Title

Few subpowers and the Constraint Satisfaction Problem

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Two versions of CSP

- Variable-Value:
  INPUT: V (variables), D (values) and
  $\{(\overline{s}_1, R_1), \ldots, (\overline{s}_k, R_k)\}$, where $\overline{s}_i \in V^{k_i}$ and $R_i \subseteq D^{k_i}$
  QUESTION: is there a $\varphi : V \rightarrow D$ such that $\varphi(\overline{s}_i) \in R_i$?

- Homomorphism:
  INPUT: two similar finite relational structures
  $\mathbf{V} = \langle V, R_1^{\mathbf{V}}, \ldots, R_l^{\mathbf{V}} \rangle$ and $\mathbf{D} = \langle D, R_1^{\mathbf{D}}, \ldots, R_l^{\mathbf{D}} \rangle$
  QUESTION: is there a homomorphism from $\mathbf{V}$ to $\mathbf{D}$?

NP-complete (interpret graph k-coloring)

Translate from one to another in PTIME.

Constraint languages; Dichotomy Conjecture

Fix D (finite)

Constraint language $\Gamma =$ any set of relations on D

CSP$(\Gamma) =$ restriction of CSP (first definition), where each $R_i \in \Gamma$.

The Dichotomy Conjecture (Feder and Vardi): CSP$(\Gamma) \in$ P $\cup$ NP-complete.

For finite Γ, CSP(Γ) is equivalent to the homomorphism version of CSP where we fix $\mathbf{D}$.

Polymorphisms and relational clones

- $Pol(\Gamma)$
- $Inv(\mathcal{C})$

Γ is a relational clone when $\Gamma = Inv(\mathcal{C})$ for some $\mathcal{C}$.

$\langle\Gamma\rangle = Inv(Pol(\Gamma))$.

$\langle\Gamma\rangle =$ closure of Γ under primitive positive formulas

Jeavons: CSP(Γ) is in the same complexity class as CSP$(Inv(Pol(\Gamma)))$.

Bulatov, Jeavons, Krokhin: CSP(Γ) is in the same complexity class as CSP$(Inv(Pol_{id}(\Gamma)))$.

NP-complete result for CSP(Γ)

Bulatov, Jeavons, Krokhin: If $\mathbf{1} \in$ typ$\{HSP(\langle D, Pol(\Gamma)\rangle)\}$ then CSP(Γ) $\in$NP-complete.

Let $w : D^k \to D$, $k > 1$, satisfy:

- $w(x, x, \ldots, x) = x$ and
- $w(y, x, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, x, \ldots, x, y)$

Then w is a weak near-unanimity (WNU) operation on D.

Maróti, McKenzie: $\mathbf{1} \notin$ typ$\{HSP(\mathbf{D})\}$ iff $\mathbf{D}$ has a WNU term.

Second Dichotomy Conjecture (Bulatov, Jeavons and Krokhin): CSP(Γ) $\in$ P when $\mathbf{1} \notin$ typ$(HSP(\mathbf{D}))$.

Some examples of tractable CSP results

Fix an instance $\langle D, V, \{(\overline{s}_1, R_1), \ldots, (\overline{s}_k, R_k)\}\rangle$ of CSP(Γ), where $\Gamma \subseteq Inv(\wedge)$.

$(\overline{s}_i, R_i) \rightsquigarrow R'_i \leq \mathbf{D}^V$.

If for any $x \in V$, $\bigcap_i proj_x(R'_i) = \emptyset$, then the instance has no solution.

Otherwise, $f(x) := \bigwedge \bigcap_i proj_x(R'_i)$. Then $f \in D^V$ is the solution of the instance.

Near-unanimity operations and projections

Let $n(x_1, \ldots, x_k)$ be a near-unanimity (NU) operation on D.

Baker and Pixley: any $R \in Inv(n)$ is characterized by $proj_I(R)$, for all $|I| < k$.

Let $(\overline{s}, R)$ be a constraint. Define $(proj_I(\overline{s}), proj_I(R))$.

NU operations and projections (continued)

The algorithm:

- Add all possible constraints of the form $(\bar{t}_j, D^k)$ to the input, for all $\bar{t}_j \in V^k$
- Remove from the constraint relations R_i all tuples for which there exists $I \subseteq V$, $|I| < k$, which are in $proj_I(R_i) \setminus proj_I(R_j)$ for some other constraint relation R_j such that I is a subset of the coordinates of both $\overline{s_i}$ and $\overline{s_j}$.
- Repeat the previous step until no such erasures are possible. If any constraint relation became the empty set, there is no solution to the instance of CSP, otherwise there is (and any tuple in the remaining constraints can be extended to a solution).

Mal’cev operations and splittings

Assume that m is a Mal’cev operation on D, $V = \{1, 2, \ldots, n\}$.

- (i, a, b)-splitting
- generating by splittings.

Define again R'_i from $(\overline{s}_i, R_i)$. Now the algorithm would go like this:

- Create a small generating set S_0 for all of D^V
- Assume that there is a small generating set S_{j-1} for $R'_1 \cap \cdots \cap R'_{j-1}$
- Use this set S_{j-1} and $(\overline{s}_j, R_j)$ to compute a small generating set S_j for $R'_1 \cap \cdots \cap R'_{j-1} \cap R'_j$.
- If S_k is empty, return ’no solutions’, otherwise return any element of S_k.

Mal’cev operations and splittings (continued)

The third step of the previous algorithm $=$ procedure `Next`.

Use `Next-beta`, replace R_i' with $proj_{s_1}(R_i')$, then with $proj_{(s_1,s_2)}(R_i')$ and so on.

The same basic algorithm by V. Dalmau can be used whenever it is possible to express a subuniverse of $\mathbf{D}^V$ with a small generating set.

Three functions

- $s_{\mathbf{A}}(n) = \log_2 |\mathrm{Sub}(\mathbf{A}^n)|$;
- $g_{\mathbf{A}}(n) = \max\limits_{B \in \mathrm{Sub}(\mathbf{A})} \min\limits_{\langle X \rangle = B} |X|$;
- $i_{\mathbf{A}}(n) =$ the maximal size of an independent subset of A^n.

Two easy observations:

- $g_{\mathbf{A}}(n) \leq i_{\mathbf{A}}(n) \leq s_{\mathbf{A}}(n) \leq \log_2(|A|) \cdot n g_{\mathbf{A}}(n)$.
- If $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ and $|B| < \infty$, then there exist constants $c_i, d_i > 0$ such that $s_{\mathbf{B}}(n) \leq s_{\mathbf{A}}(c_1 n + d_1)$, $g_{\mathbf{B}}(n) \leq g_{\mathbf{A}}(c_2 n + d_2)$ and $i_{\mathbf{B}}(n) \leq i_{\mathbf{A}}(c_3 n + d_3)$.

Few subpowers and its characterization

The first observation: when one of the three functions $\leq$ a polynomial, then all three are $=$: $\mathbf{A}$ has *few subpowers*.

The second observation: having few subpowers is a (pseudo-)varietal property.

$e(x_0, x_1, \ldots, x_k)$ is an edge term of $\mathbf{A}$ if

$$\mathbf{A} \models \begin{array}{rcl} e(y,y,x,x,x,\ldots,x) & = & x \\ e(y,x,y,x,x,\ldots,x) & = & x \\ e(x,x,x,y,x,\ldots,x) & = & x \\ & \vdots & \\ e(x,x,x,x,\ldots,x,y) & = & x \end{array}$$

BIMMVW: $\mathbf{A}$ has few subpowers iff $\mathbf{A}$ has an edge term.

Some more auxilliary terms

Let **A** be a finite algebra with a $k+1$-variable edge term e.
Then **A** also has terms $s(x_1, x_2, \ldots, x_k)$ and $m(x, y, z)$ such that

$$\begin{aligned}
m(x,y,y) &= x \\
s(y,x,x,x,\ldots,x,x) &= m(y,y,x) \\
s(x,y,x,x,\ldots,x,x) &= x \\
s(x,x,y,x,\ldots,x,x) &= x \\
&\vdots \\
s(x,x,x,x,\ldots,x,y) &= x.
\end{aligned}$$

Moreover, $m(y, y, m(y, y, x)) = m(y, y, x)$.

We will call $(a, b) \in A^2$ such that $m(a, a, b) = b$ a *minority* pair.

A nice small generating set

$X \subseteq \mathbf{R}' \leq \mathbf{D}^n$ is a *representation* of $\mathbf{R}'$ when

- For each $I \subseteq V$ and $|I| = k$, $proj_I(X) = proj_I(R')$ and
- For each minority pair (a, b) and each index (i, a, b) which has a witnessing pair in R', it also has a witnessing pair in X.

If $X \subseteq R'$ is a representation of the subpower $\mathbf{R}' \leq \mathbf{D}^n$, then $\langle X \rangle = \mathbf{R}'$.

Modification of the algorithm

Now the algorithm for the Mal'cev situation needs to be modified in the following way: we do not represent a constraint with all splittings, just with splittings where (a, b) is a minority pair, and also we include the witnesses for projections onto all small subsets of variables into our representations (similar as in NU algorithm). The overall structure of the procedures remains the same as in the Mal'cev case algorithm.

THANK YOU FOR YOUR ATTENTION!