ALGORITHMIC COMPLEXITY
and UNIVERSAL ALGEBRA

2007. 7.16. – 7.20.
Szeged, Hungary

Some Remarks on Minimal Clones

Hajime MACHIDA
(Tokyo)

Joint Work with

Michael PINSKER
(Wien)

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— A 'Minimal' Adventure to the World of Béla Csákány —

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§1 Where to Start

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◇ Béla Csákány ◇

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In 1983, B. Csákány published the following paper :

> **B. Csákány, "All minimal clones on the three element set",** *Acta Cybernet.*, **Vol. 6, 227-238, 1983**

where he determined all minimal clones on a three-element set.

Number of minimal clones = 84

§0 Just in Case ... Definitions

clone

$C \ (\subseteq \mathcal{O}_k)$ **: clone on** E_k

$\iff$

(i) C **: closed under composition**

(ii) $C \ \supseteq \ \mathcal{J}_k$

$\mathcal{L}_k$ **: lattice of all clones on** E_k

minimal clone

$C \, (\in \mathcal{L}_k)$ **: minimal clone**

$\iff$

(i) $C \neq J_k$

(ii) **for** $\forall \, C' \in \mathcal{L}_k$ **,**

$J_k \subset C' \subseteq C \implies C' = C$

§1 Where to Start ... Continued

♡ Generators of Minimal Clones on a Three-Element Set ♡

(B. Csákány, 1983)

Each of the following operations generates mutually distinct minimal clone.

(II) Binary idempotent operations

$$b_0 = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}} \qquad b_{364} = \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{matrix}} \qquad b_{728} = \boxed{\begin{matrix} 0 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_8 = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{matrix}} \qquad b_{368} = \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}} \qquad b_{80} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{36} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{matrix}} \qquad b_{40} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{matrix}} \qquad b_{692} = \boxed{\begin{matrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{10} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{matrix}} \qquad b_{280} = \boxed{\begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{matrix}} \qquad b_{458} = \boxed{\begin{matrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{20} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{matrix}} \qquad b_{448} = \boxed{\begin{matrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{matrix}} \qquad b_{188} = \boxed{\begin{matrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{11} = \boxed{\begin{array}{ccc} 0&0&0\\ 0&1&1\\ 0&2&2 \end{array}} \qquad b_{286} = \boxed{\begin{array}{ccc} 0&1&0\\ 1&1&1\\ 2&1&2 \end{array}} \qquad b_{215} = \boxed{\begin{array}{ccc} 0&0&2\\ 1&1&2\\ 2&2&2 \end{array}}$$

$$b_{16} = \boxed{\begin{array}{ccc} 0&0&0\\ 0&1&1\\ 2&1&2 \end{array}} \qquad b_{281} = \boxed{\begin{array}{ccc} 0&1&0\\ 1&1&1\\ 0&2&2 \end{array}} \qquad b_{296} = \boxed{\begin{array}{ccc} 0&1&0\\ 1&1&2\\ 2&2&2 \end{array}}$$

$$b_{47} = \boxed{\begin{array}{ccc} 0&0&0\\ 1&1&2\\ 0&2&2 \end{array}} \qquad b_{205} = \boxed{\begin{array}{ccc} 0&0&2\\ 1&1&1\\ 2&1&2 \end{array}} \qquad b_{179} = \boxed{\begin{array}{ccc} 0&0&2\\ 0&1&1\\ 2&2&2 \end{array}}$$

$$b_{17} = \boxed{\begin{array}{ccc} 0&0&0\\ 0&1&1\\ 2&2&2 \end{array}} \qquad b_{287} = \boxed{\begin{array}{ccc} 0&1&0\\ 1&1&1\\ 2&2&2 \end{array}} \qquad b_{53} = \boxed{\begin{array}{ccc} 0&0&0\\ 1&1&2\\ 2&2&2 \end{array}}$$

$$b_{38} = \boxed{\begin{array}{ccc} 0&0&0\\ 1&1&1\\ 0&2&2 \end{array}} \qquad b_{43} = \boxed{\begin{array}{ccc} 0&0&0\\ 1&1&1\\ 2&1&2 \end{array}} \qquad b_{206} = \boxed{\begin{array}{ccc} 0&0&2\\ 1&1&1\\ 2&2&2 \end{array}}$$

$$b_{26} = \boxed{\begin{array}{ccc} 0&0&0\\ 0&1&2\\ 2&2&2 \end{array}} \qquad b_{449} = \boxed{\begin{array}{ccc} 0&1&2\\ 1&1&1\\ 2&2&2 \end{array}} \qquad b_{37} = \boxed{\begin{array}{ccc} 0&0&0\\ 1&1&1\\ 0&1&2 \end{array}}$$

$$b_{33} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{matrix}} \qquad b_{122} = \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}} \qquad b_{557} = \boxed{\begin{matrix} 0 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{35} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{matrix}} \qquad b_{125} = \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}} \qquad b_{71} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{42} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{matrix}} \qquad b_{41} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}} \qquad b_{530} = \boxed{\begin{matrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{68} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}} \qquad b_{528} = \boxed{\begin{matrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{matrix}} \qquad b_{116} = \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{matrix}}$$

$$b_{178} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{matrix}} \qquad b_{290} = \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}}$$

$$b_{624} = \boxed{\begin{array}{ccc} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{array}}$$

(III) Ternary majority operations

$$m_0 = \boxed{\begin{matrix}0&0&0\\0&1&0\\0&0&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&0\\1&1&1\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&0&2\\0&1&2\\2&2&2\end{matrix}}$$

$$m_{364} = \boxed{\begin{matrix}0&0&0\\0&1&1\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&1\\1&1&1\\1&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\1&1&2\\2&2&2\end{matrix}}$$

$$m_{728} = \boxed{\begin{matrix}0&0&0\\0&1&2\\0&2&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\1&1&1\\2&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&2&2\\2&1&2\\2&2&2\end{matrix}}$$

$$m_{109} = \boxed{\begin{matrix}0&0&0\\0&1&0\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&1\\1&1&1\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&0&2\\1&1&2\\2&2&2\end{matrix}}$$

$$m_{473} = \boxed{\begin{matrix}0&0&0\\0&1&1\\0&2&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\1&1&1\\1&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\2&1&2\\2&2&2\end{matrix}}$$

$$m_{510} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix}} \quad \boxed{\begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{matrix}} \quad \boxed{\begin{matrix} 0 & 2 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

$$m_{624} = \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{matrix}} \quad \boxed{\begin{matrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{matrix}} \quad \boxed{\begin{matrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{matrix}}$$

(IV) Ternary semiprojections

$$s_0 = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&0\\1&1&1\\0&1&1\end{matrix}} \quad \boxed{\begin{matrix}2&0&2\\0&2&2\\2&2&2\end{matrix}}$$

$$s_{364} = \boxed{\begin{matrix}0&0&0\\0&0&1\\0&1&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \quad \boxed{\begin{matrix}2&1&2\\1&2&2\\2&2&2\end{matrix}}$$

$$s_{728} = \boxed{\begin{matrix}0&0&0\\0&0&2\\0&2&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&2\\1&1&1\\2&1&1\end{matrix}} \quad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_8 = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&0\\1&1&1\\0&1&1\end{matrix}} \quad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{368} = \boxed{\begin{matrix}0&0&0\\0&0&1\\0&1&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \quad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{80} = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&2\\1&1&1\\2&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{36} = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&0&2\\0&2&2\\2&2&2\end{matrix}}$$

$$s_{40} = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&1&2\\1&2&2\\2&2&2\end{matrix}}$$

$$s_{692} = \boxed{\begin{matrix}0&0&0\\0&0&2\\0&2&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{26} = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&0\\1&1&1\\2&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{449} = \boxed{\begin{matrix}0&0&0\\0&0&1\\0&2&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}}$$

$$s_{37} = \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} \qquad \boxed{\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}} \qquad \boxed{\begin{matrix}2&0&2\\1&2&2\\2&2&2\end{matrix}}$$

$$s_{76} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}} \quad \boxed{\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}} \quad \boxed{\begin{array}{ccc} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$s_{684} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{array}} \quad \boxed{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}} \quad \boxed{\begin{array}{ccc} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$s_{332} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}} \quad \boxed{\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}} \quad \boxed{\begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$s_{424} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{array}} \quad \boxed{\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}} \quad \boxed{\begin{array}{ccc} 2 & 0 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{array}}$$

§2 Tools of Our Research

◇ Polynomials over a Finite Field ◇

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$\diamond$ Polynomials over a Finite Field $\diamond$

Let $k\,(>1)$ be a power of a prime.
We consider the base set

$$E_k = \{0, 1, \ldots, k-1\}$$

as a finite field (Galois field) GF(k), and
express a function defined on E_k as a polynomial over GF(k).

QUIZ (Easy)

QUIZ (**Easy**)

Given polynomials f, g over GF(2)

(i.e., f, g : Boolean function)

$$f(x,\, y) \;=\; x\, y + 1$$

$$g(x,\, y) \;=\; x\, y + x + y$$

QUIZ **(Easy)**

Given polynomials f, g over GF(2)

(i.e., f, g : Boolean function)

$$f(x,\, y) \;=\; x\,y + 1$$

$$g(x,\, y) \;=\; x\,y + x + y$$

QUESTION :

Which is weaker, f or g ?

(with respect to the productive power of functions)

ANSWER :

g **is much weaker than** f

ANSWER :

g **is much weaker than** f

Because :

$$f(x,\, y) \,=\, \mathrm{NAND}\,(x,\, y)$$

$$g(x,\, y) \,=\, \mathrm{OR}\,(x,\, y)$$

QUIZ (Hard)

QUIZ **(Hard)**

Given polynomials u, v, w over GF(3)

$$u(x,y) = x^2y^2 + xy^2 + x^2y + 2xy + x + y$$

$$v(x,y) = x^2y^2 + xy^2 + x^2y + xy + x + y$$

$$w(x,y) = x^2y^2 + xy^2 + x^2y + 2xy + x + y + 1$$

QUIZ (**Hard**)

Given polynomials u **,** v **,** w **over GF(3)**

$$u(x,y) = x^2y^2 + xy^2 + x^2y + 2xy + x + y$$

$$v(x,y) = x^2y^2 + xy^2 + x^2y + xy + x + y$$

$$w(x,y) = x^2y^2 + xy^2 + x^2y + 2xy + x + y + 1$$

QUESTION :

Which is the weakest among u **,** v **and** w **?**

(with respect to the productive power of functions)

ANSWER :

u **is the weakest.**

ANSWER :

u **is the weakest.**

Because :

(1) $u(x, y)$ **generates a minimal clone.**

(3) $w(x, y)$ **is "Webb function" which generates all functons.**

(2) $v(x, y)$ **is inbetween.**

Our Hope !!

♣ . . . . ♢

To characterize polynomials generating minimal clones in terms of the "form of polynomials"

or

To discover "principles" or "rules" that are common to polynomials generating minimal clones

♡ . . . . ♠

§3 Basic Facts on Minimal Clones

Lemma

A minimal clone is generated by a single function,

i.e., for any minimal clone $C \in \mathcal{L}_k$ there exists $f \in \mathcal{O}_k$ such that $C = \langle f \rangle$

Proposition

For any $f \in \mathcal{O}_k$, suppose f satisfies the following :

(i) $f \notin \mathcal{J}_k$

(ii) **For** $\forall\, h \in \langle f \rangle$ **, if** $h \notin \mathcal{J}_k$ **then** $f \in \langle h \rangle$

Then f generates a minimal clone.

Type Theorem for Minimal Clones

An function f on E_k is *minimal* if

(i) it generates a minimal clone, and

(ii) every function from $\langle f \rangle$ whose arity is smaller than the arity of f is a projection

Type Theorem for Minimal Clones

An function f on E_k is *minimal* if

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Theorem (I. G. Rosenberg, 1986 ; "Inspired by Csákány's work, ... ")

Every minimal function is classified in one of the following five types:

(1) **Unary function**

(2) **Idempotent binary function**

(3) **Majority function**

(4) **Semiprojection**

(5) **If $k = 2^m$, the ternary function $f(x, y, z) := x + y + z$ where $\langle E_k; + \rangle$ is an elementary 2-group**

Note :

$f(x_1, \ldots, x_n)$ **: idempotent**

$\Longleftrightarrow$

$f(x, \ldots, x) = x$ **for** $\forall\, x \in E_k$

♡ Generators of all minimal clones of type (2) over GF(3) ♡

(Originally from B. Csákány)

$$b_{11} = xy^2$$

$$b_{624} = 2x + 2y$$
$$b_{68} = 2x + 2xy^2$$
$$b_{0} = 2x^2y + 2xy^2$$

$$b_{449} = x + y + 2x^2y$$
$$b_{368} = x + y^2 + 2x^2y^2$$
$$b_{692} = x + 2y^2 + x^2y^2$$
$$b_{33} = x + 2x^2y + xy^2$$
$$b_{41} = x^2 + xy^2 + 2x^2y^2$$
$$b_{71} = 2x^2 + xy^2 + x^2y^2$$

$$b_{26} = 2x + x^2 + 2xy + 2x^2y$$
$$b_{37} = 2x + 2x^2 + xy + 2x^2y$$
$$b_{17} = 2x + x^2 + 2xy^2 + 2x^2y^2$$
$$b_{38} = 2x + 2x^2 + 2xy^2 + x^2y^2$$
$$b_{10} = xy + 2x^2y + 2xy^2 + 2x^2y^2$$

$$
\begin{aligned}
b_{20} &= 2xy + 2x^2y + 2xy^2 + x^2y^2 \\
b_{43} &= x + xy + 2x^2y + xy^2 + 2x^2y^2 \\
b_{53} &= x + 2xy + 2x^2y + xy^2 + x^2y^2 \\
b_{35} &= x + xy + x^2y + 2xy^2 + 2x^2y^2 \\
b_{42} &= x + 2xy + x^2y + 2xy^2 + x^2y^2 \\
b_{530} &= x + y + y^2 + 2x^2y + 2x^2y^2 \\
b_{125} &= x + y + 2y^2 + 2x^2y + x^2y^2 \\
b_{116} &= x + y + xy + 2y^2 + 2xy^2 \\
b_{528} &= x + y + 2xy + y^2 + 2xy^2 \\
b_{206} &= x + 2y + y^2 + x^2y + 2x^2y^2 \\
b_{287} &= x + 2y + 2y^2 + x^2y + x^2y^2 \\
b_{215} &= x + 2y + 2xy + y^2 + xy^2 \\
b_{286} &= x + 2y + xy + 2y^2 + xy^2 \\
b_{122} &= y + x^2 + 2y^2 + 2x^2y + xy^2 \\
b_{557} &= y + 2x^2 + y^2 + 2x^2y + xy^2 \\
b_{16} &= 2x + x^2 + xy + 2x^2y + x^2y^2 \\
b_{47} &= 2x + 2x^2 + 2xy + 2x^2y + 2x^2y^2 \\
b_{178} &= 2x + 2y + x^2 + xy + y^2 \\
b_{290} &= 2x + 2y + 2x^2 + 2xy + 2y^2 \\
b_{40} &= x^2 + xy + 2x^2y + 2xy^2 + x^2y^2
\end{aligned}
$$

$$
\begin{aligned}
b_{80} &= 2x^2 + 2xy + 2x^2y + 2xy^2 + 2x^2y^2 \\
b_{364} &= x^2 + xy + y^2 + 2x^2y + 2xy^2 \\
b_{728} &= 2x^2 + 2xy + 2y^2 + 2x^2y + 2xy^2
\end{aligned}
$$

$$
\begin{aligned}
b_{448} &= x + y + xy + x^2y + xy^2 + 2x^2y^2 \\
b_{458} &= x + y + 2xy + x^2y + xy^2 + x^2y^2 \\
b_{205} &= x + 2y + xy + y^2 + xy^2 + x^2y^2 \\
b_{296} &= x + 2y + 2xy + 2y^2 + xy^2 + 2x^2y^2 \\
b_{188} &= 2x + 2y + x^2 + 2xy + y^2 + 2x^2y^2 \\
b_{280} &= 2x + 2y + 2x^2 + xy + 2y^2 + x^2y^2 \\
b_{8} &= 2x + x^2 + xy + x^2y + xy^2 + x^2y^2 \\
b_{36} &= 2x + 2x^2 + 2xy + x^2y + xy^2 + 2x^2y^2
\end{aligned}
$$

$$
\begin{aligned}
b_{179} &= 2x + 2y + x^2 + y^2 + x^2y + 2xy^2 + x^2y^2 \\
b_{281} &= 2x + 2y + 2x^2 + 2y^2 + x^2y + 2xy^2 + 2x^2y^2
\end{aligned}
$$

§4 More Properties of Minimal Clones

For $f, g \in \mathcal{O}_k$ **we write**

$$f \to g \quad \text{if} \quad g \in \langle f \rangle$$

(Note : Binary relation $\to$ **on** $\mathcal{O}_k$ **is a quasi-order)**

Lemma

Let $f \in \mathcal{O}_k^{(2)}$ **be an essentially binary function. If** f **satisfies (1) and (2) then** f **is minimal.**

(1) f **is idempotent**

(2) **For any** $g \in \mathcal{O}_k^{(m)}$ **(** $m \geq 2$ **)**

if $f \to g$ **then** g **is a projection**

or

$g \to f$

Let $m \geq 3$ **and** $g \in \mathcal{O}_k^{(m)}$**. We say** g **is a** *quasi-projection* **if** g **becomes a projection whenever two arguments of** g **are identified, i.e., if** $g(x_1, \ldots, x_i, \ldots, x_i, \ldots, x_m)$ **is always a projection.**

Lemma

Let $f \in \mathcal{O}_k^{(2)}$ **be a binary idempotent function. Then**

f **is minimal** $\iff$ **(1) and (2) hold**

(1) **For** $\forall\, g \in \mathcal{O}_k^{(2)} \setminus \mathcal{J}_k$

if $f \to g$ **then** $g \to f$

(2) **For** $\forall\, m \geq 3$ **and** $\forall\, g \in \mathcal{O}_k^{(m)} \setminus \mathcal{J}_k$

if $f \to g$ **then** g **is not a quasi-projection.**

For $f \in \mathcal{O}_k^{(2)}$ **let** $\Gamma_f^{(x,y)}$ **be :**

$$\{\ f(f(x,y),x),\ \ f(f(x,y),y),\ \ f(x,f(x,y)),\ \ f(y,f(x,y)),\ \ f(f(x,y),f(y,x))\ \}$$

Then let $\Gamma_f \ = \ \Gamma_f^{(x,y)} \ \cup \ \Gamma_f^{(y,x)}$

<u>**Lemma**</u>

Let $f \in \mathcal{O}_k^{(2)}$ **be a binary idempotent function which is not a projection. Suppose that, for any** $\gamma \in \Gamma_f$**, one of the following holds:**

$$\gamma(x,y) \approx f(x,y) \quad \textbf{or} \quad \gamma(x,y) \approx f(y,x)$$

Then f **is minimal**

(Here, by $h_1(x,y) \approx h_2(x,y)$ **we mean** $h_1(x,y) = h_2(x,y)$ **for all** $(x,y) \in E_k^2$ **)**

§5 Polynomials generating Minimal Clones

Here we consider only

"Binary Idempotent Minimal Functions"

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"Binary Idempotent Minimal Functions"

Our Strategy :

Step 1 : Take arbitrary $f(x, y) \in \mathcal{O}_3^{(2)}$ **from Csákány's list.**

Step 2 : Search for a polynomial $g(x, y) \in \mathcal{O}_k^{(2)}$ **for** $k \geq 3$ **whose counterpart for** $k = 3$ **is** $f(x, y)$

Step 3 : Examine if g **is minimal.**

(1) Linear Polynomials

For $k = 3$, the binary linear polynomial

$$f(x, y) \ = \ 2x + 2y$$

is minimal.

For $k = 5$, the binary linear polynomial

$$f(x, y) \ = \ 2x + 4y$$

is minimal, whose Cayley table is :

$f(x, y) \ = \ 2x + 4y$

$x \backslash y$	0	1	2	3	4
0	**0**	**4**	**3**	**2**	**1**
1	**2**	**1**	**0**	**4**	**3**
2	**4**	**3**	**2**	**1**	**0**
3	**1**	**0**	**4**	**3**	**2**
4	**3**	**2**	**1**	**0**	**4**

Theorem (**Á. Szendrei**)

Let k **be a prime. Let** $f(x, y)$ **be a binary linear polynomial on** E_k

Then f **is** *minimal* **if and only if**

$$f(x, y) = ax + (k + 1 - a)y$$

for some $1 < a < k$

Moreover, all such linear polynomials generate the same minimal clone.

(2) Monomials

Consider $x^s\, y^t$ **for** $1 \leq s \leq t < k$

For $k = 3$**,** $f(x,y) \;=\; x\,y^2$ **is minimal.**

For $k = 5$**,**

$$f(x,y) \;=\; x\,y^4$$

is minimal whose Cayley table is :

$f(x,y) \;=\; x\,y^4$

$x \backslash y$	0	1	2	3	4
0	**0**	**0**	**0**	**0**	**0**
1	**0**	**1**	**1**	**1**	**1**
2	**0**	**2**	**2**	**2**	**2**
3	**0**	**3**	**3**	**3**	**3**
4	**0**	**4**	**4**	**4**	**4**

Theorem

Let k be a prime.

(1) $x\,y^{k-1}$ **is a minimal function.**

(2) **For any $1 < s < k-1$,** $x^s\,y^{k-s}$ **is not a minimal function.**

(3) More Examples

(3) More Examples

Recall !!

Our Strategy :

Step 1 : Take arbitrary $f(x,y) \in \mathcal{O}_3^{(2)}$ **from Csákány's list.**

Step 2 : Search for a polynomial $g(x,y) \in \mathcal{O}_k^{(2)}$ **for** $k \geq 3$ **whose counterpart for** $k = 3$ **is** $f(x,y)$

Step 3 : Examine if g **is minimal.**

(3) <u>More Examples</u>

(3-1) <u>Example A</u>

Step 1 :

$$k = 3 \textbf{ : } x + y + 2\,x\,y^2$$

(3) More Examples

(3-1) Example A

Step 1 :

$$k = 3 \textbf{ : } x + y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } \quad \textbf{? ? ?}$$

(3) More Examples

(3-1) Example A

Step 1 :

$$k = 3 \textbf{ : } x + y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } x + y + (k-1)\,x\,y^{k-1}$$

(3) More Examples

(3-1) Example A

Step 1 :

$$k = 3 \textbf{ : } x + y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } x + y + (k-1)\,x\,y^{k-1}$$

Example. $k = 5$ **:**

$$\underline{f(x,y) \;=\; x + y + 4\,x\,y^4}$$

$x \backslash y$	**0**	**1**	**2**	**3**	**4**
0	0	**1**	**2**	**3**	**4**
1	**1**	1	**2**	**3**	**4**
2	**2**	**1**	2	**3**	**4**
3	**3**	**1**	**2**	3	**4**
4	**4**	**1**	**2**	**3**	4

(3-2) <u>Example B</u>

Step 1 :

$$k = 3 \textbf{ : } x + 2y^2 + x^2\, y^2$$

(3-2) Example B

Step 1 :

$$k = 3 \textbf{ : } x + 2y^2 + x^2 y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } \quad \textbf{? ? ?}$$

(3-2) Example B

Step 1 :

$$k = 3 \textbf{ : } x + 2y^2 + x^2\, y^2$$

Step 2 :

$$k \geq 3 \textbf{ : }$$

$$x + (k-1)y^{k-1} + x^{k-1}\, y^{k-1}$$

(3-2) Example B

Step 1 :

$$k = 3 : \; x\,y^2 + 2\,x^2 + x^2\,y^2$$

Step 2 :

$$k \geq 3 :$$

$$x + (k-1)y^{k-1} + x^{k-1}\,y^{k-1}$$

Example. $k = 5$ **:**

$$\underline{f(x,y) \;=\; x + 4\,y^4 + x^4\,y^4}$$

$x \backslash y$	**0**	**1**	**2**	**3**	**4**
0	0	**4**	**4**	**4**	**4**
1	**1**	1	**1**	**1**	**1**
2	**2**	**2**	2	**2**	**2**
3	**3**	**3**	**3**	3	**3**
4	**4**	**4**	**4**	**4**	4

(3-3) Example C

Step 1 :

$$k = 3 \textbf{ : } x\,y^2 + 2\,x^2 + x^2\,y^2$$

(3-3) Example C

Step 1 :

$$k = 3 \textbf{ : } x\,y^2 + 2\,x^2 + x^2\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } \quad \textbf{? ? ?}$$

(3-3) Example C

Step 1 :

$$k = 3 \textbf{ : } x\,y^2 + 2\,x^2 + x^2\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : }$$

$$x\,y^{k-1} + (k-1)x^{k-1} + x^{k-1}\,y^{k-1}$$

(3-3) Example C

Step 1 :

$$k = 3 \textbf{ : } \; x\,y^2 + 2\,x^2 + x^2\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ :}$$

$$x\,y^{k-1} + (k-1)x^{k-1} + x^{k-1}\,y^{k-1}$$

Example. $k = 5$ **:**

$$\underline{f(x,y) \;=\; x\,y^4 + 4\,x^4 + x^4\,y^4}$$

$x \backslash y$	**0**	**1**	**2**	**3**	**4**
0	0	**0**	**0**	**0**	**0**
1	**4**	1	**1**	**1**	**1**
2	**4**	**2**	2	**2**	**2**
3	**4**	**3**	**3**	3	**3**
4	**4**	**4**	**4**	**4**	4

(3-4) Example D

A bit more difficult example !

(3-4) <u>Example D</u>

Step 1 :

$$k = 3\ \textbf{:}\ 2\,x + 2\,x\,y^2$$

(3-4) Example D

Step 1 :

$$k = 3 \textbf{ : } 2\,x + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } \quad \textbf{? ? ?}$$

(3-4) Example D

Step 1 :

$$k = 3 \textbf{ : } 2\,x + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : }$$

$$(k-1)\,x + (k-1)\,x\,y^{k-1}$$

is not good !

(3-4) Example D

Step 1 :

$$k = 3 \textbf{ : } 2\,x + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : }$$

$$(k-1)\,x + 2\,x\,y^{k-1}$$

(3-4) Example D

Step 1 :

$$k = 3 \textbf{ : } 2\,x + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : }$$

$$(k-1)\,x + 2\,x\,y^{k-1}$$

Example. $k = 5$ **:**

$$\underline{f(x,y) \;=\; 4\,x + 2\,x\,y^4}$$

$x \backslash y$	**0**	**1**	**2**	**3**	**4**
0	0	**0**	**0**	**0**	**0**
1	**4**	1	**1**	**1**	**1**
2	**3**	**2**	2	**2**	**2**
3	**2**	**3**	**3**	3	**3**
4	**1**	**4**	**4**	**4**	4

(3-5) Example E

This example requires better skill even to find a candidate of generalization !!

(3-5) <u>Example E</u>

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2y + 2\,x\,y^2$$

(3-5) <u>Example E</u>

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2 y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ : } \quad \textbf{? ? ?}$$

(3-5) <u>Example E</u>

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ :}$$

$$(k-1)\,x^{k-1}y + (k-1)\,x\,y^{k-1}$$

is not good !

(3-5) Example E

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2 y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ :}$$

$$(k-1)\,x^{k-1} y + (k-1)\,x\,y^{k-1}$$

is not good !

$$2\,x^{k-1} y + (k-1)\,x\,y^{k-1}$$

is also not good !!

(3-5) <u>Example E</u>

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2 y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ :}$$

$$(k-1)\sum_{i=1}^{k-1} x^{k-i}\,y^i$$

(3-5) <u>Example E</u>

Step 1 :

$k = 3$ **:** $2\,x^2 y + 2\,x\,y^2$

Step 2 :

$k \geq 3$ **:**

$$(k-1) \sum_{i=1}^{k-1} x^{k-i}\, y^i$$

Step 3 (Sketch of Proof):

◇ $f(x, x) = x$ **since** $(k-1)^2 = 1$

◇ **For** $x \neq y$**, let** $D = \Sigma_{i=1}^{k-1}\, x^{k-i}\, y^i$

Then

$$xy^{-1} D = D$$

Hence $xD = yD$ **which implies** $D = 0$

Therefore $f(x, y) = x$ **if** $x = y$

and $f(x, y) = 0$ **if** $x \neq y$

It is easy to see that f **is minimal**

(3-5) Example E

Step 1 :

$$k = 3 \textbf{ : } 2\,x^2 y + 2\,x\,y^2$$

Step 2 :

$$k \geq 3 \textbf{ :}$$

$$(k-1) \sum_{i=1}^{k-1} x^{k-i}\, y^i$$

Example. $k = 5$ **:**

$$f(x,y) \;=\; 4\,x^4\,y + 4\,x^3 y^2 + 4\,x^2 y^3 + 4\,x\,y^4$$

$x \backslash y$	**0**	**1**	**2**	**3**	**4**
0	0	**0**	**0**	**0**	**0**
1	**0**	1	**0**	**0**	**0**
2	**0**	**0**	2	**0**	**0**
3	**0**	**0**	**0**	3	**0**
4	**0**	**0**	**0**	**0**	4

§ ∞ More and More Examples

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You will hear at the conference celebrating the 80th Birthday of Béla Csákány !!

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Thank you