Generalized Associative Spectra

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Outline

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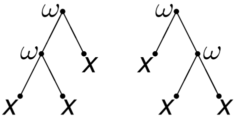
Introductory Example

$p = 2$:

$((xx)x)$ $(x(xx))$



Introductory Example

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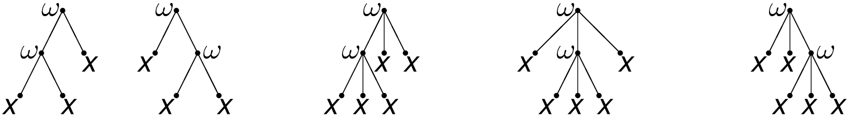
$p = 3$: $((xxx)xx)$ $(x(xxx)x)$ $(xx(xxx))$



Definition of Bracketings

Definition (Bracketings)

Term algebra: $\boldsymbol{T^{(p)}} := \left(T_\omega(x), \omega^{\boldsymbol{T^{(p)}}} \right)$ with

- $p \in \mathbb{N}_{\geq 2}$
- alphabet $\{x\}$
- signature $\{\omega\}$, ω p-ary operation symbol

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Definition (Occurence number)

The occurence number $|t|_\omega$ of a bracketing $t \in T_\omega(x)$ is the number of occurences of the symbol ω in t.

$$\Longrightarrow |x|_\omega = 0, \qquad |\omega t_1 t_2 \ldots t_p|_\omega = 1 + \sum_{k=1}^{p} |t_k|_\omega$$

Definition of Bracketings - 2

Notation

The set of bracketings with occurence number n:

$$B_n^{(p)} := \{t \in T_\omega(x) \mid |t|_\omega = n\}$$

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Example

- $B_0^{(2)} = \{x\}$
- $B_1^{(2)} = \{\omega xx\} = \{(xx)\}$
- $B_2^{(2)} = \{\omega\omega xxx, \omega x\omega xx\} = \{((xx)x), (x(xx))\}$

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Definition of p-ary Groupoids and Regular Operations

Definition (p-ary groupoid)

$\boldsymbol{G} = \langle G, f \rangle$ p-ary groupoid $:\Longleftrightarrow f : G^p \longrightarrow G$ p-ary operation

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For a bracketing $t \in T_\omega(x)$, ε enumerates the symbols x in t beginning with 1.

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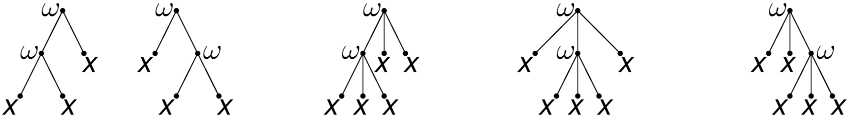

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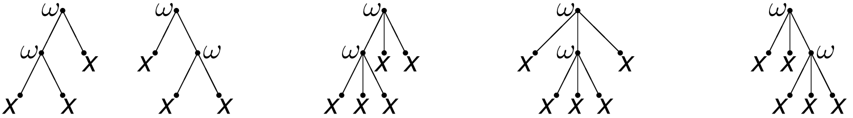

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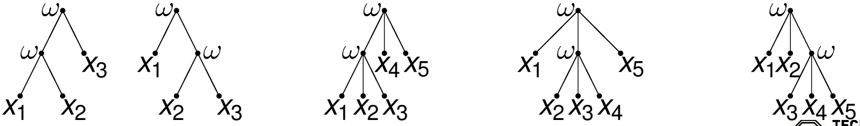
$((x_1x_2)x_3)$ $(x_1(x_2x_3))$ $((x_1x_2x_3)x_4x_5)$ $(x_1(x_2x_3x_4)x_5)$ $(x_1x_2(x_3x_4x_5))$



Definition of p-ary Groupoids and Regular Operations

Definition (p-ary groupoid)

$\boldsymbol{G} = \langle G, f \rangle$ p-ary groupoid $:\Longleftrightarrow f : G^p \longrightarrow G$ p-ary operation

Definition (Enumeration)

For a bracketing $t \in T_\omega(x)$, ε enumerates the symbols x in t beginning with 1.

Definition (Regular operation)

For a bracketing $t \in T_\omega(x)$ and a p-ary groupoid $\boldsymbol{G}$, the regular operation $t^{\varepsilon;\boldsymbol{G}}$ is the term operation of $\varepsilon(t)$.

Outline

Definition of the Generalized Associative Spectrum

Example

$$
\begin{aligned}
((xx)x)^{\varepsilon;\langle\mathbb{R},-\rangle}(a_1,a_2,a_3) &= (a_1-a_2)-a_3 \\
(x(xx))^{\varepsilon;\langle\mathbb{R},-\rangle}(a_1,a_2,a_3) &= a_1-(a_2-a_3)
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Definition (Associative spectrum)

For a p-ary groupoid $\boldsymbol{G}$, the n-th element of the associative spectrum of $\boldsymbol{G}$ is the number of different regular operations of bracketings of occurence number n:

$$s_{\boldsymbol{G}}(n) := \left| \left\{ t^{\varepsilon;\boldsymbol{G}} \;\middle|\; t \in B_n^{(p)} \right\} \right|.$$

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General Associative Law

Example

$$\neq \begin{cases} ((xx)x)^{\varepsilon;\langle\mathbb{R},-\rangle}(a_1,a_2,a_3) &= (a_1-a_2)-a_3, \\ (x(xx))^{\varepsilon;\langle\mathbb{R},-\rangle}(a_1,a_2,a_3) &= a_1-(a_2-a_3) \end{cases}$$

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Proposition (General associative law)

For a p-ary groupoid $\boldsymbol{G}$ it holds:

- $\boldsymbol{G}$ is associative (i.e. $s_{\boldsymbol{G}}(2) = 1$) $\iff \forall n \in \mathbb{N}: \ s_{\boldsymbol{G}}(n) = 1$

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- $\boldsymbol{G}$ is associative (i.e. $s_{\boldsymbol{G}}(2)=1$) $\iff \forall n\in\mathbb{N}:\ s_{\boldsymbol{G}}(n)=1$
- $s_{\boldsymbol{G}}(n)=1$ for $n\in\mathbb{N}_{\geq 2} \implies \forall m\in\mathbb{N}, m\geq n:\ s_{\boldsymbol{G}}(m)=1$

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Estimations of the Generalized Associative Spectrum

Definition (Generalized CATALAN numbers)

Generalized CATALAN numbers $\left(C_n^{(p)}\right)_{n\in\mathbb{N}}$:

$$C_n^{(p)} := \frac{1}{(p-1)\cdot n+1} \cdot \binom{p\cdot n}{n}$$

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Proposition

For a p-ary groupoid $\boldsymbol{G}$ it holds:

- $\forall n \in \mathbb{N}:\ 1 \leq s_{\boldsymbol{G}}(n) \leq C_n^{(p)}$
- $\forall n \in \mathbb{N}_{>0}:\ s_{\boldsymbol{G}}(n) \leq \sum_{i:\{1,\dots,p\}\to\mathbb{N},\ \sum_{k=1}^{p} i(k)=n-1} \left(\prod_{j=1}^{p} s_{\boldsymbol{G}}(i(j))\right).$

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Substructures, Homomorphic Images and Isomorphisms

Proposition

For two p-ary groupoids $\boldsymbol{G}$, $\boldsymbol{H}$ it holds:

- if $\boldsymbol{H}$ and $\boldsymbol{G}$ are isomorphic or antiisomorphic then

$$\forall n \in \mathbb{N} : \ s_{\boldsymbol{H}}(n) = s_{\boldsymbol{G}}(n).$$

Remark (antiisomorphic)

$$\varphi\left(f_{\boldsymbol{H}}\left(g_1, \ldots, g_p\right)\right) = f_{\boldsymbol{G}}\left(\varphi(g_p), \ldots, \varphi(g_1)\right)$$

Substructures, Homomorphic Images and Isomorphisms

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- if $\boldsymbol{H}$ and $\boldsymbol{G}$ are isomorphic or antiisomorphic then

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- if $\boldsymbol{H}$ is a subgroupoid or homorphic image of $\boldsymbol{G}$ then

$$\forall n \in \mathbb{N}: \; s_{\boldsymbol{H}}(n) \leq s_{\boldsymbol{G}}(n).$$

Remark (antiisomorphic)

$$\varphi\left(f_{\boldsymbol{H}}\left(g_1, \ldots, g_p\right)\right) = f_{\boldsymbol{G}}\left(\varphi(g_p), \ldots, \varphi(g_1)\right)$$

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A Binary Example with a Quadratic Spectrum

Example

The associative spectrum of the groupoid $\boldsymbol{G} := \langle \mathbb{Z}_6[Y], \oplus \rangle$ with the operation

$$\begin{array}{rccc} \oplus : & (\mathbb{Z}_6[Y])^2 & \longrightarrow & \mathbb{Z}_6[Y] \\ & (X_1, X_2) & \longmapsto & 3Y \cdot X_1 + 2Y \cdot X_2 \end{array}$$

is quadratic:

$$\forall n \in \mathbb{N}_{\geq 2} : \; s_{\boldsymbol{G}}(n) = \frac{n^2 + n - 2}{2}.$$

(The operations occurring in the definition of $\oplus$ are addition and multiplication in the polynomial ring $\langle \mathbb{Z}_6[Y], +, \cdot \rangle$.)

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Congruence Relations with the Invariance Property

Proposition

For a p-ary groupoid $\boldsymbol{G}$ the bracketing congruence

$$Id_{\boldsymbol{G}} := \left\{ (s,t) \in (T_\omega(x))^2 \;\middle|\; s^{\varepsilon;\boldsymbol{G}} = t^{\varepsilon;\boldsymbol{G}} \right\}$$

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$$Id_{\mathbf{G}} := \left\{ (s,t) \in (T_\omega(x))^2 \;\middle|\; s^{\varepsilon;\mathbf{G}} = t^{\varepsilon;\mathbf{G}} \right\}$$

is a congruence relation in $\mathbf{T^{(p)}}$ with the invariance property:

- $\forall (s,t) \in Id_{\mathbf{G}} : |s|_\omega = |t|_\omega$
- $\forall (s,t) \in Id_{\mathbf{G}}\ \forall t_1, \ldots, t_k \in T_\omega(x)$:

$$\left(s^{\varepsilon;\mathbf{T^{(p)}}}(t_1, \ldots, t_k), t^{\varepsilon;\mathbf{T^{(p)}}}(t_1, \ldots, t_k) \right) \in Id_{\mathbf{G}}$$

where k is the number of symbols x in s (and t).

Congruence Relations with the Invariance Property Are All You Need

Remark

For a p-ary groupoid $\boldsymbol{G}$ it holds:

$$\forall n \in \mathbb{N}: \; s_{\boldsymbol{G}}(n) = \left| \left\{ [t]_{Id_{\boldsymbol{G}}} \in T_\omega(x)_{/Id_{\boldsymbol{G}}} \;\middle|\; t \in B_n^{(p)} \right\} \right| .$$

Congruence Relations with the Invariance Property Are All You Need

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Theorem (No need for groupoids)

For a congruence relation Σ in $\mathbf{T}^{(p)}$ with the invariance property there exists a p-ary groupoid $\mathbf{G}$ with:

$$\Sigma = Id_{\mathbf{G}}.$$

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Examples with Polynomial Spectra of Arbitrary Degree

Theorem

With the concept of the congruence relations with the invariance property it is possible to find the following spectra:

$$s_k^{(p)}(n) = \begin{cases} C_n^{(p)} & \text{for } n < k \\ \dfrac{(p-1)\cdot(n-k)+1}{k!} \cdot \displaystyle\prod_{\ell=1}^{k-1}((p-1)\cdot n + k + 1 - \ell) & \text{for } n \geq k \end{cases}$$

with $k \in \mathbb{N}, k \geq 1$.

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Summary

- Associative spectra can be generalized to p-ary operations in a natural way.
- The concept of the congruence relations with the invariance property is a powerful tool to study associative spectra.

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Thank you

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