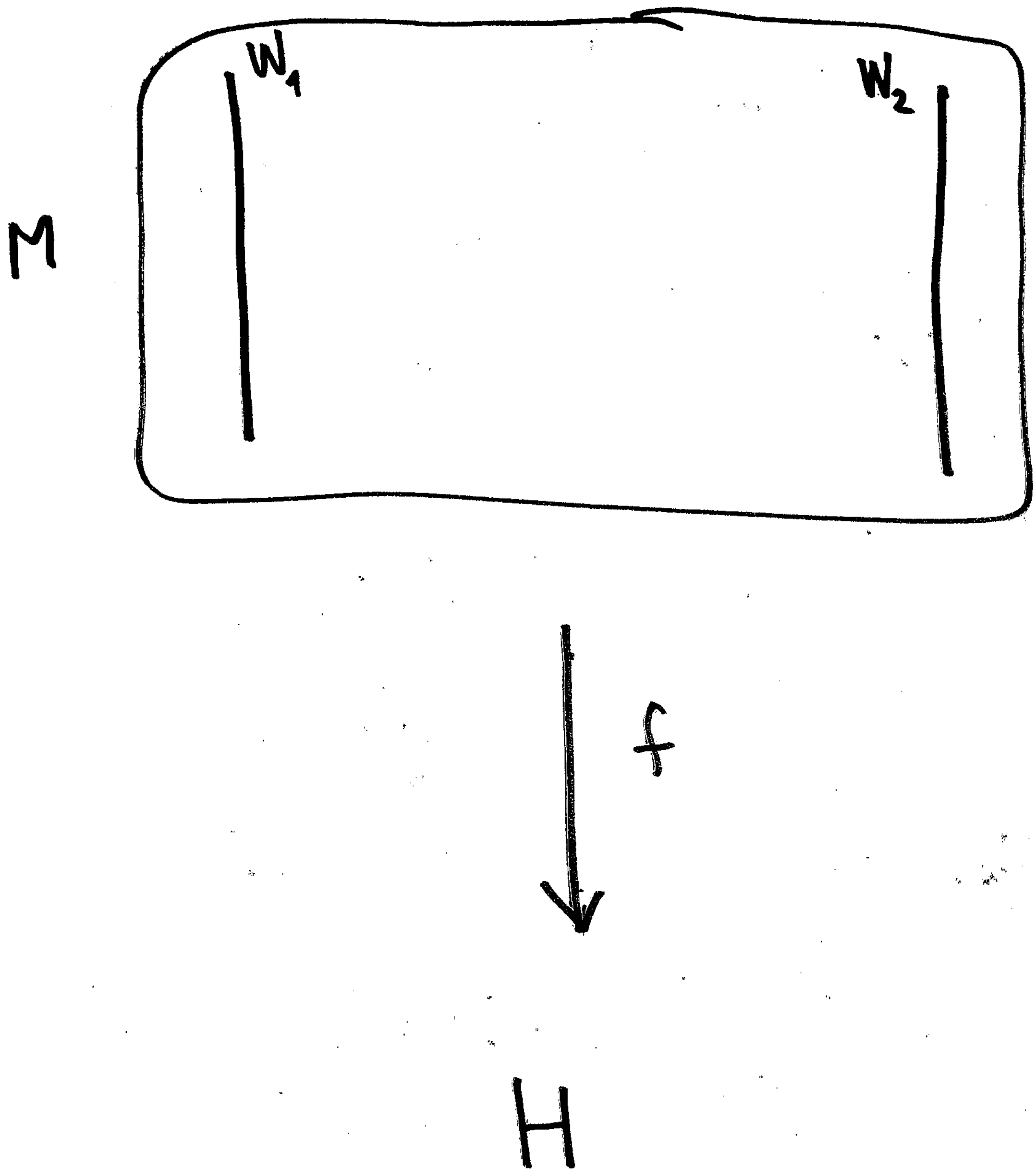
$f|W^1$ AND $f|W^2$ ARE INDIFFERENT SETS $\mathcal{P}_i$

(4) THERE ARE $P_i \in \mathcal{P}_i$ (REPRESENTATIVE PATTERNS)

SUCH THAT FOR EVERY $i \neq j$

THERE EXISTS $f_{ij}: M \longrightarrow H$ $f_{ij}|W^1 = P_1$, $f_{ij}|W^2 = P_j$

FIBRE GADGET



$$f|W_1 \in \mathcal{P}_i \implies f|W_2 \in \mathcal{P}_j \qquad j \neq i$$

$f|W_1 = P_1$, $f|W_2 = P_2$ CAN BE EXTEDED

THM (N., SIGGERS '07)

FIBRE GADGET + FIBRE CONSTRUCTION

$\Downarrow$

IF H HAS FIBRE GADGET
THEN CSP(H) IS NP-COMPLETE.

PARTICULARLY:

FIBRE GADGET EXISTS FOR
BLOCK PROJECTIVE STRUCTURES

$\boxed{\text{BLOCK PROJECTIVE}}$ H

$S \subseteq V(H)$ IS BLOCK PROJECTIVE
IF THERE EXIST DISJOINT SETS
$\{H_s \; ; \; s \in S\}$ $\quad H_s \subseteq V(H)$
SUCH THAT:

FOR EVERY $\phi : H^d \longrightarrow H$
THERE EXISTS i SUCH THAT
FOREVERY $(s_1, \dots, s_d) \in S^d$

$$\phi(s_1, \dots, s_d) \in H_{s_i}$$

H BLOCK PROJECTIVE
IF H IS A CORE AND
IT CONTAINS A NON-TRIVIAL
BLOCK PROJECTIVE SET.

$\boxed{\text{BULATOV, JEAVONS, KROKHIN} \quad \text{CONJ}}$

☆ H IDEMPOTENT, $\mathcal{A}_H$ HAS SUBALGEBRA $\mathcal{B}$ WITH HOMOMORPHIC IMAGE $\mathcal{C}$ ALL OF WHOSE TERM OPERATORS ARE PROJECTIVE.

$$\Updownarrow$$

CSP (H) NP-COMPLETE

LAROSE - ZADORI VIA TAYLOR OPER.

☆ $\Longrightarrow$ H IS BLOCK PROJECTIVE

LZ
BJK

$\bigstar \subseteq$ BLOCK PROJECTIVE $\subseteq$ FIBRE GADGET

$\wr$ CSP(H) NP-COMPLETE

$\Updownarrow$

$\exists$ FIBRE GADGET FOR H

COMBINATORIAL PROOF $\Rightarrow$ BOUNDED DEGREES

GIRTH