Free (*m*+*k*, *m*)*–bands*

Valentina Miovska and Dončo Dimovski

Abstract. *The subject of this presentation is the class of* (*m*+*k*, *m*)*–bands, i.e. the class of vector-valued* (*m*+*k*, *m*)*–semigroups that are direct products of p–zero* (*m*+*k*, *m*)*–semigroups* (*an* (*m*+*k*, *m*)*–groupoid* (*Q*; [ ]) *is said to be a p–zero* (*m*+*k*, *m*)*–groupoid*, $0 \le p \le m$, *if* $[x_1^{m+k}] = x_1^p x_{p+k+1}^{m+k}$ *for any* $x_1^{m+k} \in Q^{m+k}$). *Two characterizations of* (*m*+*k*, *m*)*–bands are given and free* (*m*+*k*, *m*)*–bands are described.*

0. Introduction

We will introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the *s*–th Cartesian power of *Q*, will be denoted by x_1^s. If $x_1 = x_2 = ... = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

2) The symbol x_i^j denotes the sequence $x_i x_{i+1} ... x_j$ for $i \le j$, and the empty sequence when $i > j$.

3) The set $\{1, 2, ..., s\}$ will be denoted by $\mathbb{N}_s$.

Let m and k be positive integers. An $(m+k, m)$–*groupoid* is a pair $\mathbf{Q}=(Q; [\ ])$ where $Q\neq\emptyset$, $[\ ]$ is an $(m+k, m)$–*operation*, i.e. a map $[\ ]: Q^{m+k}\rightarrow Q^m$.

Let $[\ ]$ be an $(m+k, m)$–operation on a set Q. We can associate a sequence of m $m+k$–ary operations $[\ ]_1, [\ ]_2,..., [\ ]_m$ ($[\ ]_i: Q^{m+k}\rightarrow Q$, $1\leq i\leq m$) to $[\ ]$ by

$$[x_1^{m+k}]_i = y_i \Leftrightarrow [x_1^{m+k}] = y_1^m \qquad (1)$$

for every $1\leq i\leq m$.

An $(m+k, m)$–groupoid $\mathbf{Q}=(Q; [\ ])$ is called an $(m+k, m)$–*semigroup* if for each $x_1^{m+2k}\in Q^{m+2k}$, $1\leq i\leq k$

$$[x_1^i[x_{i+1}^{i+m+k}]x_{i+m+k+1}^{m+2k}] = [[x_1^{m+k}]x_{m+k+1}^{m+2k}]. \qquad (2)$$

1. *p*–zero $(m+k,m)-$semigroups

Definition 1.1 An $(m+k,m)-$groupoid **Q**=$(Q;[\ ])$ is said to be a *projection* $(m+k,m)-$*groupoid* if there are $1\le \alpha_1 < \alpha_2 < ... < \alpha_m \le m+k$, such that

$$[x_1^{m+k}] = x_{\alpha_1} x_{\alpha_2} ... x_{\alpha_m}, \qquad (3)$$

for any $x_1^{m+k} \in Q^{m+k}$.

The left-zero $(m+k,m)-$groupoid (a pair **A=**$(A;[\ ])$, where [ ] is an $(m+k,m)-$operation defined by $[x_1^{m+k}] = x_1^m$) and the right-zero $(m+k,m)-$groupoid (**B=**$(B;[\ ])$, where [ ] is defined by $[x_1^{m+k}] = x_{k+1}^{m+k}$) are examples for projection $(m+k,m)-$groupoids which are also $(m+k,m)-$semigroups. In general, projection $(m+k,m)-$groupoids are not necessarily $(m+k,m)-$semigroups. For example, the $(4,2)-$groupoid **Q**=$(Q;[\ ])$ where [ ] is defined by $\left[x_1^4\right] = x_2^3$, is a projection $(4,2)-$groupoid, but not a $(4,2)-$semigroup.

Definition 1.2 Let $0 \le p \le m$. An $(m+k,m)-$groupoid **Q**$=(Q;[\ ]^p)$ is said to be a p–zero $(m+k,m)-$ groupoid if

$$[x_1^{m+k}]^p = x_1^p x_{p+k+1}^{m+k}, \qquad (4)$$

for any $x_1^{m+k} \in Q^{m+k}$.

Proposition 1.3 Any p–zero $(m+k,m)-$groupoid **Q**$=(Q;[\ ]^p)$ is an $(m+k,m)-$semigroup.

Proposition 1.4 If **Q**$=(Q;[\ ])$ is a projection $(m+k,m)-$groupoid which is also an $(m+k,m)-$semigroup, then **Q** is a p–zero $(m+k,m)-$semigroup, for some $0 \le p \le m$.

Propositions 1.3 and 1.4 imply that there are exactly $m+1$ projection $(m+k,m)-$semigroups.

2. $(m+k,m)-$bands

Let $\mathbf{A}_i=(A_i;[\ ]^i), i=1,2,...,t$ be $(m+k,m)-$semigroups. Their direct product is an $(m+k,m)-$semigroup, where the $(m+k,m)-$operation [ ] is defined by

$$[x_1^{m+k}]=y_1^m \Leftrightarrow x_i=(x_{i,1},x_{i,2},...,x_{i,t}),\ y_j=(y_{j,1},y_{j,2},...,y_{j,t}),$$

$$y_{j,r}=\left[x_{1,j}x_{2,j}...x_{m+k,j}\right]^r,\ i\in\mathbb{N}_{m+k}, j\in\mathbb{N}_m, r\in\mathbb{N}_t. \qquad (5)$$

Definition 2.1 Let $\mathbf{A}_p=(A_p;[\ ]^p)$ be p–zero $(m+k,m)-$semigroups, $0\leq p\leq m$. The direct product of $A_m,A_{m-1},...,A_0$ is called $(m+k,m)-$*band*.

If $(A_m\times A_{m-1}\times...\times A_0;[\ ])$ is an $(m+k,m)-$band then its $(m+k,m)-$operation [ ] is of the form

$$[x_1^{m+k}]=y_1^m \Leftrightarrow x_i=(x_{i,1},x_{i,2},...,x_{i,m+1}),$$

$$y_j=(x_{j,1},x_{j,2},...,x_{j,m+1-j},x_{j+k,m+2-j},...,x_{j+k,m+1}), \qquad (6)$$

$$i\in\mathbb{N}_{m+k}, j\in\mathbb{N}_m.$$

Proposition 2.2 An $(m+k,m)-$semigroup $\mathbf{Q}=(Q;[\ ])$ is an $(m+k,m)-$band if and only if the following conditions are satisfied in $\mathbf{Q}$:

(I) $\left[x_1^{m+k}\right]_i = \left[y_1^{i-1} x_i y_{i+1}^{i+k-1} x_{i+k} y_{i+k+1}^{m+k}\right]_i$, $i \in \mathbb{N}_m$;

(II) $$\left[\overset{j-1}{a}\left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\ y\ \overset{m-i}{a}\right]_i \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j = \left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\left[\overset{j-1}{a}\ y\ \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j \overset{m-i}{a}\right]_i,$$

for a fixed element of Q and $j \le i$;

(III) $\left[\overset{i-1}{a}\left[\overset{j-1}{a}\ x\ \overset{k-1}{a}\ y\ \overset{m-j}{a}\right]_j \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i$, for a fixed element of Q and $j \le i$;

(IV) $\left[\overset{j-1}{a}\ x\ \overset{k-1}{a}\left[\overset{i-1}{a}\ y\ \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j = \left[\overset{j-1}{a}\ x\ \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j$, for a fixed element of Q and $j \le i$;

(V) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.

3. A characterization of $(m+k,m)-$bands

In the sequel we will give a characterization of $(m+k,m)-$bands using the usual rectangular bands, where a rectangular band is a semigroup $(Q;*)$ that satisfies the following two identities

$$x*y*z = x*z \text{ and } x*x = x, \qquad (7)$$

for each $x,y,z \in Q$.

Proposition 3.1 $\mathbf{Q} = (Q;[\ ])$ is an $(m+k,m)-$band if and only if there are rectangular bands $(Q;*_i)$, $i \in \mathbb{N}_m$, such that

(i) $(x *_i y) *_j z = x *_i (y *_j z)$, $j \le i$;

(ii) $(x *_j y) *_i z = x *_i z$, $j \le i$;

(iii) $x *_j (y *_i z) = x *_j z$, $j \le i$;

and $[x_1^{m+k}]_i = x_i *_i x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbb{N}_m$. (8)

4. Free $(m+k, m)-$bands

Let $B \neq \emptyset$. We define a sequence of sets B_0, B_1, …, B_p, … by induction as follows:

$B_o = B$;

Let B_p be defined and let $C_p = \{xy \mid x, y \in B_p\}$. Then we take $B_{p+1} = B_p \cup (\mathbb{N}_m \times C_p)$ and $\overline{B} = \bigcup_{p \geq 0} B_p$.

We define a *length* for elements of $\overline{B}$, i.e. a map $|\ | : \overline{B} \to \mathbb{N}$, in the following way:

For each $a \in B$, $|a| = 1$;

Let $|u|$ be defined for each $u \in B_p$, then for $(i,\ xy) \in B_{p+1}$, we put $|(i, xy)| = 1 + |x| + |x||y|$.

By induction on the length we define a map $\varphi : \overline{B} \to \overline{B}$ as follows:

If $a \in B$ then

$$\text{(D0)}\ \varphi(a) = a$$

Let $u = (i,\ xy) \in \overline{B}$ and suppose that for each $v \in \overline{B}$, with $|v| < |u|$, $\varphi(v)$ be defined and:

i) $\varphi(v) \neq v \Rightarrow |\varphi(v)| < |v|$

ii) $\varphi(\varphi(v)) = \varphi(v)$.

Let $\varphi(x) \neq x$ or $\varphi(y) \neq y$. Then

$$\text{(D1)}\ \varphi(i, xy) = \varphi(i, \varphi(x)\varphi(y)).$$

Let $\varphi(x) = x$ and $\varphi(y) = y$.

If $u = (i, xx)$ then

$$\text{(D2)}\ \varphi(u) = \varphi(x).$$

If $u = (i, (j, zw)y)$, $j \leq i$ then

$$\text{(D3)}\ \varphi(u) = \varphi(i, zy).$$

If $u = (i, x(j, zw))$, $i \leq j$ then

$$\text{(D4)}\ \varphi(u) = \varphi(i, xw).$$

If $u = (i, (j, zw)y)$, $i < j$ then

$$\text{(D5)}\ \varphi(u) = (j, z(i, wy)).$$

If $u = (i, x(j, xz))$, $j < i$ then

$$\text{(D6)}\ \varphi(u) = \varphi(j, xz).$$

If $\varphi(u)$ can not be defined by (D1), (D2), (D3), (D4), (D5) or (D6) then

$$\text{(D7)}\ \varphi(u) = u.$$

Proposition 4.1 φ is a well defined mapping.

Proposition 4.2 a) For each $u \in \overline{B}$, $|\varphi(u)| \le |u|$;

b) For $u \in \overline{B}$, if $\varphi(u) \ne u$ then $|\varphi(u)| < |u|$;

c) For each $u \in \overline{B}$, $\varphi(\varphi(u)) = \varphi(u)$.

Proposition 4.3 Let $u = (i, xy) \in \overline{B}$. Then:

a) $\varphi(u) = \varphi(i, \varphi(x)\varphi(y))$;

b) $\varphi(u) = \varphi(i, \varphi(x)y) = \varphi(i, x\varphi(y))$.

Proposition 4.4 Let $u = (i, xx) \in \overline{B}$. Then $\varphi(u) = \varphi(x)$.

Proposition 4.5 (I) Let $u = (i, (j, zw)y)$, $j \le i$. Then $\varphi(u) = \varphi(i, zy)$.

(II) Let $u = (i, x(j, zw))$, $i \le j$. Then $\varphi(u) = \varphi(i, xw)$.

(III) Let $u = (i, (j, zw)y)$, $i < j$. Then $\varphi(u) = (j, z(i, wy))$.

(IV) Let $u=(i,x(j,xz))$, $j<i$. Then $\varphi(u)=\varphi(j,xz)$.

Let $Q=\varphi(\overline{B})$. If $u\in Q$ then there is $v\in\overline{B}$ such that $\varphi(v)=u$, and by Proposition 4.2 c) we have $\varphi(u)=\varphi(\varphi(v))=\varphi(v)=u$. It is clear that if $\varphi(u)=u$ then $u\in\varphi(\overline{B})=Q$. Hence, $Q=\{u \mid u\in\overline{B},\ \varphi(u)=u\}$.

We define maps $*_i : Q\times Q\to Q$, $i\in\mathbb{N}_m$ by

$$x *_i y=\varphi(i,xy). \tag{9}$$

Proposition 4.6 For each $i\in\mathbb{N}_m$, $(Q;*_i)$ are rectangular bands that satisfy (*i*), (*ii*) and (*iii*) from Proposition 3.1.

Let [ ] be the $(m+k,m)-$operation on Q defined by

$$[x_1^{m+k}]_i = x_i *_i x_{i+k},\ x_1^{m+k}\in Q^{m+k},\ i\in\mathbb{N}_m. \tag{10}$$

Theorem 4.7 $(Q;[\ ])$ is a free $(m+k,m)-$band with a basis B.